Measure & Integration
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Lecture - 04 A
Monotone Class

Welcome to lecture 4 on measure and integration. I recall we have been looking at classes of subsets of a set X with various properties; we will started with the collection called is semi algebra or subsets of a set X, then we looked at what is called the algebra of subsets of a set X and today, we will start with looking at some more classes of subsets of set X.

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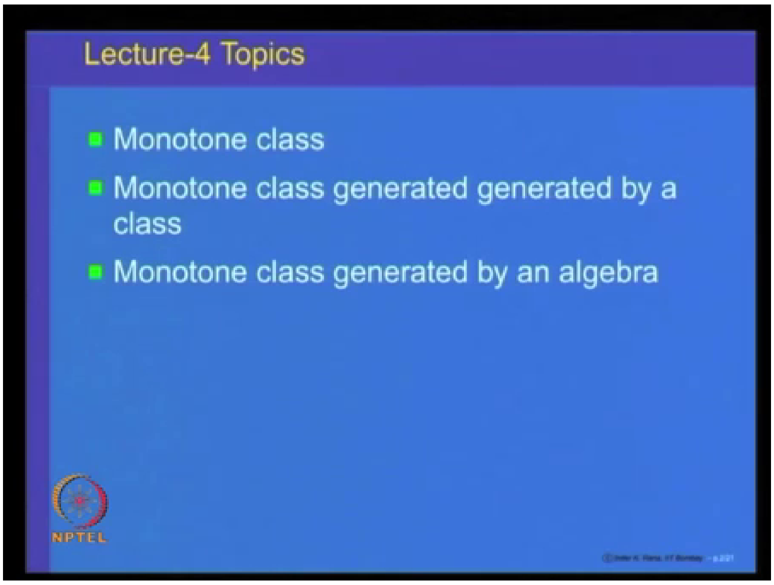

So, we will start with what is called a monotone class and then we look at the monotone class generated by a collection of subsets of a set X and then go over to describe; what is the monotone class generated by an algebra and that is an important relation which will be using again and again.

So, let us start with describing; what is a monotone class a monotone class is a collection of subsets of a set X.

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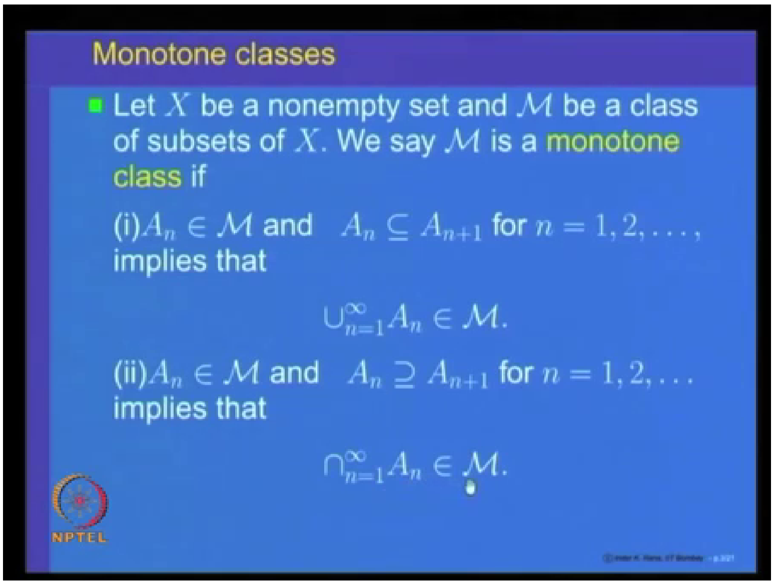

So, let us denote that collection of subsets by M. So, M is a collection of subsets of X and we say it is a monotone class, if it has the following 2 properties; one whenever there is a sequence of sets A n belonging to M the collection M such that the sequence is increasing; that means, for every n, A n is a subset of A n plus 1, then we demand that the union of these sets A ns also belongs to M. So, the first property is that the collection M of subsets of set X is closed under unions of increasing sequences and the second property that we expect from this collection is that whenever a sequence A n is in M and A n is decreasing; that means, A n is a subset of A n plus 1 for every n, then the intersection of the sequence of sets A n should also an element of n.

So, let us recall once again; what is a monotone class a monotone class is a collection of subsets of a set X with 2 properties one for every sequence of sets A n in M such that A ns is increasing their union also belongs to M and secondly, whenever A n is a sequence of sets in M such that A n is decreasing then the intersection of the sets is also in M.

