Measure & Integration
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Lecture – 31 A
Lebesgue Integral on R^2

In the previous few lectures, we have been looking at the lebesgue measure on the space R 2 and its various properties in the previous lecture. We have started analyzing how does a lebesgue measure of a set change, when we apply a linear transformation to it. So, we had started analyzing it. Let us recall what we have done and then will continuing analyzing this problem and some more properties of lebesgue measure under other transformation.

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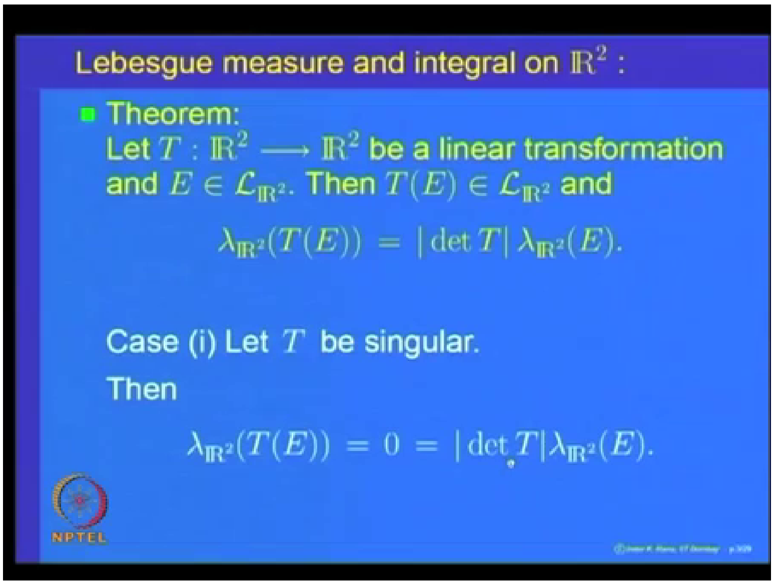

So, let us just recall what we have done last time was that we started looking at the theorem namely if T is a linear transformation from R 2 to R 2 such that and E is a lebesgue measurable subset, then we showed that we wanted to show that the transform set T of E the image of E and T is again a lebesgue measurable set and the lebesgue measure of the transform set is obtained by multiplying the lebesgue measure of the original set with the constant called determinant of E. So, lebesgue measure of T of E is equal to determinant of T times lebesgue measure of the original set E.

So, this property this theorem we had started analyzing we had analyze the proof of this theorem in the first case when T is a singular transformation and there we argued that if T is a singular transformation then T of E the image set is going to be a lebesgue null set and for a singular transformation determinant of E T is also equal to 0, that is how singular transformation are characterize. So, in that case both the terms the lebesgue measure of the translated set is equal to 0 is same as determinant of T times lebesgue measure of the original set.

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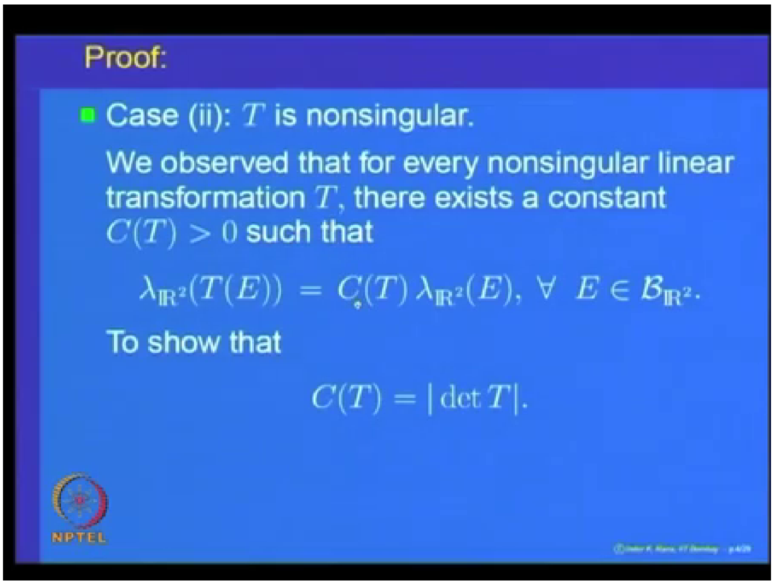

In the case when T is a, on-singular transformation when T is nonsingular. So, nonsingular means T is invertible and here are a few facts about linear algebra saying that T is nonsingular is same as saying as determinant is not equal to 0 and it is equivalent to saying that as map T is a one to one on to map and the inverse of course, is also a linear transformation.

So, in a analyzing the proof of that we are already shown that if E is a Boral set. So, we first visit our self to Boral subset of R 2. We showed that if E is a Boral set then for every non singular linear transformation T of E T, the transform set T of E also is a boreal set that basically follows from the effect that every linear transformation is a continuous map and if it is non singular then the universe also is a continuous map.

