Measure & Integration
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Lecture – 31 B
Lebesgue Integral on R^2

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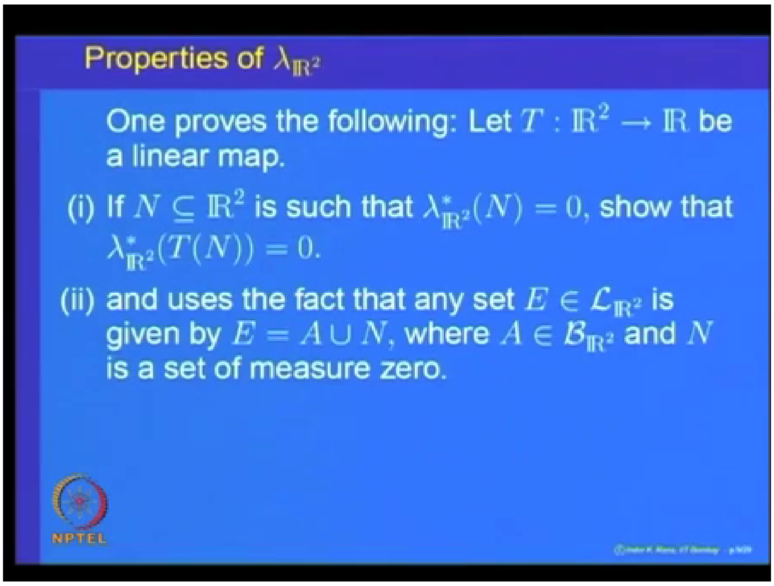

And the second fact between Lebesgue measure and Lebesgue measure and Lebesgue measurable sets and the Borel measurable sets is the following. Every Lebesgue measurable set E can be represented as A union of 2 sets - one A Borel sets A and a null sets N and the Lebesgue measure of E is same as the Lebesgue measure of the set A N is the set of measure 0.

So, that is the second fact one has.

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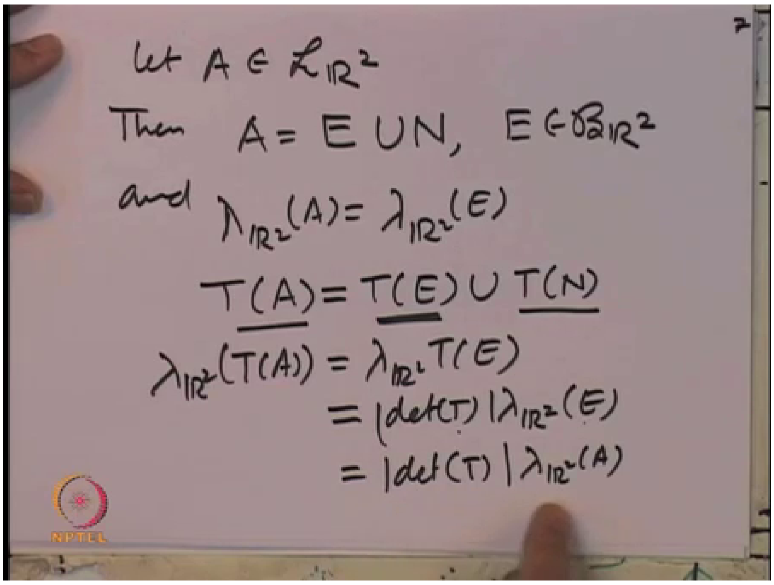

So, let us take, so let A B A Lebesgue measurable. Let us take Lebesgue measurable set in R 2 then the set A can be return as E union N where E is a Borel set, in R 2 and the Lebesgue measure. So, lambda R 2 of A, A is same as lambda R 2 of E.

Now, if we look at T of A. So, if you look at the set T of A then that will be equal to T of E union T of T of N. If T is say non singular and now this is Borel set and this is again a null set. So, Lebesgue outer measure of T of A is equal to Lebesgue outer measure of T of E which is equal to determinant of because this is a Borel set. So, determinant of T times Lebesgue outer measure. Lebesgue measure of the set E and which is same as the Lebesgue measure of the set A.

So, this is same as Lebesgue measure of T times Lebesgue measure of A. So, we have used the fact that if A is a Lebesgue measurable set then A can be written as E union N where E a Borel set and N is a null set; that means, the Lebesgue measure of R 2 of the set A is same as the Borel Lebesgue measure of the Borel component of it that is R 2 of E. So, now, if I apply transformation T to it and say T is non singular, then T of A will be equal to T of E union, T of N. And just now observed that T of N is a null set and T of E is a Borel set.

So, Lebesgue measure of T of a will be nothing, but the Lebesgue measure of T of E which by the earlier case is determinate of T times Lebesgue measure of E and Lebesgue measure of E is same as the Lebesgue measure of A. So, that proves that for A Lebesgue measurable set the Lebesgue measure of the transforms set T of A , A same as determinate of T times the Lebesgue measure of the set A itself.

