Measure & Integration
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Lecture – 30 B
Properties of Lebesgue Measure on R^2

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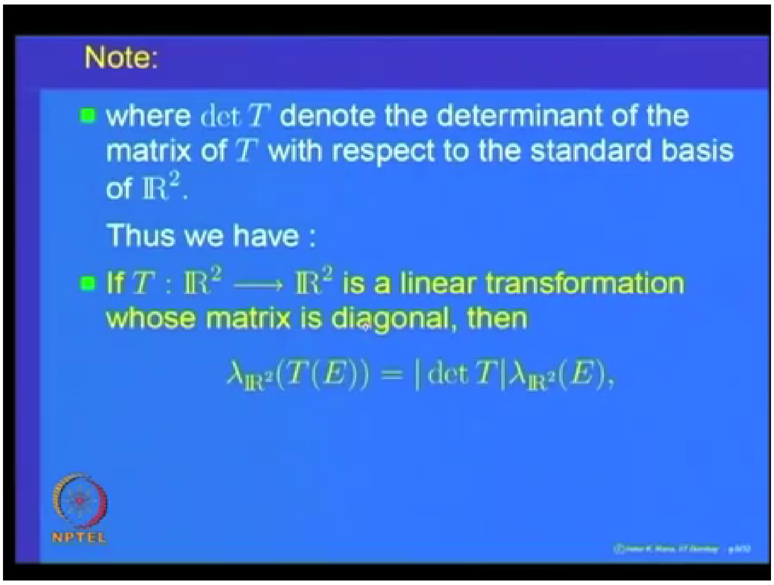

So we are got a result for diagonal transformations, namely for if T is a diagonal transformation from R 2 to R 2. And we transform a Lebasque measurable set according to this. Then the Lebasque measure of the transform set is absolute value of the determinant of T times the original measure. And the question is can we say this result is also true for arbitrarily in the transformations of the plane. And we are going to prove yes that is true it is this result holds for all linear transformations in R 2.

So, that is what we want to prove the theorem. So, the theorem says for all linear transformations in R 2, one can say that the Lebasque measure of the transform set is determinant of the absolute value of the determinant of T times the Lebasque measure of E.

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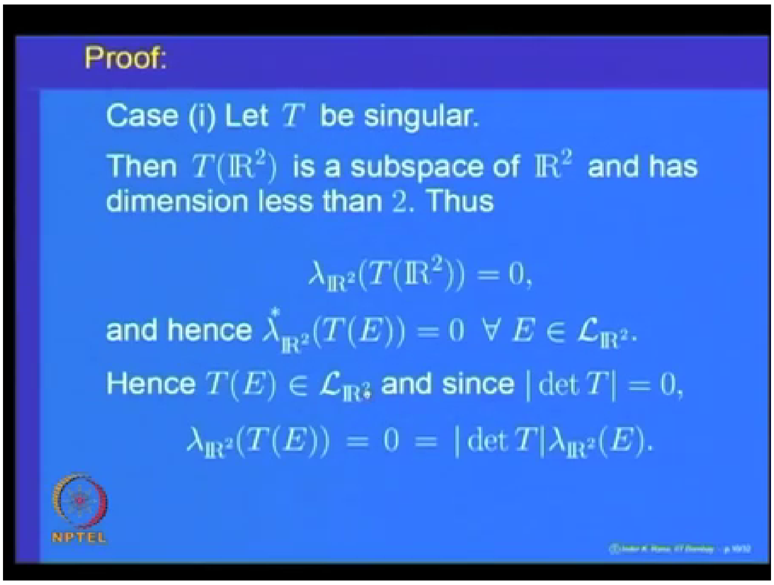

So, as before we will let assume T is a singular. And that we have just now observed when T is singular this result is true because T of R 2 is a subspace of R 2 of dimension less than 2. So, it will be either a line or a single point the origin vector. So, in either case Lebasque measure of the transform set is equal to 0. So, it is an outer measurable set of measure 0. So, it will belong to the Lebasque measurable is a Lebasque measurable set. So, what we are saying is that if E is a Lebasque measurable set and T is a singular transformation then T of E is an set of outer Lebasque measure 0 in the plane and hence it is Lebasque measurable and since determinant of T is also 0. So, the required property holds for required property holds when T is a singular transformation.