So, essentially saying that for every boreal set E T of E is boreal the 1 analyze is that when E is open set, T of E is an open set and NZ is a boreal set and then one shows that the collection of all sets for base. This property is true namely the image is a boreal set is a sigma algebra including open sets and hence, one can includes that for every set E the transform set the image set of E is also a boreal set. And we also analyze that if you consider this as a measure for all boreal sets then it is translation invariant because T is linear and; that means, by the uniqueness property we got that for every linear transformation T which is non singular the lebesgue measure of the transform set namely T of E must be a constant multiple of the original measure lambda or R 2 of E and that constant will depend on the transformation T.

So, this is a stage we had reached and then we wanted to analyze further and the claim we want to prove is that this C of T is nothing, but determinant of T. So, this is a stage we are reached. So, let us continue the proof, let us observe that this map T to C of T. See for every transformation T, we are associating a number C of T which is non negative. So, we get a map T going to C of T.

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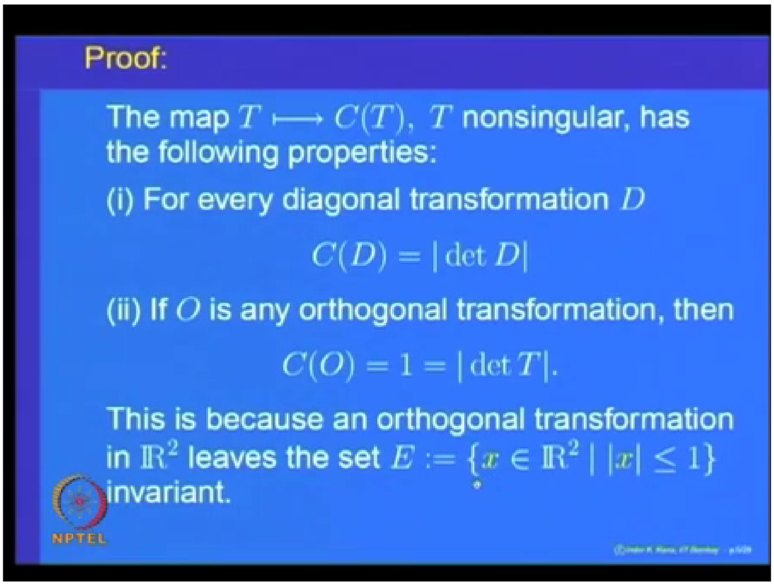

So, there is a map for every nonsingular transformation T. We are associating number C of T. This association, this map as the following properties namely if T is a diagonal transformation, then we observed that is, that was beginning of our analysis of this theorem that C of T is nothing, but the determinant of T.

So, for diagonal transformation C of T is equal to determinant of T, determinant and the second property is that if the linear transformation is orthogonal transformation then this

C of O is equal to 1 is equal to determinant of T and this is because of the fact that a orthogonal transformation on R 2 leaves the set namely the units circle you can think of it has all x in R 2 says that norm of x is less than equal to 1 invariant.

So, let us just look at the property is straightly more some of you may not be knowing about linear transformations.

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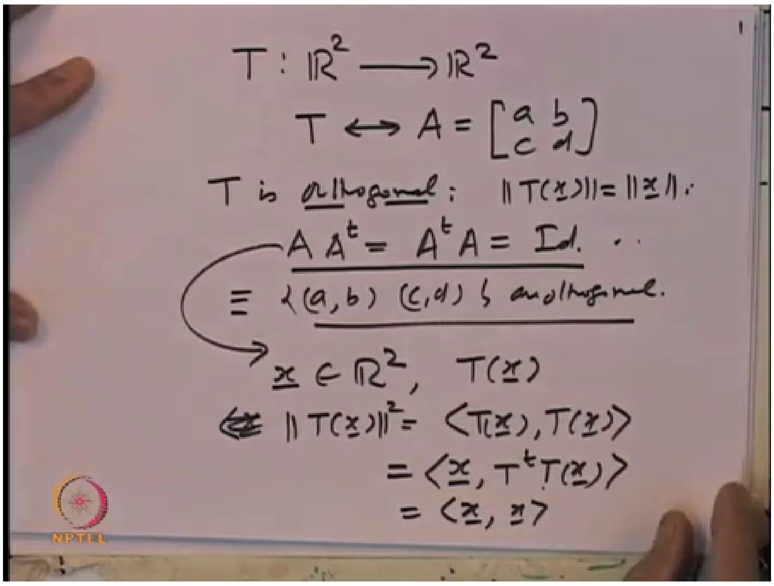

So, T is a linear transformation from R 2 to R 2 and let us as every linear transformation is given by a matrix A. So, it is A 2 by 2 matrix sort is a b c and d. So, saying that T is orthogonal is same as saying that if it look A and look at A transpose that is same as A transpose A and that is equal to identify.