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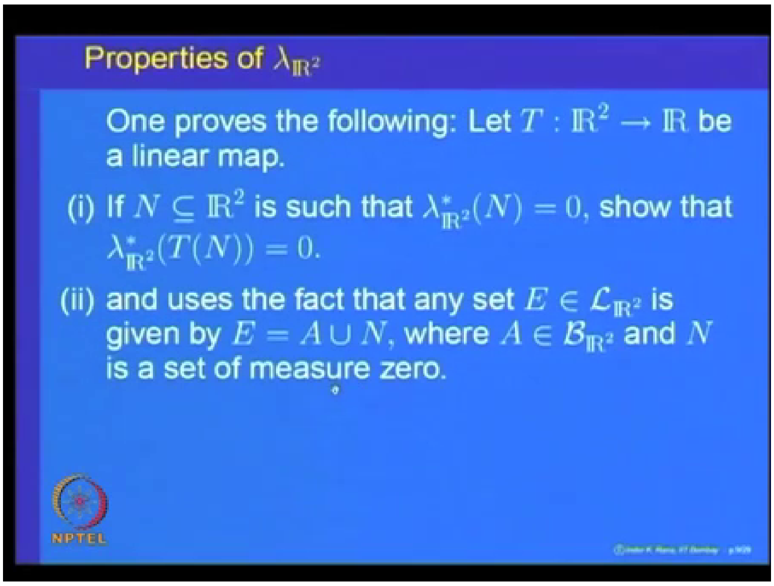

So, the proves the theorem completely namely, so that proves the theorem completely namely that if A, A, if E is any Lebesgue measurable set then T of E is also Lebesgue measurable and the Lebesgue measure of T of E is equal to determinate of T times the Lebesgue measure of E. So, this is how the Lebesgue measure of a set E in the plain changes with respect to linear transformation will give some application of this. Now, because there are many nice linear transformations in the plain.

So, let us look at the first application of this namely.

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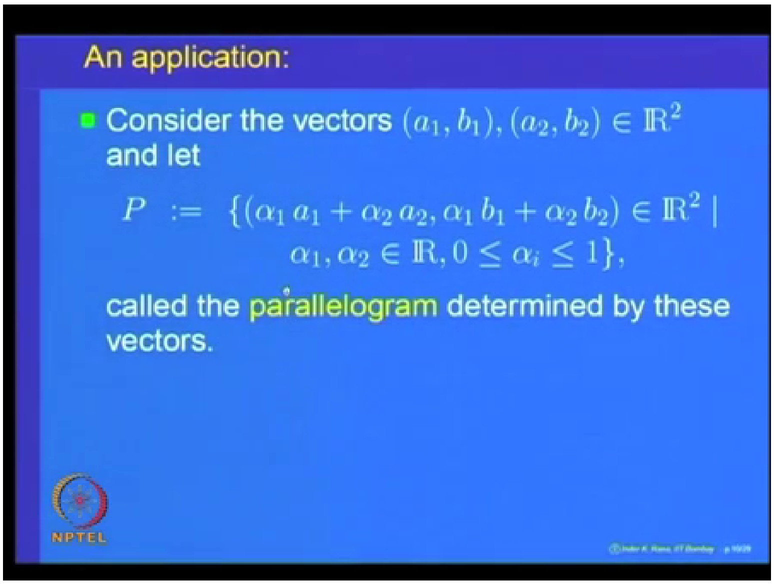

So, let us take a 2 vector a and b a 1 1 and a 2 b 2 in R 2 and look at the set which are all sets set P of all vector in the plain of the type, where the first component is alpha 1 a 1 plus alpha 2 a 2 and the second component is alpha 1 a 1 plus alpha 2 b 2 where alpha 1 and alpha 2 are numbers between 0 and 1.

This is called the parallelogram determinate by the vectors a 1 b 1 and a 2 b 2 in the picture it is the following.

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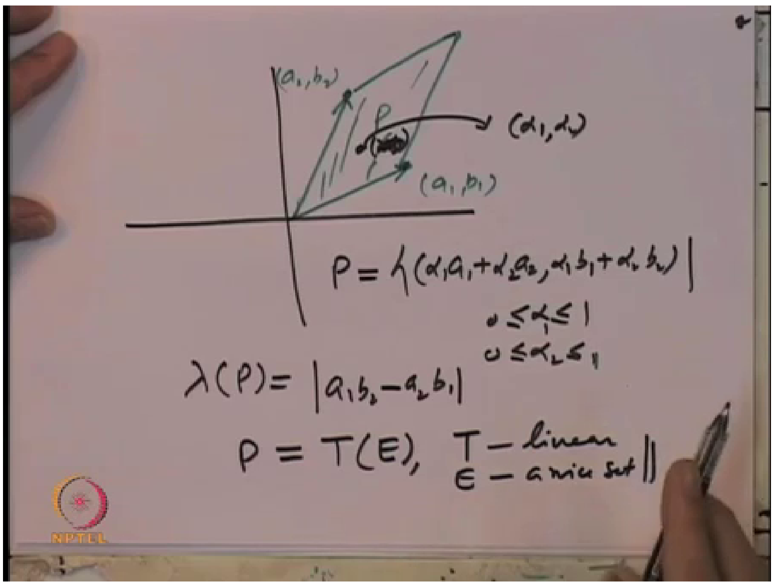

So, if we have the plain let us take 2 vectors 1 vector is this vector and other vector is this. So, this is the vector which is a 1 b b 1 and this is the vector which is a 2 b 2, then they determine a parallelogram the geometric object. So, let us see what is the parallelogram? So, that is nothing, but this parallelogram.