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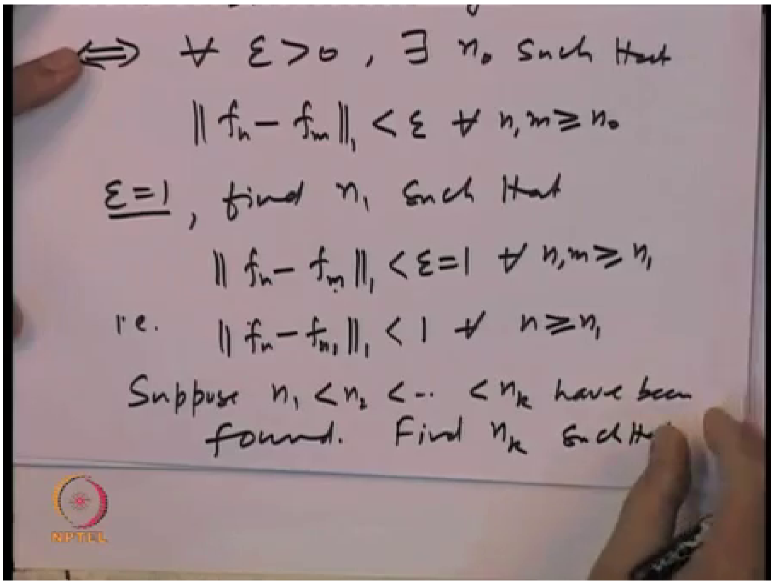

Now, second case is when T is non-singular. So, let us look at the case when T is non-singular and we want to prove.

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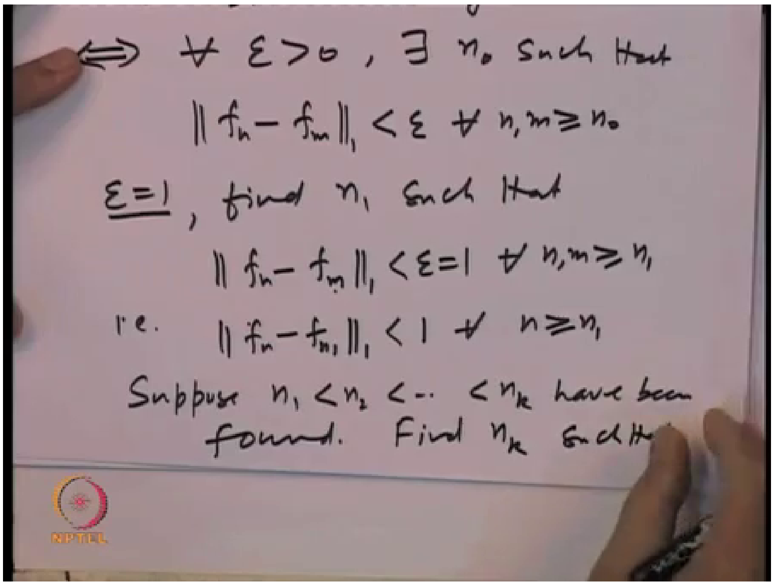

So, T R 2 to R 2 non-singular. So, claim for every E belonging to l Lebasque measurable set T of E is also Lebasque measurable. And the Lebasque measure of the transform set T of E is absolute value of determinant of T times Lebasque measure of E. So, that is what we want to check.

Later we will first do it. So, we will the let us assume for the time being that the set E is a borel subset in R 2. So, when E is a borel subset of R 2 we want to check the T of E is also a borel subset of R 2. So, that is the question we want to first analyse.

So, for that we observe that if T is a linear transformation from R 2 to R 2, T linear and non-singular that implies the nonsigularity implies T is bijective. Every linear transformation which is invertible of course, has to be bijective that is a and secondly and on the on the plane, we have got the notion of topology convergence one can easily check that every linear transformation is continuous.

So, we are using the 2 things here namely a linear transformation is non-singular if and only if actually it is bijective. And every linear transformation on R 2 to R 2 is continuous. Likely it is to for rn to rn, but we ares only constructing it for R 2 to R 2. So, T is continuous. So, not only T is continuous the inverse map bijective T inverse is also linear and hence continuous.

So, any non-singular linear transformation T from R 2 to R 2 is a continuous map and the inverse also is a continuous map. Once that is true let us.