So, saying that T is orthogonal is characterized by this property about the matrices is matrix of the linear transformation namely A times A transpose is same as A transpose times A and that is equal to identity and which is also equivalent to the property that if you look at the row vectors and the column vectors. So, the row vector is ab and cd. So, these 2 vectors are orthogonal are orthogonal namely the dot product is equal to 0 and each is a unit vector. So, that is call orthogonal, but we are not you can to uses property. So, let us look at this property which says A, A transpose is equal to A transpose.

So, this property gives us the following fact namely, let us look at the dot product. So, let us take any vector x belonging to R 2 and look at the dot product of. So, let us look at the image the image is T of x. So, x goes T of x. So, let us look at the dot product. So, norm of T x square, that is given by the dot product of T x with itself. So, that is the definition of the magnitude the dot product in R 2. But this dot product can also be written as T times. This T can be written as T transpose T of x. So, T transpose is a same as a transpose basically. So, you can think it as matrices and that being identity. So, this is same as x, x.

So, orthogonal transformations are also characterize by the property that the norm of the image of any vector is equal to norm of the original vector that is another way of characterizing a orthogonal transformation we can take that as a definition of the transformation if you like.

Now from both this properties that A transpose A is equal to identity. So, that implies the following fact that A transpose A equal to identity this implies that the determinant of A transpose determinant of A is equal to 1 and that implies determinant of A transpose is same as determinant of A. So, that say determinant of a determinant, A times determinant of A. So, this square is equal to 1. So, that implies that the absolute value of determinant of A is equal to 1.

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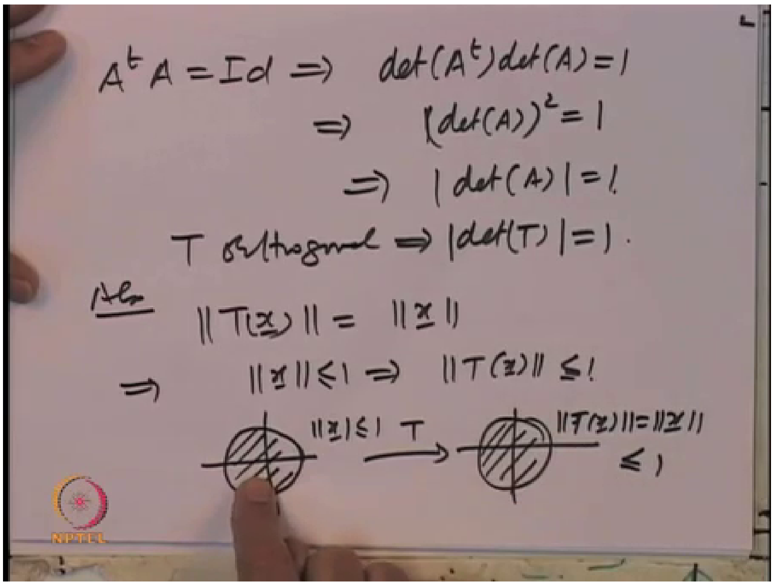

So, for A, so T orthogonal implies that determinant of T absolute value is equal to 1, so that is 1 fact. Also the fact that norm of T x is equal to norm of x implies that if norm of x is less than or equal to 1. So, that implies norm of T x is also less than or equal to 1 so;

that means, if in the plain we look at the unit circle, but this is the set where norm of x is less than or equal to 1 and if you take the transform set the transform set is same. So, that. So, under T this gives back to the same thing. So, norm of T x equal to norm x less than or equal to 1.

So, that is saying that T leaves if T is orthogonal then it leaves the unit circle the region inside the unit circle invariant and; that means, that essentially means that the lebesgue measure of the set is same as lebesgue measure of that set. So, that implies that the lebesgue measure of the transform set, so that circle, so mode x less than or equal to 1 is same as the lebesgue measure of mode x less than or equal to 1 and this being equal to determinant of T, this is being determinant of T times this being, this is same as that. So, this is sorry, this is same as. So, that implies that, but this can be written as determinant of T, because that is equal to 1 times lebesgue measure of norm x less than or equal to 1.