So, that is a parallelogram P determinant by these 2 vectors a 1 b 1 and a 2 b 2 and any vector in between. So, this is vector. So, this is parallelogram P is characterized by that, this any vector inside here. So, call it as X y. So, P is then X looks like alpha 1 a 1 plus alpha 2 a 2 and second looking looks like alpha 1 b 1 plus alpha 2 b 2, where this alpha 1 alpha 2 has a property. They are numbers between 0 and 1.

So, alpha 1 alpha 2 between 0 and 1. So, this is the point with component alpha 1 and alpha 2. So, if I take. So, this is how, this is what the parallelogram. So, one can check geometric fact that if I take 2 vectors a and b 1 and look at the geometric picture of this parallelogram, then if I take any alpha 1 alpha 2 between 0 and 1 and look at this, then this is nothing, but the parallelogram given by these 2 vectors. The claim we want to is that, we want to show that the Lebesgue measure of parallelogram is same as the absolute value of a 1 b 1 a 1 b 2 minus a 2 b 1. Say these are the components a 1 b 1 and a 2 b 2 are the components of the vectors which we started with.

So, the claim is the Lebesgue measure of P is equal to the absolute value of a 1 b 2 minus a 2 b 1. So, to prove this we are going to show that this P is equal to T of a set E. We are T is linear transformation T linear and E is a nice set and it is not difficult to guess what is T and what is E. So, let us just look at that. So, the claim, so let us observe that; if T is the matrix with the component a 1 a 2 b 1 b 2.

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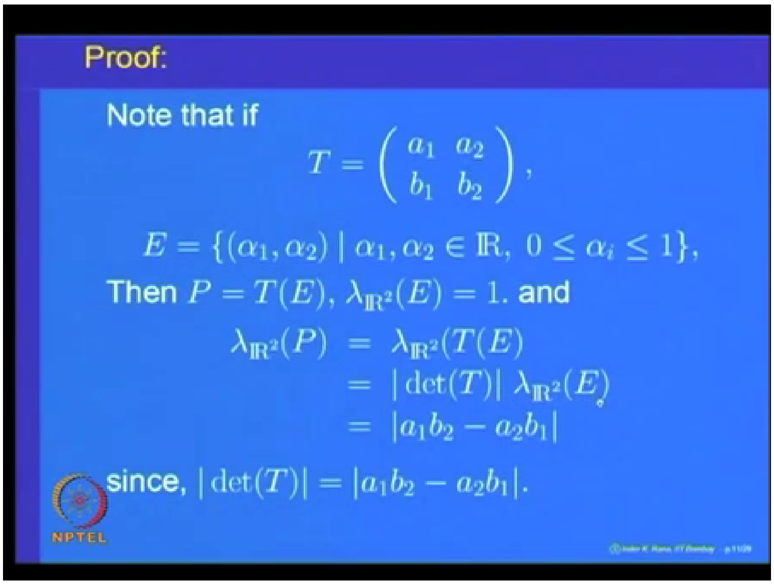

So, those are the vector, which are given vector are a 1 b 1 and a 2 b 2. So, the first column is the vector a 1 b 1. Second column is the vector components of the vector a 2 b 2. If I look at this transformation T and look at the set E with components alpha 1 alpha 2 in where alpha 1 alpha 2 are real line and there are between 0 1 on. So, what is this set E. This set E is nothing, but the set E is nothing, but a rectangle actually square in the plain with sides 0 1 to 0 1.

And if I look at T of E. So, any set alpha 1 alpha 2. So, what will be this? So, what we are saying is a following that if I look at the set E.

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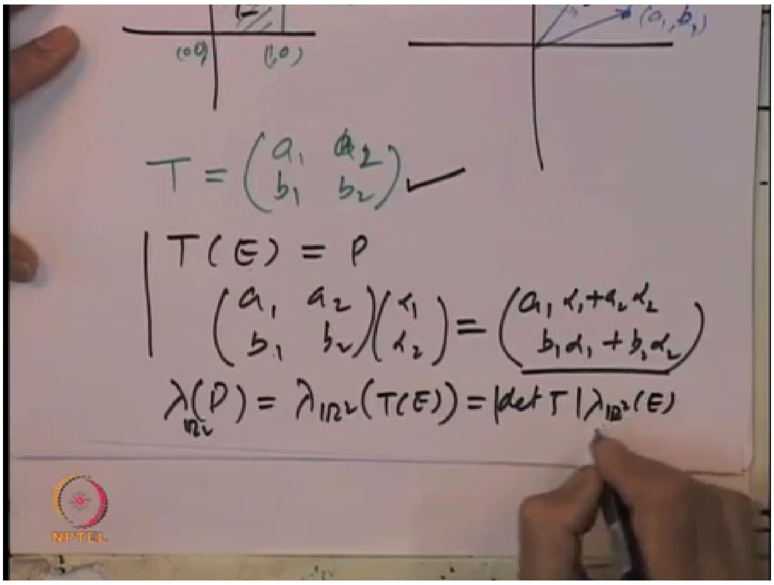

So, here is the set E which looks like. So, this is the set E 0 0. This is 1 0 1 1 and 0 1. So, if I look at this set and look at the transformation given by T, where T is equal to a 1 b 1 a 1 b 1 and a 2 b 2 if I look at this, then the image of this, under this is precisely that a parallelogram P where this is a 1 and a 2 and this is a 1 b 1 and this is a 2 b 2.