So, that is why the name monotone comes; that means, now this class M of subsets of M is closed under monotone sequences whenever a sequence A n is increasing in M their union belongs to M and whenever a sequence A n is decreasing in M, then their intersection also belongs to M. So, such a collection of subsets of a set X is called a monotone class let us look at some examples of such collections firstly.

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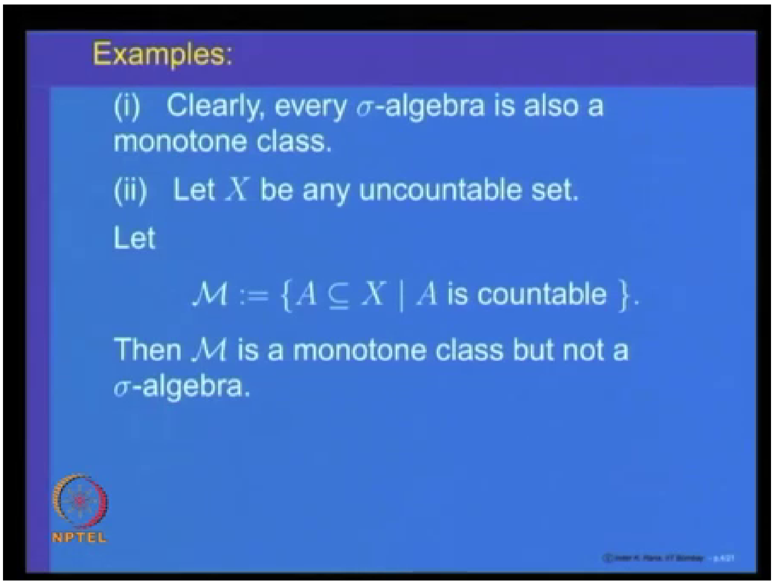

Let us observe every sigma algebra is also a monotone class why is that true because a sigma algebra is a collection of subsets of X which is closed under any countable unions because it is closed under countable unions. So, it will also be closed under increasing unions. So, first property will be true and. Secondly, if a sequence A n is decreasing in a sigma algebra, then look at the compliments of that sequence. So, that sequence of compliments of the sets will be an increasing sequence of sets and because it is a sigma algebra, it is also closed under compliments.

So, A n compliments will belong to it. So, union of A n compliments to it and; that means, the intersections of A ns compliment belong to it and; that means, intersection of A ns belong to it.

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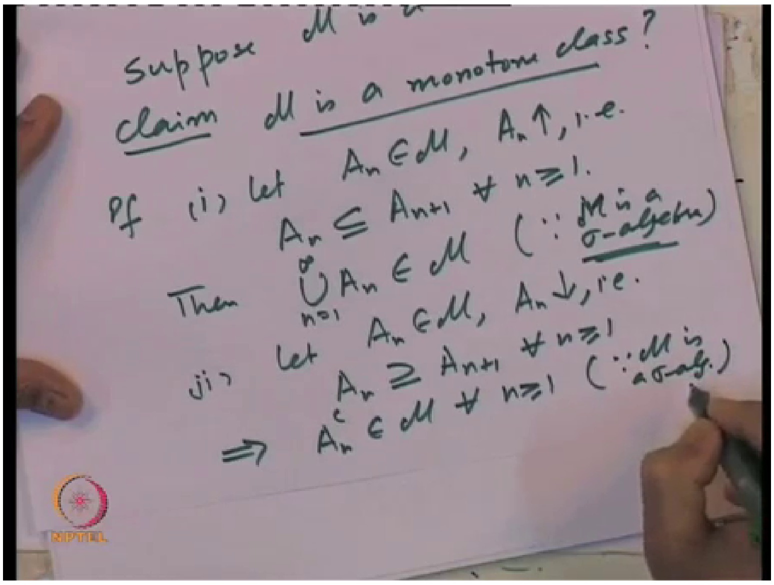

So, let us look at this property how do we write it. So, let us look at suppose M is a sigma algebra claim M is monotone class. So, to prove this how do we go ahead. So, one let A n belong to M and A ns increase. So, that is A n is a subset of A n plus 1 for every n bigger than or equal to 1, then union of A ns n equal to 1 to infinity belong to M because M is a sigma algebra because it is a sigma algebra it is closed under all unions and hence such types also and secondly, let us take let A n belong to M and A ns decrease. So, that is A n includes A n plus 1 for every n bigger than or equal to 1.