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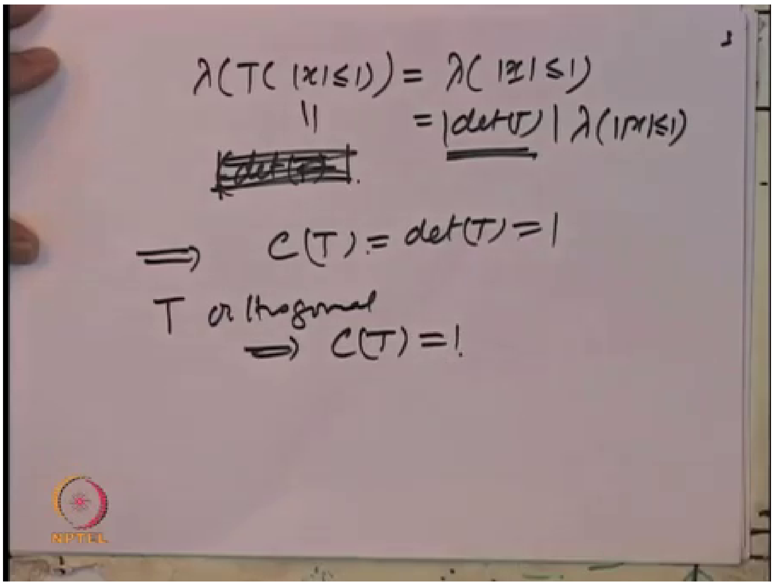

So, this implies that. So, this is a constant. So, C of T which is determinant of T is equal to. So, C of T is one which is same as determinant of T so; that means, for orthogonal transformation. So, T orthogonal implies C of T is equal to 1. So, that is a second fact that we wanted to proof.

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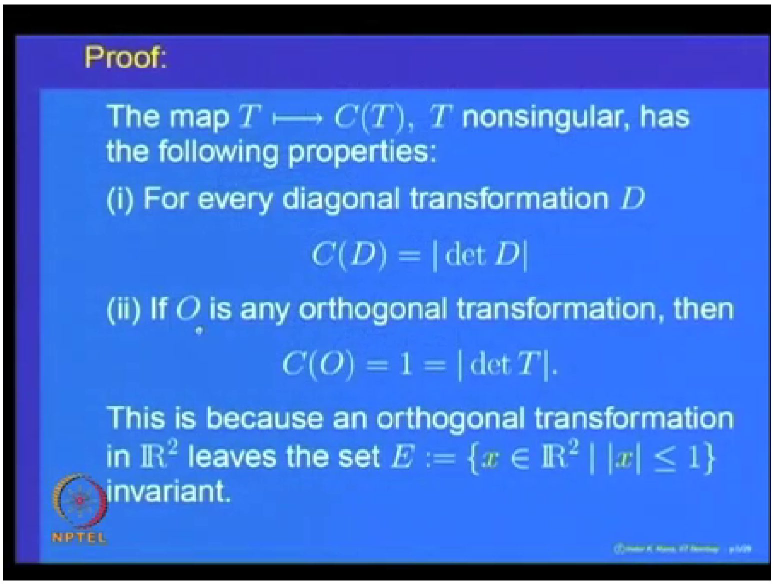

Namely if O is a orthogonal transformation, then the C of O the constant this way type of here, this should have been determinant of O is because the unit circle with in inside unit circle of is left invariant by the orthogonal transformation.

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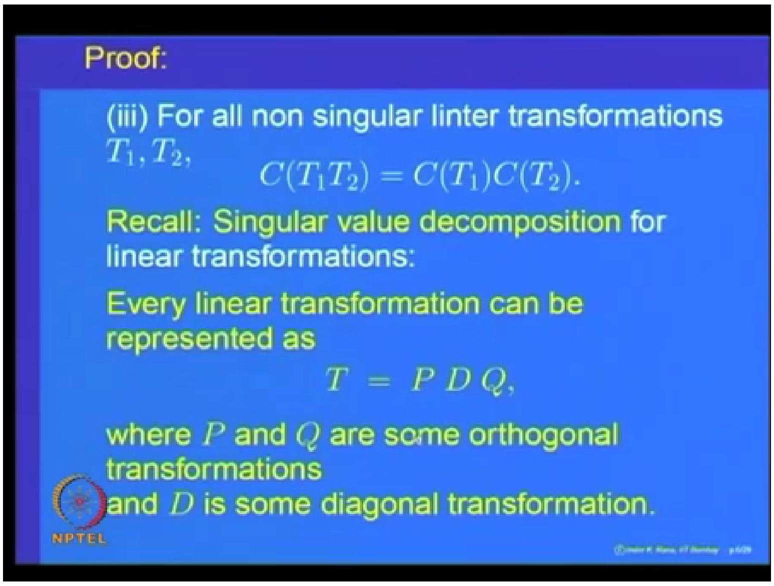

So, now and the next property, we want to analyze is that for all non singular transformation T 1 and T 2 C of T 1 T 2 is equal to C of T 1 times c of T 2; that means, is map is multiplicated the map T going to C of T is a multiplicating map, namely if I look to transformation T 1 and T 2 then c of T 1 T 2 is same as c of T 1 times c of T 2.