So, this transforms to this parallelogram. So, if T is this and E is this set. Then T of E is equal to P that is because what is so, let us look at a 1 a 2 b 1 b 2 and a vector is alpha 1 alpha 2. So, what is that? So, that is a 1 alpha 1 plus a 2 alpha 2 and that is gives b 1 alpha 1 plus b 2 alpha 2. So, that says the vector here alpha 1 alpha 2 goes to the vector there given by this and that is precisely the parallelogram.

So, under the and this is the linear transformation. So, under linear transformation T given by this matrix by unit square changes to the parallelogram and once that is true. So, this will imply that the Lebesgue measure. So, this will imply the Lebesgue measure of P is same as the Lebesgue measure of in the plain of T of E and that is equal to determinant of T absolute value times Lebesgue measure of E, but Lebesgue measure of E that is a area there is a rectangle. So, it is a area that is equal to 1 and determinant of T is a 1 b 2 minus a 2 b 1.

So, that gives us the result. So, that gives us the result that the Lebesgue measure of P is same as the Lebesgue measure of the transform set T of E which is equal to determinant of T times the Lebesgue measure of E and that Lebesgue measure of E being equal to 1 that gives us determinate of T which is nothing, but a 1 b 2 minus a 2 b 1. So, this gives us that Lebesgue measure of the parallelogram is the determinant of the given by the vector. So, that is a 1 b 2 minus a 2 b 1.

So, these are, this is one of the result that one proves normally in geometric and linear algebra that determinant is nothing, but a measure of the parallelogram area of the parallelogram determinant by the vectors. Let us look at another application of this formula how the linear transformation changes.

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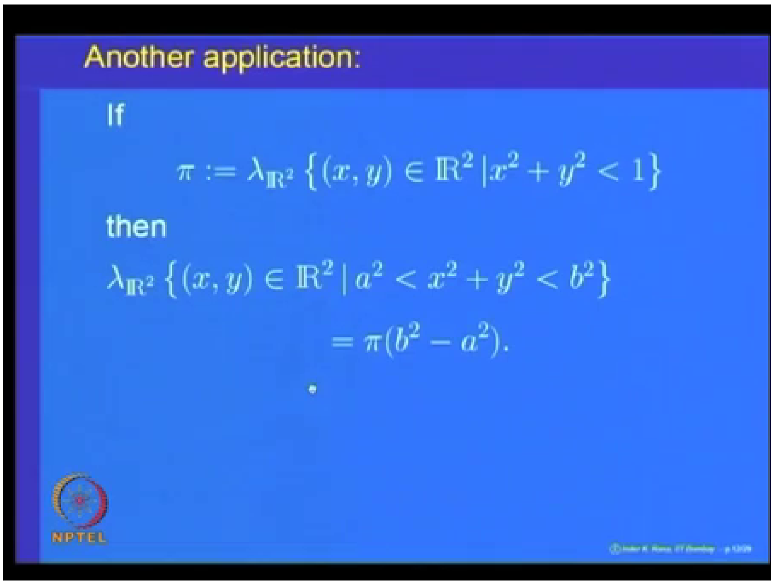

So, let us look at the Lebesgue measure of the unit circle area reason enclose by the unit circle. So, Lebesgue measure of all vectors x, Y in R 2 such that X square plus Y square is less than 1.

So, let us look at that. So, let us call the Lebesgue measure of this to be equal to a number pi and we are not assuming anything about pi, where as saying that the Lebesgue measure of this unit circle is a finite quantity. So, let us see, it is finite.

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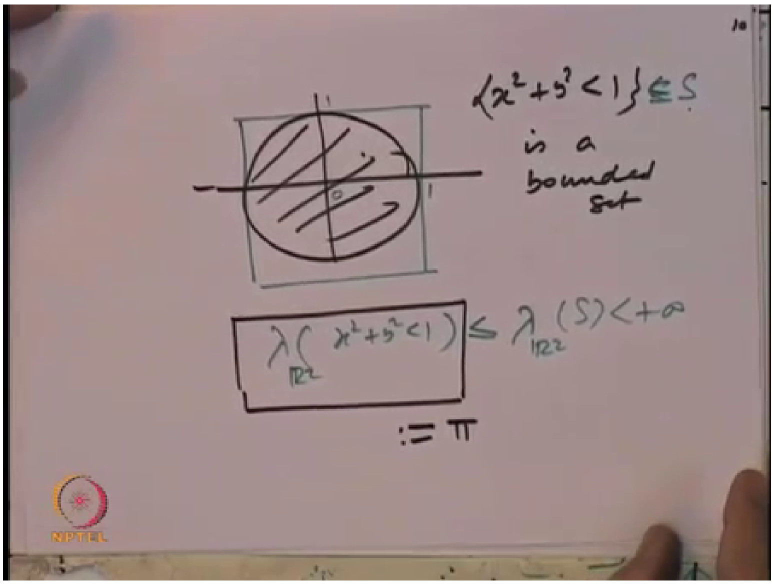

So, here is the unit, it is a bounded. So, this is X square plus Y square less than 1. So, that is that set. So, this is a is a bounded set. So, for example, this is enclose inside, this rectangle inside, this square of, this is 1 and this is 1. So, this is 0 this is 1 and this is 1. So, it is enclose inside this square of a length so; that means, that the area. So, this is less than equal to it is the subset of the square.