So, in that case; so, that implies because A ns belong to M and M is. So, A n compliments belong to M for every A n compliments belong to M for every n bigger than or equal to 1 because M is a sigma algebra because it is a sigma algebra; it is closed under compliments.

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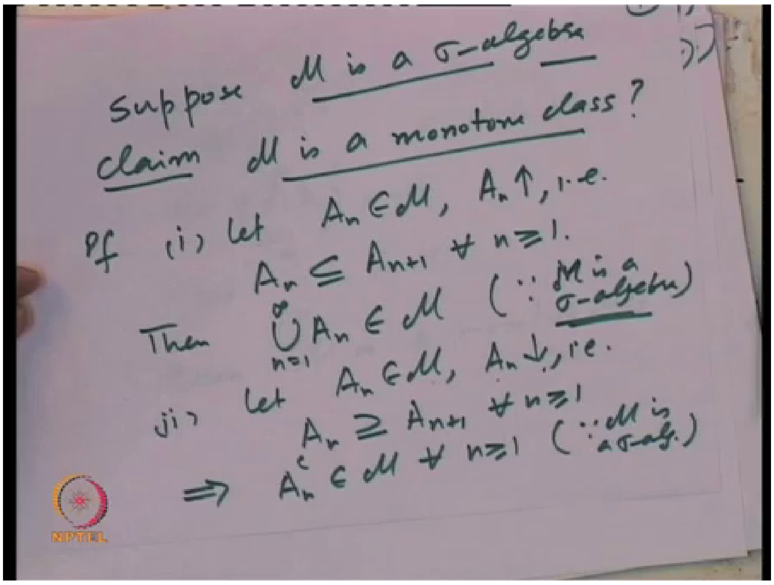

So, once that is true. So, that implies that A n compliment union n equal to 1 to infinity belongs to M because A n decreasing implies the sequence of A n compliments will be increasing and just now we saw that whenever a sequence is increasing their union belongs to m, but that implies, but what is the set that is intersection of n equal to 1 to infinity A n compliment that is by de Morgan's law. So, this belongs to M and now M is a sigma algebra. So, that implies the intersection n equal to 1 to infinity A n belongs to M. So, we have shown whenever a sequence A n is in M and A ns are decreasing that implies the intersection also belongs to M. So, hence M is a monotone M is a monotone class. So, the first proposition first observation is that every sigma algebra is also a monotone class; let us go on to some more properties more examples.

Let X be any uncountable set and let us look at the collection of all subsets A of X such that A is a countable set the claim is that this collection M is a monotone class, but it is not a sigma algebra.

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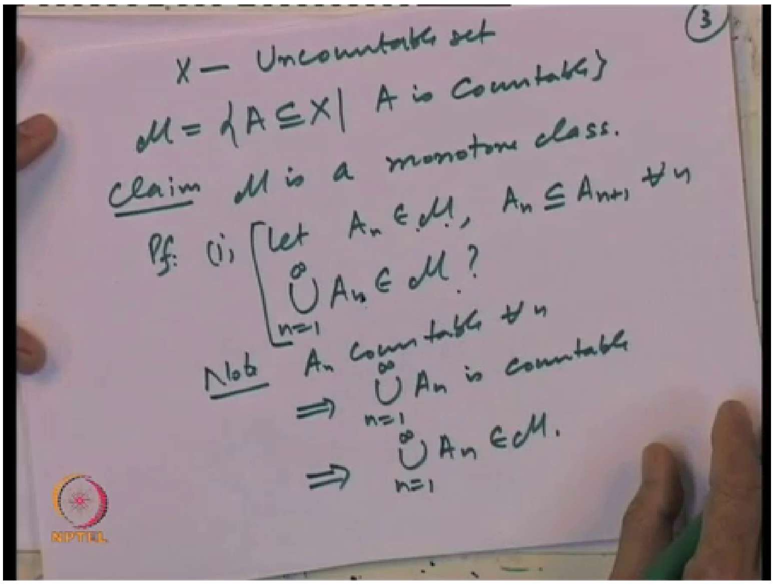

So, let us look at M X is uncountable and we are looking at the collection M of all those subsets of X such that A is such that a is countable. So, claim M is a monotone class. So, let us see; how do we prove it. So, first property; so, let us take a sequence A n belonging to M A ns increasing A n subset of A n plus 1 for every n.