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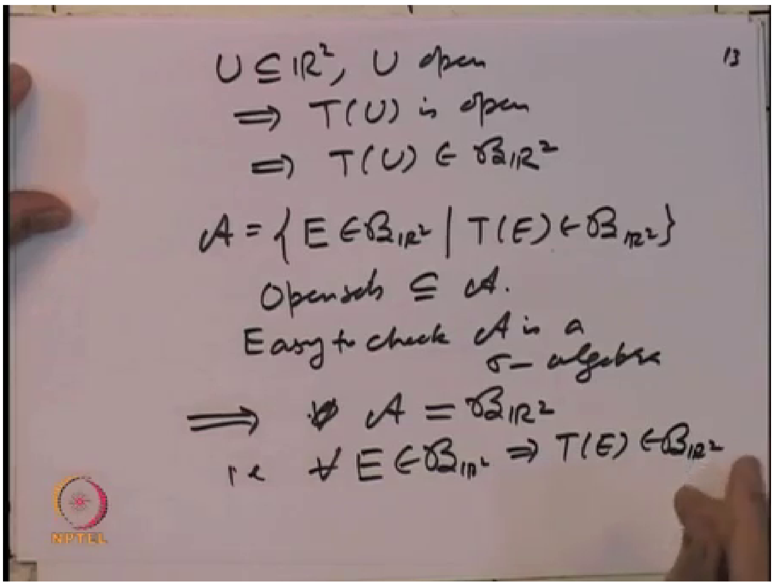

So, let us take a set U contained in R 2 and U open. So, that will imply T of U is open because T is a T inverse is continuous so; that means, that mean this implies the T of U is a borel set in R 2.

So, what we are saying is, if I look at the collection of all subsets all borel subsets in R 2, such that the image is a borel subset in R 2. Then this collection a collection a includes. So, open sets are inside this collection a and it is easy to check that. So, easy to check that this collection a is a sigma algebra basically. Because br 2 is a sigma algebra and T is a bijective map. So, a is a sigma algebra. So, that will imply that for that this this collection a is actually equal to br 2; that means, So, that is for every set E which is a borel set in R 2 implies T of E is also a borel subset of R 2. So, T preserves the collection of all borel sets.

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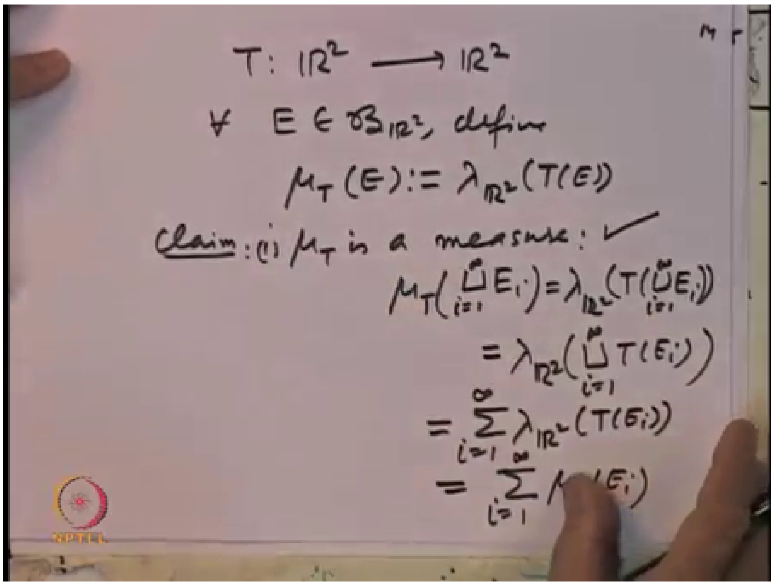

So, now let us T is a map from R 2 to R 2. So, let us define. So, for every set E which is a borel subset of R 2, let us define a new measure mu of T as follows mu T of E is equal to Lebasque measure of the set T times E. So, let us define.

So, the claim one that mu T is a measure. That is easy to check because so what is that is easy to check because if I take the disjoint union of sets ei want to infinity and then look at mu T of that. So, that is going to be lambda of T of the disjoint union of the sets ei of the sets. And now let us observe that T is a one to one on to map. So, T of the union is going to be union of. So, it is a disjoint union of T of ei i equal to 1 to infinity and lambda being a measure this is equal to sigma I equal to 1 to infinity lambda R 2 of T of ei and that is same as saying this is same as the summation I equal to 1 to infinity lambda R 2 of T of ei is mu T of ei. So, that proves the fact mu to is a measure.

The second property we want to check for this measure mu, T is that mu to is translation invariant. So, let us check that second property.