And square is a bounded thing so; that means, the Lebesgue measure of the point. So, that X square plus Y square less than 1 will be less than or equal to the Lebesgue measure in the plain of the square of the finite quantity so; that means, that the Lebesgue measure of the region enclose by the unit circle is the finite number and this finite number. We are just calling it by the number, by where denoting it by the symbol pi. So, pi is the Lebesgue measure of the region enclose by the unit circle, then the claim is that if you look at the annulus region, if you look at the annulus region that is X square plus Y square bigger than a square and less than b square then it is Lebesgue measure is pi of b square minus a square.

So, what we want to prove is that if I look at.

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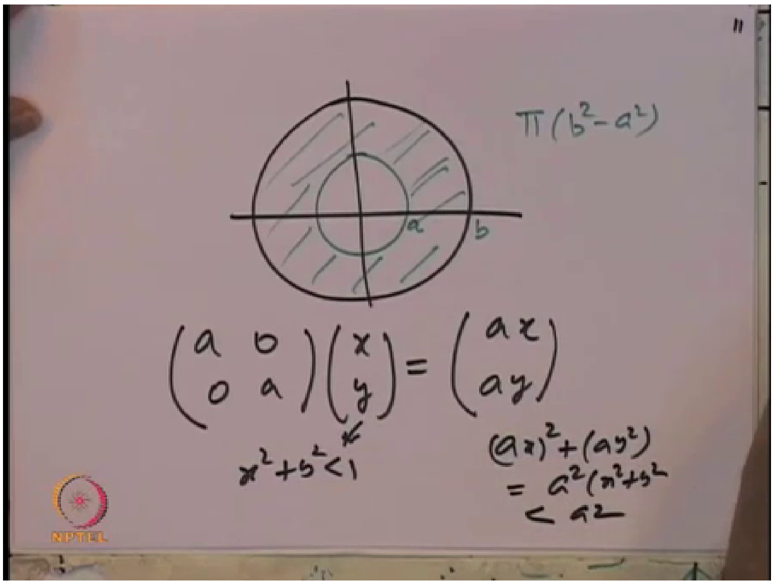

If I look at here is bigger circle and here is the smaller one and this radius is a in this radius is b. So, we are saying the Lebesgue measure of this portion is nothing, but pi b square minus a square that is what now we should be expecting from our ordinary geometric that we are been learning in school namely the area of the circle is equal to pi r square.

So, will first prove that the area of a circle of radius r is equal to pi r square and from there we will reduce this fact. So, let us observe that. So, the first thing is let us take the linear transformation T, which is diagonal which is given by a 0 0 a. So, will looking at the diagonal transformation a 0 0 a, and look at the unit circle area enclose by the unit circle. So, that is e. So, E is the set of a all at vectors X coma Y in out. So, X square by Y square is the less than one, then if we look at any point here, and transformate according to this T that will look like.

So, let us look at what will that look like. So, let us look at the transformation a 0 0 a, and let us look at a vector X Y. So, that gives the vector ax a y. So, if this vector had the property that X square plus Y square less than one, then the transform vector ax a Y has a property a X square plus a Y square is equal to a square times X square plus Y square, which is less than a square. So, that shows that the unit circle.

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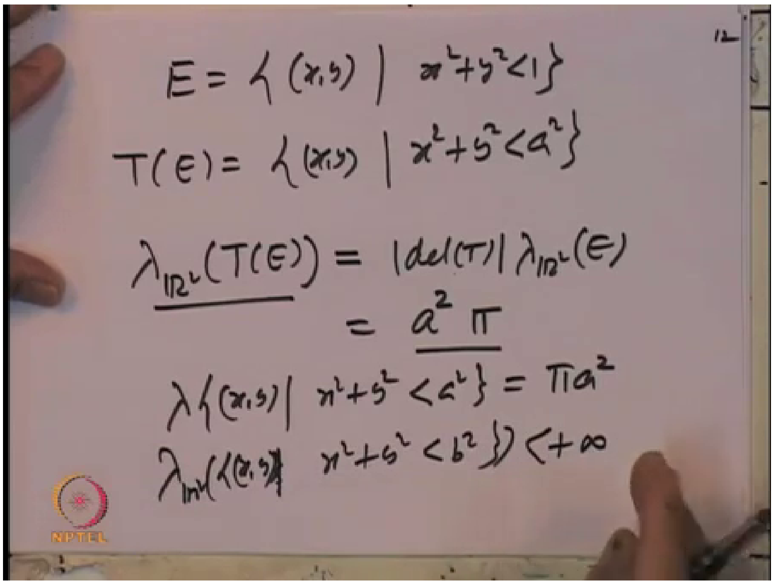

So, if E is the unit circle; that is XY, X square plus Y square less than one, then T of E is the circle the region enclose by the circle X square plus Y square less than a square. So, that is what we know. So; that means, now we have apply our theorem of linear transformation. So, look at the Lebesgue measure R 2 of the transform set e. So, that is equal to by the property of that the theorem; that is terminate of T times Lebesgue measure of the set E. The determinant of T is equal to that is a diagonal transformation. So, that is a square and Lebesgue measure of E, which is unit circle is pi. So, Lebesgue measure of the transform set is equal to pi a square. So; that means, the Lebesgue measure of all the point X Y such that X square plus Y square is less than a square is nothing, but pi a square.