So, what we have to prove we have to check that union of A ns n equal to 1 to infinity also belongs to M that is what we have to check, but let us now note to check that the union belongs to M, we have to show it is a countable set. Now each A n is given to be an element of m; that means, each A n is countable A n countable for every n implies union A n is also countable. So, here we are using a fact that the countable union of countable sets is again a countable set and hence this implies that union A ns n equal to 1 to infinity belongs to M. So, the first property we have checked is that if A ns belong to M and A ns is increasing sequence then the union A n belongs to M.

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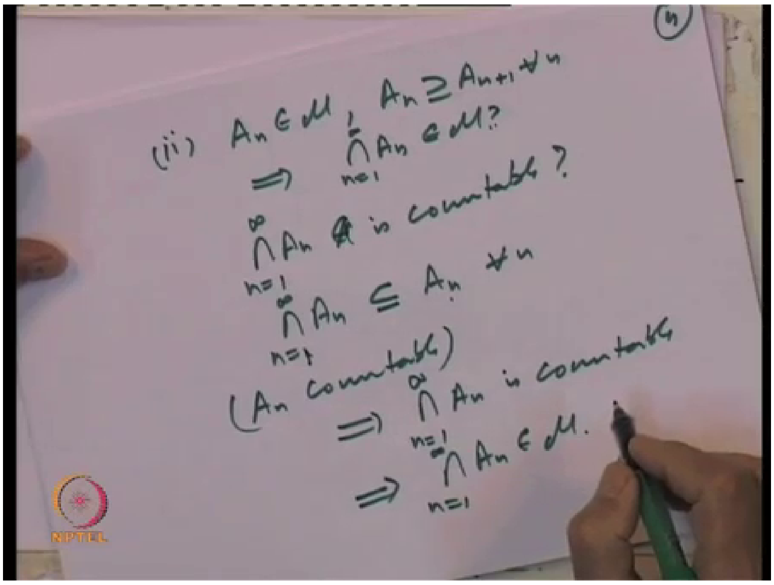

Let us check the second property namely. So, let us check the second property that A n belonging to M A ns decreasing for every n should imply that the intersection A ns belongs to M. So, what we have to check we have to check that intersections A n n equal to 1 to infinity belongs to m; that means, is countable. So, that is what we have to check, but let us observe that intersection A ns this is a subset of A n for every n because it is intersection and each A n is countable A n countable implies and this is the subset of it. So, intersection A n is countable and hence implies intersection A n belongs to M. So, we have shown that if you take the collection of if X is a countable or uncountable set does not matter actually it. So, far; what we have we not used the fact that it is uncountable set if M is a set which is if M is the collection of all countable subsets of a set X then that forms a monotone class why it is not A.

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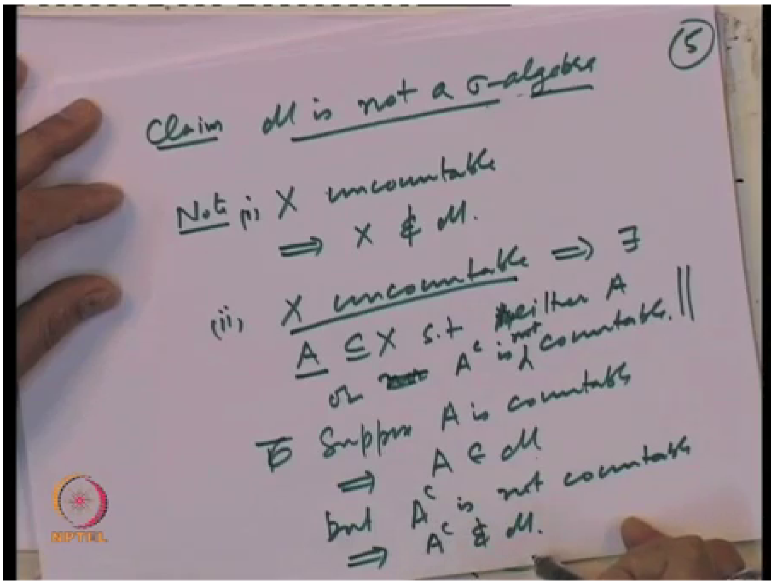

So, claim. So, finally, we want to prove that M is not a sigma algebra. Let us observe a few things here. So, note here we will be using X uncountable implies first of all X does not belong to X does not belong to M. So, the very first property of a collection being a sigma algebra namely the whole space belong to it is violated because X is not countable it is uncountable another way of looking at this is the following. So, this is one observation. Secondly, X uncountable implies there exists a subset A in X such that neither a nor a compliment is countable and that is obvious.