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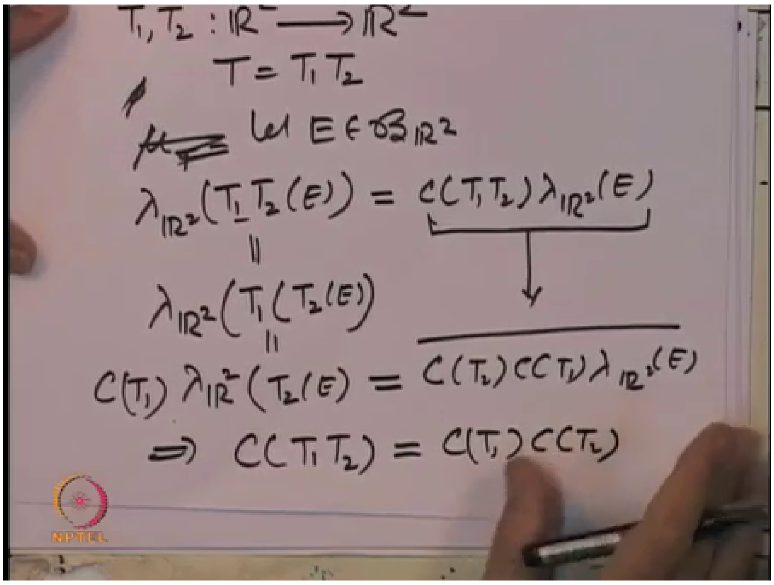

So, let us look at a proof of that T 1 and T 2 are transformations form R 2 to R 2. So, we are going to look at. So, let us write T is equal to T 1 T 2, then we to compute C of T 2. So, we are going to look at the measure mu of T T which is equal to. So, let us take any set. So, so let Eb any set which is the boreal set.

So, let us look at lambda of lambda R 2 of T 1 T 2 applied to E. So, let us look at this by definition of the constant C of T is C of T 1 T 2 because T 1 T 2 is a linear transformation applied. So, this composite T 1 T 2 is applied to E. So, by definition of C of T 1 T 2 that should be equal to C T 1 T 2 of Lebesgue measure of the set E, on the other hand we can also think of this as lambda Lebesgue measure of R 2 of T 1 applied to T 2 of E. So, this composition T 1 T 2 is same as saying the linear transformation T 1 is applied to the set T 2 of E, but if you do that then you know that this is equal to lambda R 2 of it is lambda R 2 of T 1 of a set. So, it is CT 1. So, this is equal to CT 1 times lambda R 2 of T 2 of E and now, that once again lambda R 2 of C of 2 of E gives you C T 2 and the original name C T 1 into lambda R 2 of E.

So, we get that C of T 1 T 2 and the Lebesgue measure of any set E is same as is same as C T 2 into C T 1 of Lebesgue measure of E. So, this happens for every set E. So, that implies. So, this implies that C of T 1 compose it T 2 is same as C of T 1 times C of T 2. So, this map is a multiplicative a map. So, this is the property we wanted to prove. And now, we need another fact from a linear algebra namely what is called a singular value D composition for linear transformations.

So, in case we have not come across this theorem called singular value D composition for linear transformations. Please, look into the text book that we have says it namely an introduction to measure and integration and look at the appendix of that book. You will find a proof of this singular valued, singular value decomposition for linear transformations.

So, let us take what is the singular value decomposition; it says the at every linear transformation T can be represented as a product of 3 transformations were the first one P and the last one Q are both orthogonal transformation. P and Q are orthogonal transformations and this D is a diagonal transformation. So, every linear transformation T can be represented as P times, D times, Q were P and Q are some orthogonal transformation and D is some diagonal transformation. So, this is a theorem called the singular value decomposition in linear algebra. So, please have a look at a proof of this in case you have not come across this theorem in the text book mention.

So, once we know that for every linear transformation T can be represented as P times D times Q. So, and the property 3, just know ready to says that the constant C of any composite is a product. So, we apply that property through this. So, we get C of T will be equal to C of P times D times Q which is nothing, but the product. So, the A C of transformation T will be C of P into C of D into C of Q.