So, that is magnification that, where getting and. So, as a consequence of this, let us reduce for the annulus region, the area is the required Lebesgue measure is pi of B square minus a square. So, let for that we have to just observe that, if I look at the circle. So, the set of points X Y X Y; such that X square plus Y square is less than b square is a set, which is the bounded set I will Lebesgue R 2 of that is finite. So, is a set of finite outer measure of finite Lebesgue measure. So, I can write that the Lebesgue outer measure of X square plus Y square, bigger than a square, and less than b square is nothing, but Lebesgue measure of the set X square plus Y square less than b square set minus the inner circle. So, that is X square plus Y square less than a square.

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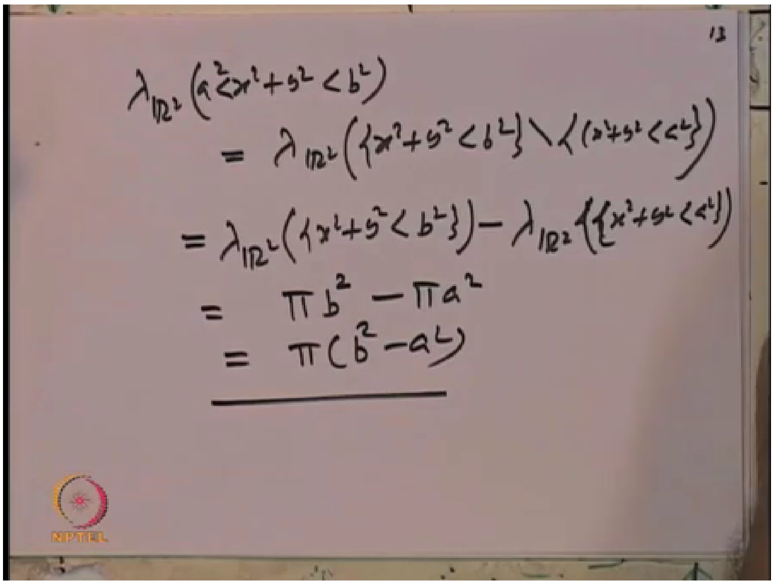

And now, everything being finite in, can write this as the Lebesgue measure of the region enclose by the outer circle. So, that is X square plus Y square less than b square minus Lebesgue measure of X square plus Y square of Lebesgue measure of X square plus Y square less than a square. So, this is possible, because everything is a finite quantity. So, the measure of the difference a minus b measure of a minus b is measure of a minus measure of b, whenever b is the subset of a, and everything is finite.

So, that property gives is this, and this is pi of b square. Just now we saw pi of a square. So, that is pi of b square minas a square. So, that proves, so what we saying is, that is simple properties help us to conform that the Lebesgue measure on the plain that we have define is a essentially, the extending the notion of area in the plain to a bigger class of subset, and the usual formula as for the area that you have been using a priory without any justification are now being a justify by the Lebesgue measure. I just point out and more excision of this result namely that the area Lebesgue measure annulus region is pi b square minus a square to something in, in integration which is looks like the change of a variable formula in multiple integrals; namely if you have a double integral then, and you change to Cartesian to polar coordinate, then the dx dy. normally we have that formula that when you change d X dy it will be r dr tta d theta.

So, more reargues we have saying that for a particular class of functions I want to state, and given outline of the proof will not proving it fully, will given outline of the proof. So, let us go to the next application of.

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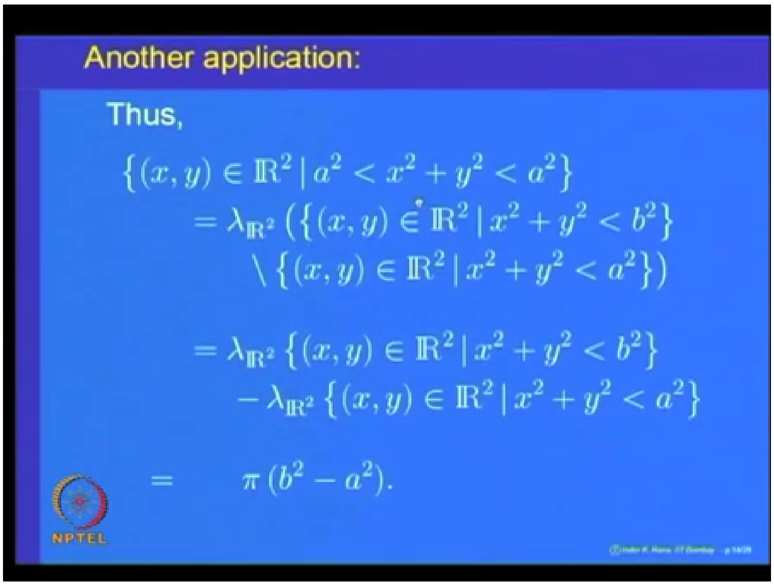

So, this is what I just now said that the Lebesgue measure of annulus region is Lebesgue measure of outer circle minus Lebesgue measure of the inner circle; that is pi b square minus a square .