Because if let us see is that X is a uncountable set then the claim that there exists. Now this is let us; this is not required and this may not be true let us take a subset A of X such that either a or a compliment is not countable and that is possible because if for every set a and a compliment are countable then X will be a countable set. So, let us choose a set A such that either A or a compliment is not countable then suppose A is countable a is countable. So, that will imply that A belongs to M.

But a compliment is not countable and that implies a compliment does not belong to M. So, when X is uncountable you can have. In fact, for every set which is countable a will belong to M, but compliment will not belong to M. So, this collection M is not also is also not going to be closed under compliments. So, X does not belong to it and that it is not going to be closed under compliments. So, and that is because X is uncountable. So, when X is uncountable collection M of all countable sets of it; it is a monotone class, but it is not a sigma algebra. So, every sigma algebra is a monotone class, but the converse need not be true. So, this is what we have shown just now that if M is a monotone class every monotone class is sigma algebra, but there exists the examples of sigma algebras examples of monotone class is which are not sigma algebras.

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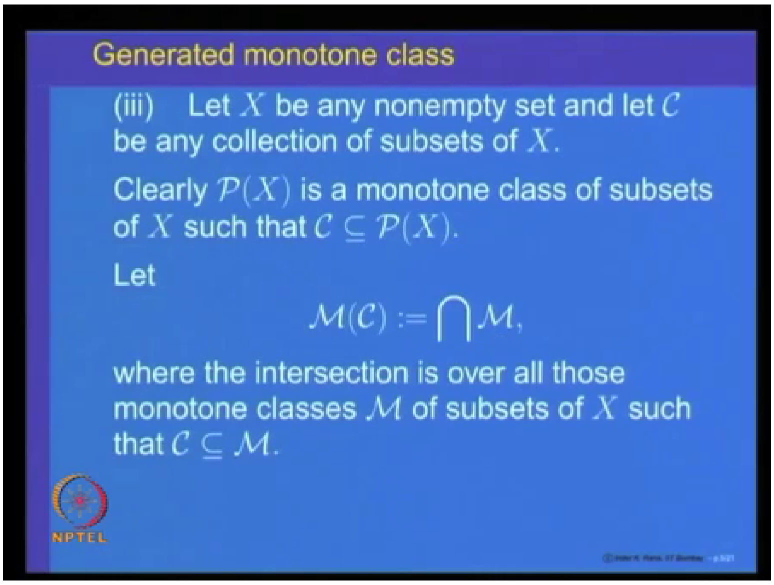

Let us look at the next scenario. Let us start with a collection C of subsets of set X C is any collection it may or may not be a monotone class. So, in we would like to find a monotone class of subsets of X which includes C and this is smallest. So, first of all let us observe that all subsets of the set; X is a monotone class of subsets of X and C is because it is a subset is a sub collection. So, C is sub collection. So, given any collection of subsets of a set X C it is always included in a collection namely the power set of X which is monotone class. So, given a collection there always exists a monotone class of subsets of X including it, but this is too large we want to have the smallest monotone class including C weather such a thing exists or not.

So, that the proof is something similar to what we have shown is algebra generated by a class the sigma algebra generated by a class. So, let us look at M of C. So, this is the notation for the intersection of all monotone classes M of subsets of X which include C. So, look at the collection of all monotone classes of subsets of X which include C and take their intersection and call this as M of C. So, what we want to prove is that M of C is a monotone class it includes C and it is the smallest.

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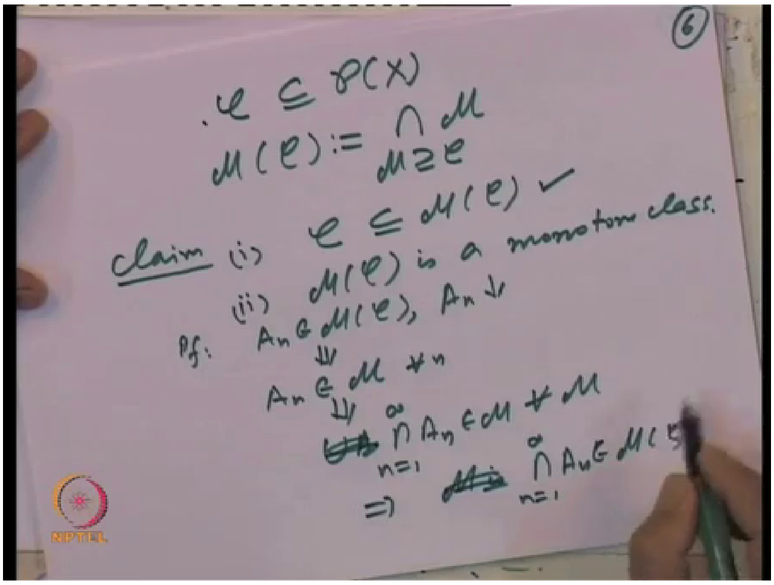