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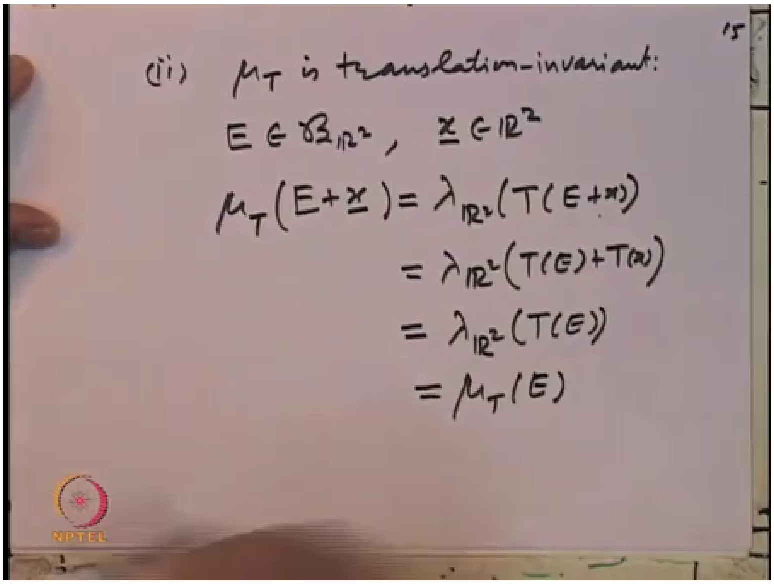

That mu T is translation is translation invariant on R 2 so; that means, what let us take a borel set in R 2. And let us take a vector x in R 2 and look at the vector E plus look at the set E plus x. So, that is a translated set. So, mu T of this set is equal to by definition lambda R 2 of T of E plus x, but T is a linear transformation. So, that implies T of E plus x is a T of E plus T of x. So, that is a that is a consequence of the fact that T is linear. So, that will imply and Lebasque measure being translation may be variant that says this is a lambda R 2 of T of E. And that is same as mu T of E. So, that is same as mu T of E. So, that it proves that mu T is a translation invariant measure.

Another fact about this prob this measure, let us write as a third property.

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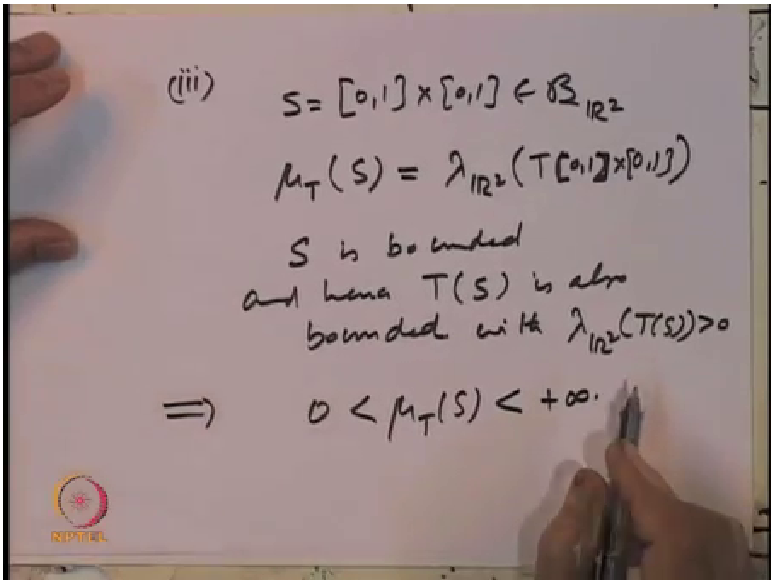

Let us take the set 0 1 cross 0 1 which is a borel set let us call this set as a s the square in the plane that is of course, a closed set cross a closed set. So, that is a closed set. So, it is a borel subset in R 2. And mu T of this set s is of is equal to lambda R T of the Lebasque measure of T of T of this, U T of 0 1 cross 0 1.

Now, let us observe T of 0 1 cross 0 1. And now let us observe that T of 0 1 cross 0 1 the observation is that this is a bounded s is bounded. And hence T of s is also bounded. It is also a bounded set. And of course, it is Lebasque measure is positive with positive measure with Lebasque measure Lebasque with Lebasque measure of R 2 ts being positive. And being bounded it has to be finite. So, that implies that Lebasque the measure mu T of s the measure of mu T of s is the measure mu T of s is positive is bigger than 0 and less than infinity.

So, these 3 properties of the measure mu, T let us look at what are 3 properties of the measure mu T that we have proved one. So, we defined the measure mu T of E to be equal to lambda R 2 of T of E and we said first of all it is a measure. That is one property that we proved the second property proved it is translation invariant and the third property we proved that there is a set of finite the positive measure with respect to mu of T.

So, all these 3 properties by the uniqueness of the Lebasque, measure in R 2 implies that mu T of E set every set E has to be equal to a constant multiple.