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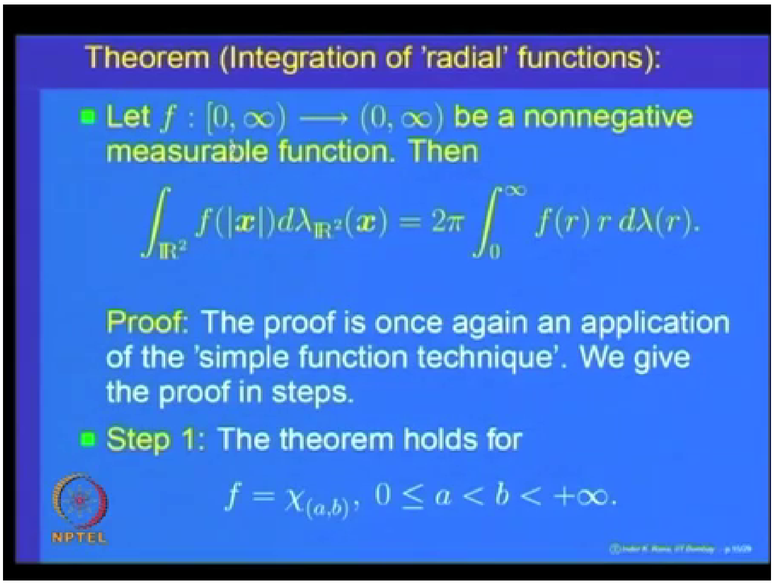

So, this is another application or extension of the result, this now proved, is called the integration of the redial function.

. So, let us look at the theorem says, let us look at a function define from 0 infinity on the positive, on the non negative part of the real line, taking values in nonnegative values. So, that is 0 to infinity. So, its a non negative measurable function defined on the non negative part of the real line, then the claim is that if I look at the double integral, integral of R 2 of F absolute value of x, X is the vector. So, absolute values the norm, the magnitude of the vector x. So, look at this. So, this is like a composite function. So, the vector X goes to the magnitude; that is a non negative real number, and F valuated at that.

So, the double integral with respect to R 2 is given by 2 pi times fr dr d theta. So, that is the clime that this integral is equal to this integral. So, and what is meaning of nonnegative radial function F is the nonnegative. I should have said here, it is a radial function; that means, it depends only on the absolute value of the function a does not. So, F is a nonnegative. So, this is a radial function. So, this we can think it as F composite. The magnitude is the radial function is the radial function, it the value of composite function depends only upon the magnitude of the vector, and not on the position of the vector.

So, to prove such a result, the proof a typical application of simple function technique. So, one tries to proof that for a simple measurable function this is true and then apply monotone convergence theorem and so on. So, I will just outline the steps for a detail proof, you may consult the text book. So, let us look at the first step, let us look at the first step when this function F is the indicator function of a interval ab, when F is the integrator function of a interval ab, where ab is a interval in the non negative part of the real life as, so a less than b bigger than 0. So, when F is the indicator function, let us compute this both sides, and what is the look like.

So, when F is the indicator function of the ab. So, here is the indicator function. So, this is 0 to infinity. So, 0 to infinity means this will give you indicator, function will give you only a to b. So, this will be a to b of the function fr in function is indicator function is values is 1. So, X rdr. So, when you integrate rdr we get r square by 2 between a and b.

So, when you put the values, we get b square minus X square by 2. So, that is equal to pi of b square minus a square.

So, this side is nothing, but pi of b square minus a square and what is this f. So, the indicator function of ab evaluated at the absolute value, means you are integrating in the annulus region between the limits a and b. So, it is pi b square minus pi a square. So, then it is just equal to pi of b square minus a square. So, this both sides are nothing, but the result that we did discuss, now that the area of the annulus region is equal to pi b square minus a square. So, step one is for indicator function, it is that result.

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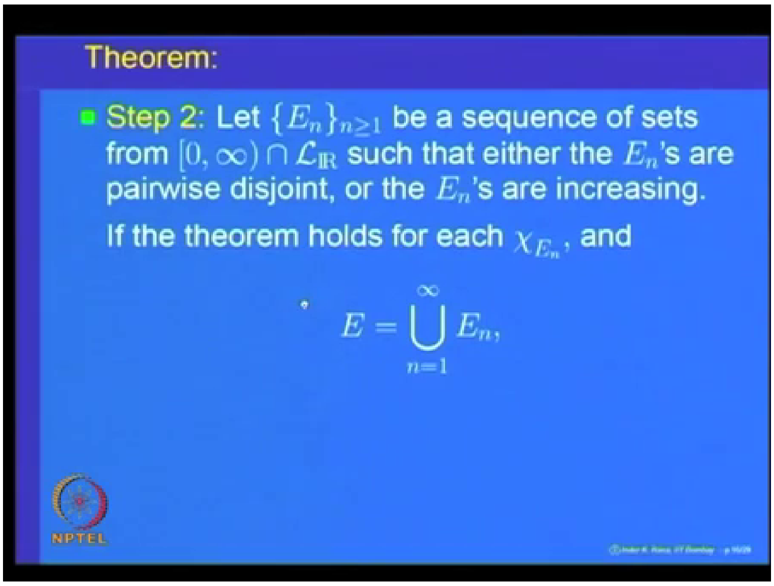

The next thing is will look at a sets, which are either sequence is E ns with sequences of sets which are Lebesgue measure of course, either pair wise disjoint or a increasing sequence.