So, let us look at a proof of that; so, C is any collection of subsets of set X M of C is the intersection of all monotone classes M, m including C claim 1 C is inside M of C that is obvious because C is inside every collection M and M of C is the intersection. So, this property is obvious second claim that M of C is a monotone class and the proof is goes on the same lines as that of algebra and semi algebra and sigma algebra. So, to prove this let us look at A ns belong to M of C A ns decreasing, but this implies that each A n also belongs to M and also belongs to M for every collection M which include C for every n and that implies that union of decreasing.

So, let us we want to show that intersection A ns n equal to 1 to infinity belong to M for every M and that implies that M is every M and hence that implies that intersection A ns belong to M of C. So, essentially saying that if A n is sequence in M of C which is decreasing, then this is also a sequence which is decreasing in each M and hence the intersection belongs to each M and hence belongs to the intersection M of C.

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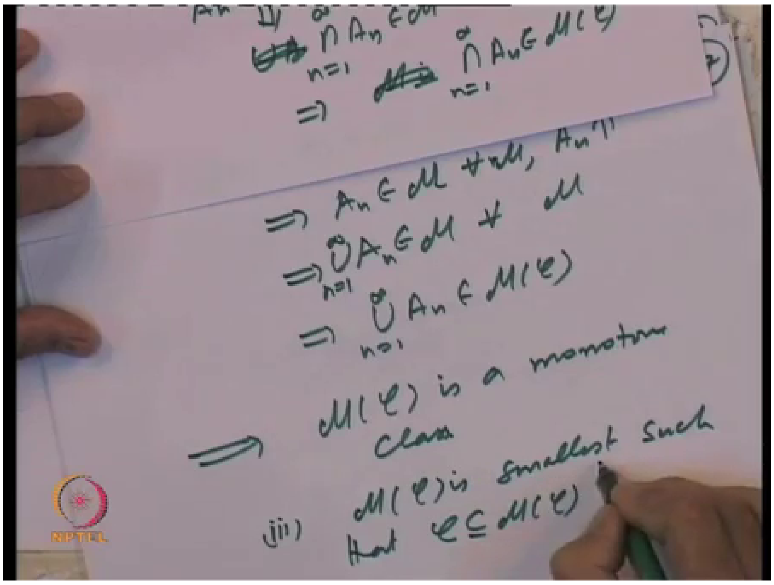

And the same proof similar proof X for the union. So, let us look at second part; A ns belongs to M of C and A ns increasing, but; that means, implies A ns belong to M for every M for every M and A n is increasing; that means, A ns union because each M is a monotone class. So, this belongs to M the union belongs to M for every M and that implies that the union A ns n equal to 1 to infinity belongs to M of C and that implies. So, hence M of C is a monotone class. So, it is a monotone class that includes the collection C and third M of C is smallest such that C is M of C smallest monotone class and that is obvious because it is the intersection of all monotone classes so; obviously, it is going to be the smallest. So, what we have shown is that given a collection C of subset of a set of X there exist a monotone class M of C which includes C and which is the smallest.