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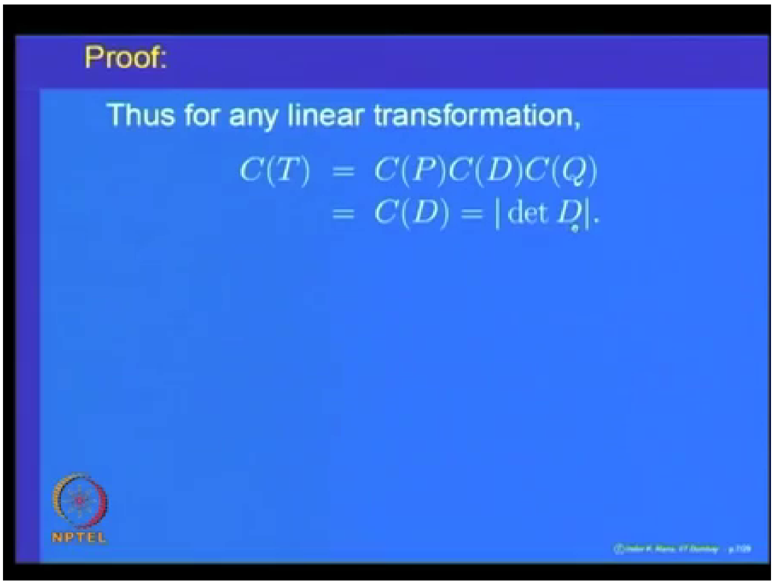

So, we get that the constant for any linear transformation T is equal to the constant for some orthogonal transformation P times the constant for a diagonal transformation D and the constant for another orthogonal transformation Q.

But just now, we have observed that for orthogonal transformations the C of orthogonal transformation the constant associated is 1. So, C of P is 1 and C of Q is 1. So, that gives you C of T is equal to C of D, because both first and the last multiplicative things are 1 that is C of D and for diagonal transformation, we have already shown. This is equal to determinant of D. So, for the linear transformation T the constant C of T is equal to determinant of D, where D is the diagonal transformation which appears in the similar decomposition.

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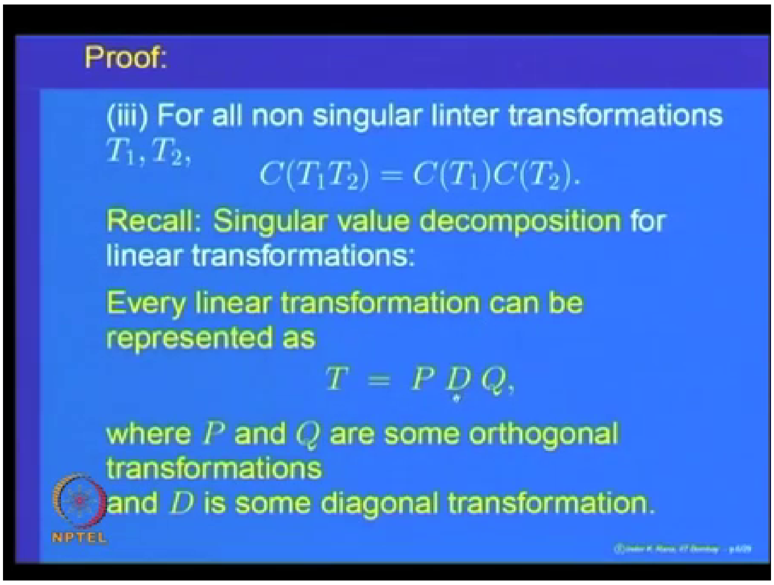

T is equal to P D Q, but on the other hand, we can also look at the determinant of T from this, so determinant of T. We call that determinant is also a multiplicative map. So, determinant of T will be equal to determinant of P into determinant of D into determinant of Q, but determinant of P and determinant of Q both are equal to 1. So, that says determinant of T is equal to determinant of D. And just now we said determinant of D is equal to C of T.

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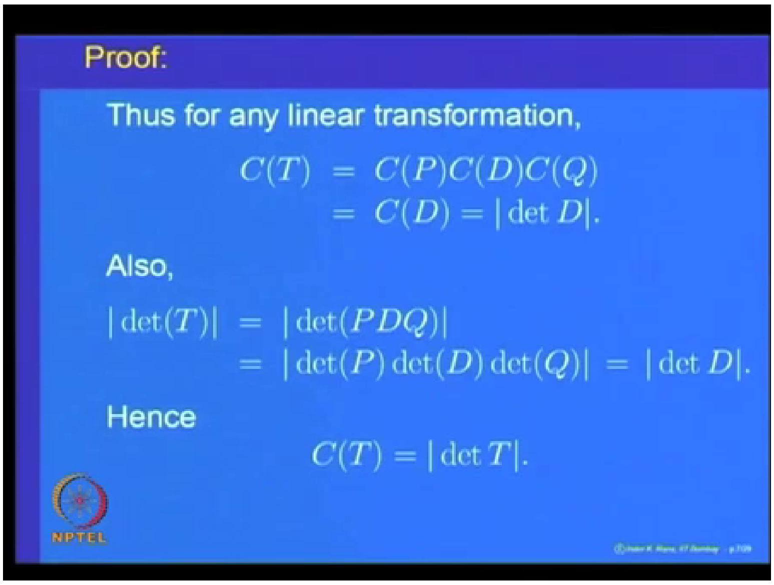

So, combining these 2 we get determinant of T is equal to determinant of D. So, sorry we get determinant of they should be C of T is determinant of D.