And closing for the indicator function of each set en, this result holds, then the claim is, it also holds for the union of E ns. So, if each E n the result holds than the result for each indicator function of each E n. It holds, it also holds for the indicator function of set E, and that essentially is an application of the monotone convergence theorem to the earlier result.

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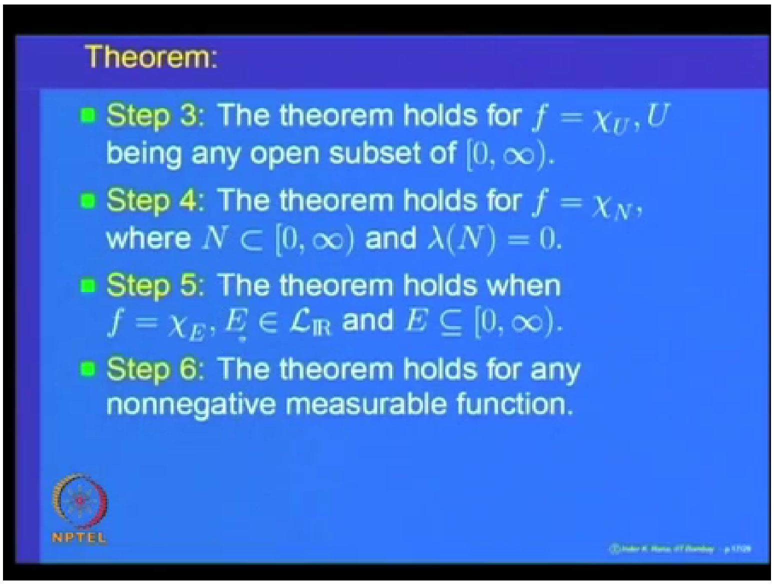

So, another step three from such saying one comes open sets by the fact that, each open sets is a countable disjoint union of countable disjoint union of intervals. So, for intervals that property holds. So, this holds for every open set.

And from the open sets and null sets. So, one shows the corresponding property also holds for null sets. So, open sets and null sets; one goes to the indicator function of any Lebesgue measure set, because any Lebesgue measure set can be return in terms of a open sets and null sets. So, and then from this, one apply usual monotone convergence theorem technique from the indicator function to nonnegative measurable function. So, these are the steps one follows to prove theorem of this kind. I just want to conclude today's lecture by say.

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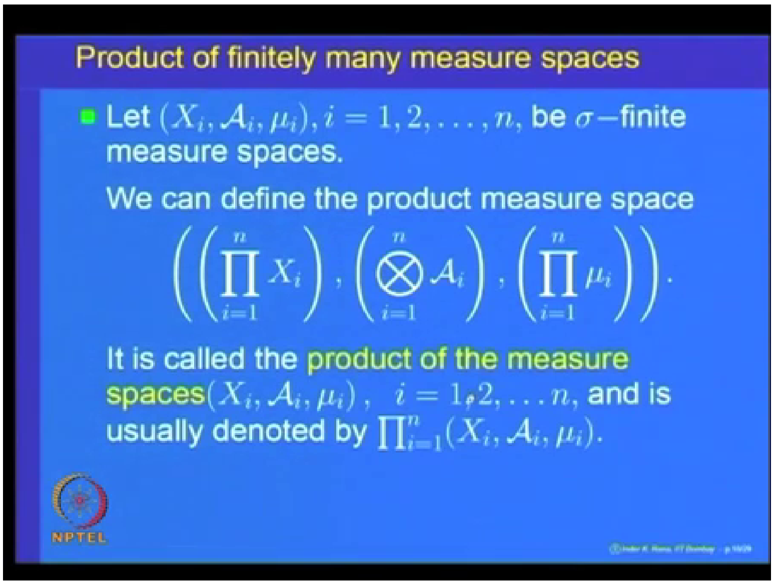

What we are done it for product space is of two product spaces, can also be extended to any finite number of product spaces. So, namely, you are given a finite number of measure spaces X I, A i mu i. we define the product of two of N can be extended by instead of being one at a time iteratively, you can define the product of the space this X spaces is xi 1 to N. you can define the product sigma algebra A i is 1 to N, and you can also define the product measure inductively one can, so that and.

.So, this is called the product space, I will not going to the details of it, but this is useful, and one can also show that if you take product of some finite number, m number and take product of some N number of copy, and then take the product name; that is same as to the product of the, then put to together. So, it is same as the product of X I from m 1 to N plus m. So, these are same. So, one can, same I with usual identity.

So, basically saying that what we are done it for two product of two, major spaces can be done it for a finite number of them. So, as a consequence one can define the notion of Lebesgue measure subset in rn, and the notion of Lebesgue measure in rn. So, this can be done. So, this again, those who are interested should refer the text book for more details. So, what we are done today is, we have completed the study of product measure spaces, and with that we have a essentially completed what is called the basic concepts in measure theory; namely we have done the extension of measure, then integration of measure, then measure and integration on product spaces. This is the core of the subject, and from now onwards I will be looking at some special topics in our subject of measure theory.

Thank you.