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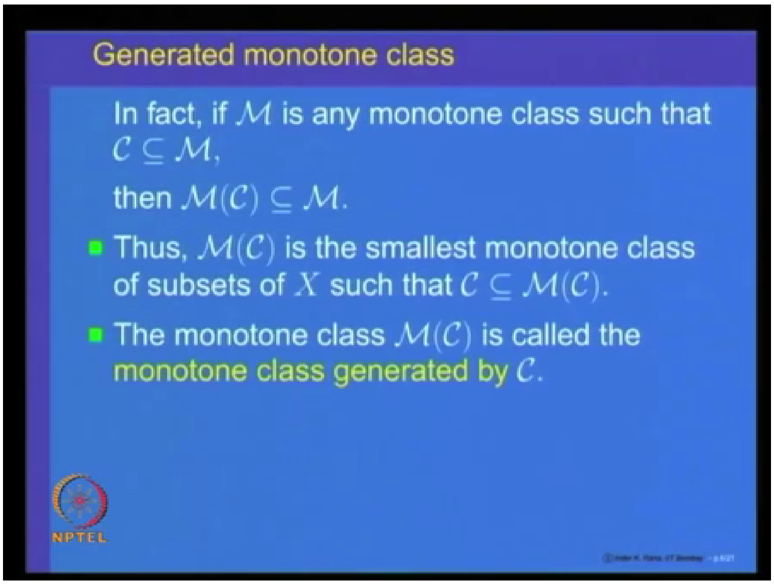

So, this monotone class is called. So, this is what we have proved. So, thus M of C is the smallest monotone class of subsets of X. So, is that C is inside M of C and this collection is called the monotone class generated by C. So, every collection C has got the smallest monotone class in which includes it. So, that is called the monotone class generated by it. So, given a collection C of subsets of a set X we are able to generate an algebra out of it we are able to generate a monotone class. So, out of it we are able to generate a sigma out of it and the next question that we want to analyze is; what is the relation between these collections.

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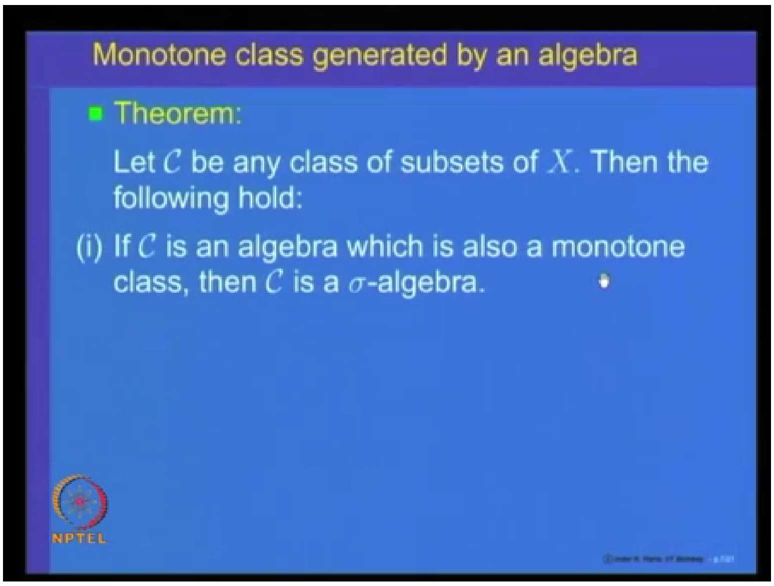

So, we want to prove theorem which relates these concepts. So, first of all let us start with any collection of subsets of a set X then the first observation is if C is an algebra which is also a monotone class then C is a sigma algebra.

So, let us first prove this fact that if C is C.

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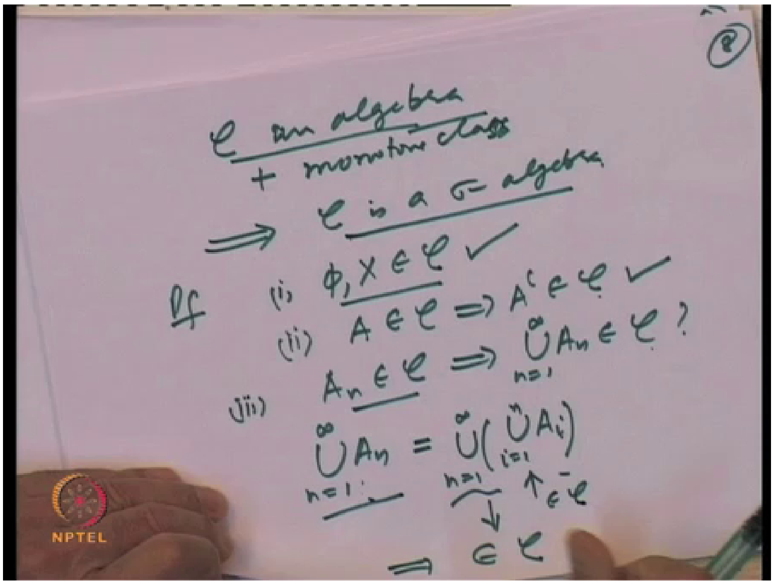

So, we want to prove an algebra plus monotone class implies C is a sigma algebra. So, proof; what we have to prove; to prove C is a sigma algebra wait prove first empty set the whole space belongs to C that is true because C is an algebra. So, this property is true second we should show that if a set a belongs to C implies a complement belongs to C and that is again obvious because it is the collection C is a algebra. So, these 2 properties are true the third property is only property to be checked that if A n belong to C then that should be imply union of A ns n equal to 1 to infinity also belongs to C. So, C is closed under countable unions that is what we have to prove and what we are given is C is a monotone class.