So, that is says that C of T should be equal to determinant of D. So, here it should be this is redundant. So, C of T is determinant of D, and determinant of D is equal to determinant of T. So, this to combine together gives you C of T is equal to determinant of T. So, that completes the proof of the fact that for a linear transformation. So, we have completed the proof that for a linear transformation.

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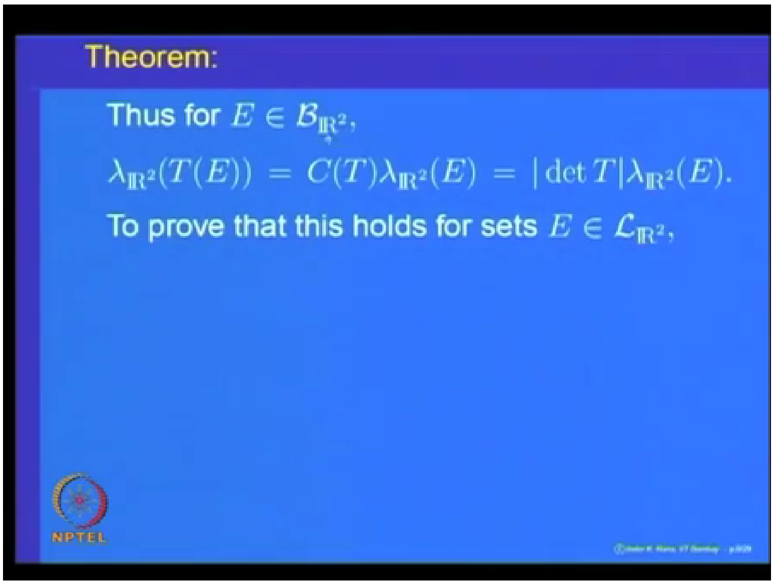

If you take the set E and change it transform it by a linear transformation that is same as determinant of t times the lebesgue measure of E.

So, the lebesgue measure of transform set is determinant of t times the lebesgue measure of the set e. So, that proves the theorem for all sets which are boreal measurable sets. So, till now we have proved the theorem only for boreal measurable sets would like to extend this theorem for lebesgue measurable sets. So, for that let us observe the following how are the lebesgue measurable sets in R 2 obtained from lebesgue from boreal measurable sets what is the relationship between lebesgue measurable sets and the boreal measurable sets.

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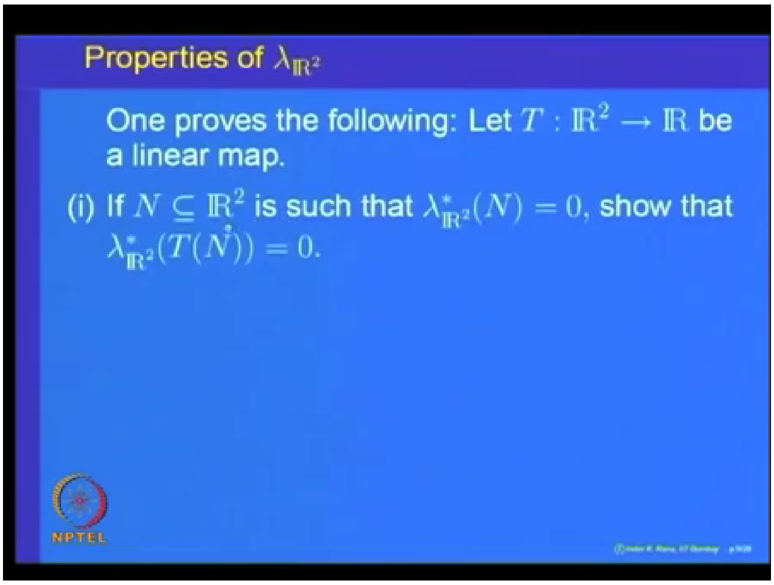

So, the first property is that if let us take any set N, which is in R 2 and says that its lebesgue outer measure is 0. We look at sets of lebesgue outer measure 0 in the plain. So, we first claim that under any linear transformation T, the image is also a lebesgue measurable set of measure 0; that means, linear transformations take sets of measure 0 to sets of measure 0 in plain.