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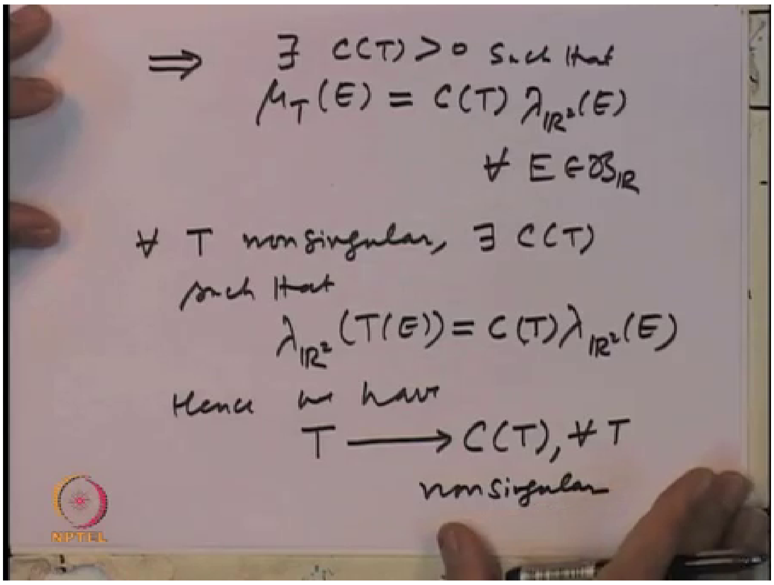

So, that is c of T times constant multiple of the Lebasque measure of E for every. So, implies there exist a constant c of T bigger than 0 such that for every set E which is a borel set.

So, this is where we are using the Lebasque measure is essentially the only translation invariant measure on the plane. So, there is a con, so any other translation invariant measure has to be constant multiple of the Lebasque measure. So, what we have got an is for every T non-singular, for every T non-singular there exists a constant c of T such that Lebasque measure mu T which is nothing, but the Lebasque measure of the transform set T of E is equal to is equal to the constant multiple c times T of the Lebasque measure of the set E. So, this is the property that we have established that, for every linear transformation the transform set T of E will be a borel set and it is Lebasque measure will be a constant multiple of this.

So; that means, this gives us a map. So, hence we have for every non-singular linear transformation we have got a constant c of T for every T non-singular. And what we want to do is to prove. So, to show that c of T is equal to absolute value of determinant of T for every T for every T non-singular. So, that is what we want to show. So, once we do that we will be through with our construction because c of T being determinant that will prove that it is non-singular.

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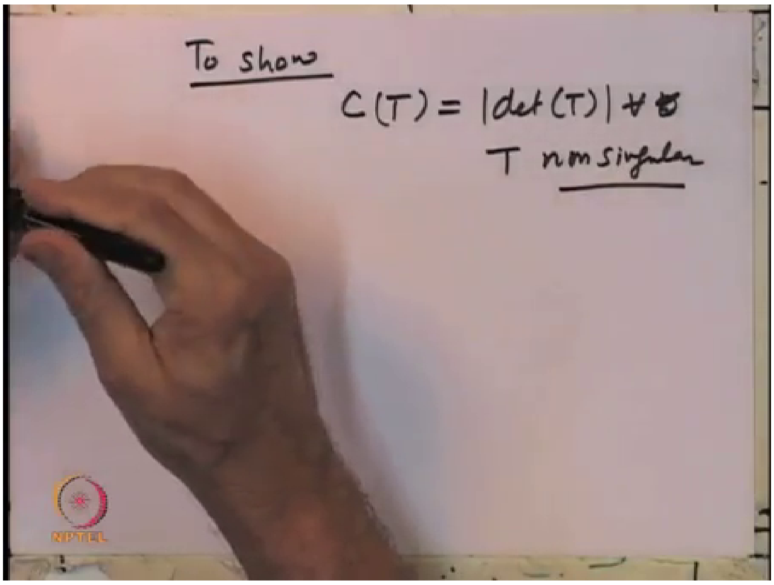

So, at this stage to prove that I need some more facts about linear algebra and it will, so to prove that we need some more facts about linear algebra. And we will not be able to complete the proof the remaining part of today's lecture. So, I will continue the proof next time. So, we will start from here saying that we have got for every non-singular linear transformation T a constant c of t; that means, there is a map T going to c of T, and what we want to show that this map does not this map actually is nothing, but the determinant of t. So, we will prove this next time.

Thank you.