So, a monotone class is a collection which is closed under only increasing and decreasing. So, let us look at union of A ns can we represent this as a union of increasing sets well the one possibility is let us take union of A is i equal to 1 to n, then this collection as n increases will be an increasing sequence and their union n equal to 1 to infinity will be an increasing union will be a union of increasing sequence of sets which are namely union A is and now observe that each one of the sets union a i i equal to 1 to n that is a finite union of elements in the algebra A ns belongs to C and that is an algebra so; that means, each one of them belong to C. So, we have written the union A ns as union of sets in C and this is the increasing sequence and C is a monotone class. So, that implies that this right hand side this set belongs to C.

So, note we have used we have represented any union as a increasing union of increasing sequence of sets and each set here is a finite union of elements of the algebra C. So, hence it belongs to it. So, this becomes a increasing union of sets and hence because it is a monotone class this belongs to C. So, every A n belonging to C implies the union one to infinity also belongs. So, C is. In fact, it closed under countable unions. So, it becomes a sigma algebra. So, this proves the first property namely if C is an algebra which is also a monotone class then it is a sigma algebra.

Let us look at the next property that if C is any collection then C is content in M of C right that is obvious because M of C is the smallest monotone class including C. So, this is obvious and now C is also subsets of S of C because S of C is the sigma algebra generated by it. So, C is inside S of C and just now we proved S of C is also a monotone class. So, S of C is a monotone class including C and hence the smallest one must come inside. So, that will prove this C is inside M of C is inside S of c; that means, given any collection of subsets of X it is always included in the monotone class generated by it and the monotone class generated by it is also inside the sigma algebra generated by it. So, let me repeat these arguments.

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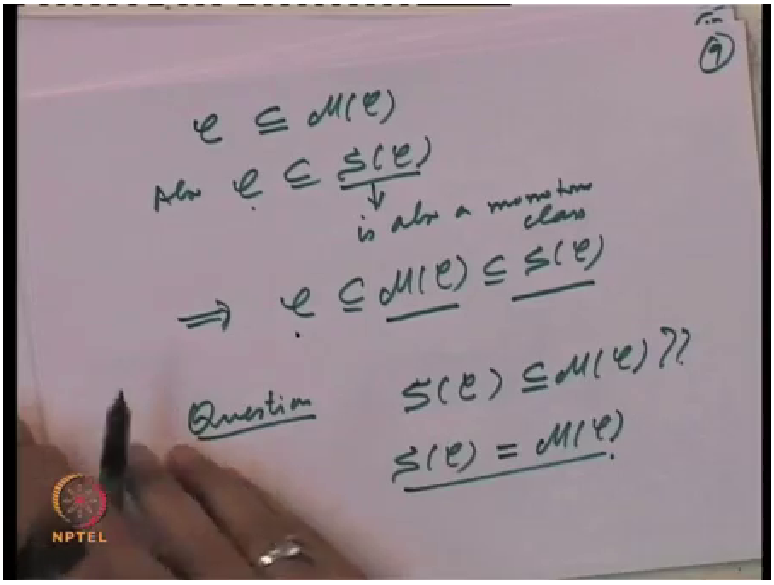

So, first of all C is contained in M of C that is because M of C is the smallest monotone class including C also C is included in S of C because S of C is the smallest sigma algebra of subsets of X which include C and this just now I proved is also a monotone class. So, this is a monotone class this including C. So, that implies the smallest one must come inside it and the smallest one is the M of C that comes inside S of C.

So, what we have shown is for every collection C of subsets of a set X. The monotone class generated by it, this is the subset of the sigma algebra generated by it we want to analyze the question when can we say S of C is also a subset of M of C when is this true that is same as saying that when is S of C the sigma algebra generated by a collection can I say that is equal to M of C and the answer is given by the next theorem which says that f C is an algebra, then this is true.