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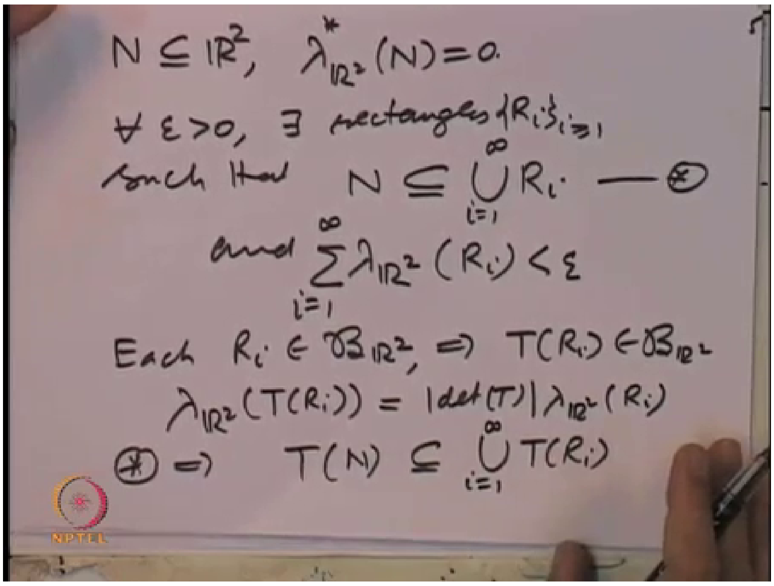

So, to prove that let us observe the following thing. So, let us take a N, a subset of R 2 and lebesgue measure of N equal to 0, but saying lebesgue measure of N is equal to 0 is same as saying for every epsilon bigger than 0. There exist rectangles; say R i I bigger than or equal to 1, such that the set N is contained in the union of this rectangles R i and the lebesgue measure of the rectangle R i added together is less than absolute.

So, saying set is a null, set is same as saying. It can be covered by rectangles such that the total measure of the rectangles put together is less than epsilon, but note now. So, each R i is a rectangle. So, it is a boreal set is a boreal sub set of R 2. So, that implies the T of R i also is a boreal sub set of R 2 for non singular linear transformation T. If T is non singular and the lebesgue measure of T of R i by what we have proved just now is equal to determinant of T times, lebesgue measure of lebesgue measure of R i. Statist this now were prove for boreal sets this property holds.

So, now the fact that N is covered is covered by union of RIS is implies. So, this fact star implies the T of N is covered by the union of T of R i. I equal to 1 2 infinity. So, this is contained by T of R i.

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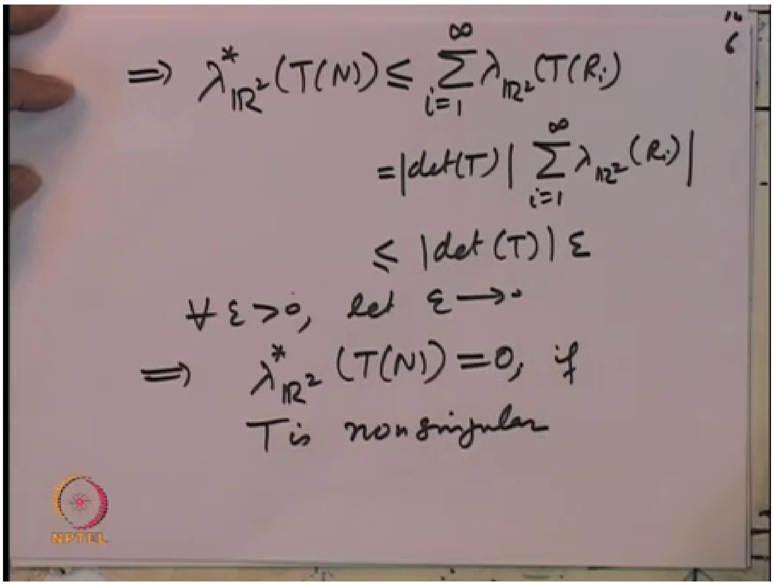

So, that implies by the countable subordity property of the lebesgue measure, outer of lebesgue measure R 2 of T N is less than or equal to summation I equal to 1 to infinity. Lebesgue outer measure of T of R i and lebesgue measure of T of R i is determinant. So, this is equal to determinant of T absolute value time's summation of I equal to 1 to infinity lebesgue measure of R i and that is less than epsilon. So, it is less than or equal to absolute value of determinant of T times epsilon.

And since this property holds. So, this holds for every epsilon bigger than 0. So, let epsilon go to 0. So, that will imply lebesgue measure of outer measure of T of N is equal to 0. When T is, if T is non singular and for a singular transformation we know T of R 2 itself is 0. So, T of N will be 0. So, this proves the fact that for every set N, which is of lebesgue outer measure 0, lambda the image T of N under any linear transformation is again a set of measure 0.