Measure & Integration
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Lecture - 03 B
Sigma Algebra Generated by a Class

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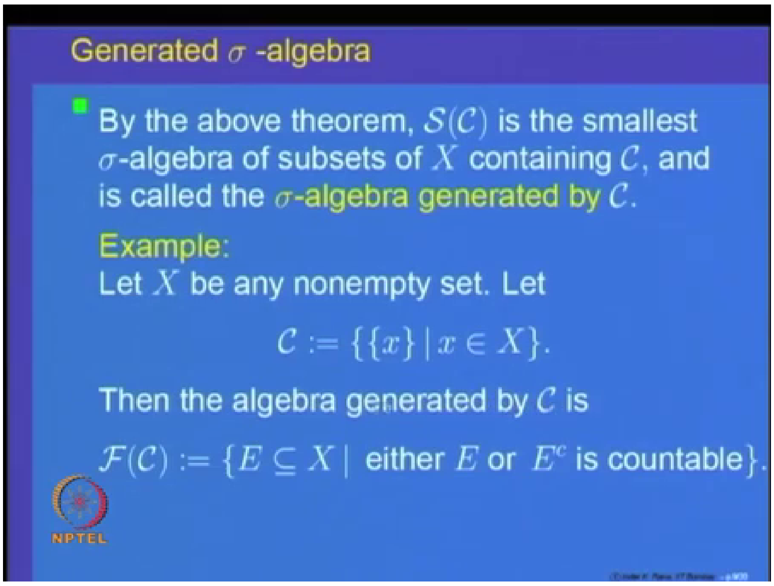

Let us look at some examples of sigma algebras generated. So, let us look at X, the collection of all X is any nonempty set and let us look at all singleton sets all singleton subsets of this set X. So, let this call that collection as C. So, C is the collection of all singletons; where singletons are elements of the set X.

So, the claim we want to find out, what is the sigma algebra generated by it? For a if you take a sequence of elements say x 1, x 2, x 3, x n in X and look at those singletons; then their union is going to be an element in C so; that means, all countable sets must be elements of C. And similarly all sets whose complements are countable also must be elements of the set e. So, as a consequence, we expect that this answer is nothing, but the algebra, so F of C the sigma algebra generated by so this is not correct. So, what we should have is? That the sigma algebra generated by C. So, this is nothing. So, this is correction here that then the sigma algebra generated by this must be equal to this collection F of C; where either E or E compliment is countable. So, let us prove this fact:

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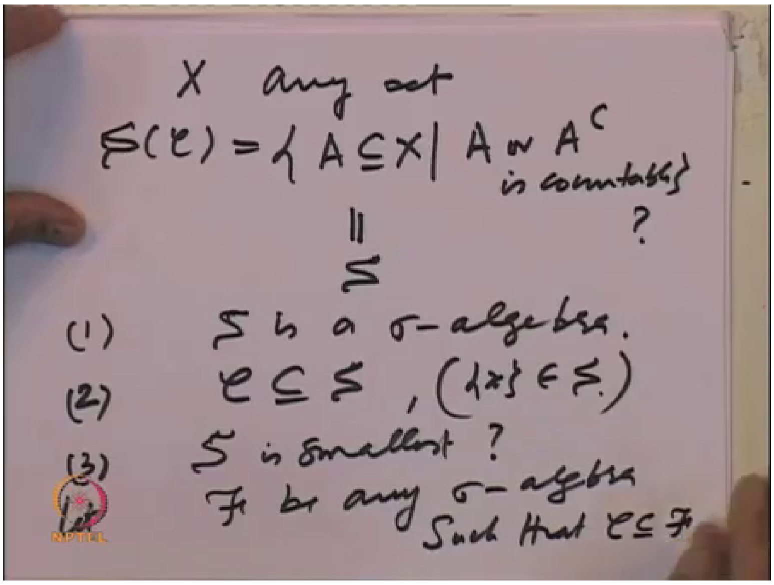

So, X is any set. We are taking S of C to be equal to all those subsets A contained in X such that, A or A compliment is countable. So, this is what we want to prove. So, let us call this collection as S. So, sets those which are countable or their compliments are countable, that collection is called S; and the claim is S of C the sigma algebra generated by the singletons is nothing, but this collection ok.

So, let us observe one first observation, that S is a sigma algebra. That we are just now proved: this collection S is a sigma algebra. Second that C is inside S right, because what a elements of C they look like singletons, and singleton is countable. So, this belongs to S. So, because of this reason C is a subset of S. So, this is sigma algebra which includes and the third we want to show it is a smallest S is smallest. So, let us take let us take any other algebra. So, let us let us look at let F be any sigma algebra, such that C is a subset of such that C is a subset of F. So, what we want to show? We have to show.

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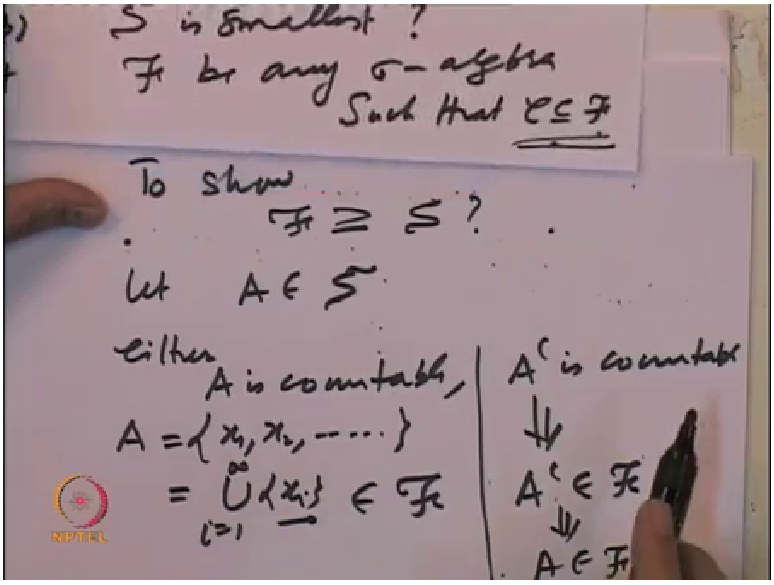

So, to show that F includes this collection S.

So, let us take; let A belong to S. So, either took case is arise either A is countable. So, in that case A can be written as x 1, x 2 and so on. So, that can be written as a union of singletons x i; i equal to 1 to infinity. And each x i is an element in C. So, and C is and C is inside F and C inside F. So, every singleton belongs to F. So, an F is algebra. So, this sigma algebra so; that means, this implies this belongs to F. So, if A is countable then it belongs to F; if not what is the second possibility? That A compliment is countable. By the same argument this will imply that A compliment belongs to F. And F is a sigma algebra. So, this implies that A belongs to F.

So, in either case; we have shown that if F is any sigma algebra which include C then this F must include S. So, that proves the fact; that S of C the sigma algebra generated by the singletons is nothing, but all those sets such that either the set is countable or its complement is countable. So, we are able to give a description. So, as a consequence we are able to give a description of the sigma algebra generated by a collection of subsets in this case; when the collection C consist of singleton sets, but let us be very careful in general for a set X given a collection C of subsets of X it is not always possible to describe the elements of S of C explicitly in terms of elements of C it is not possible always. And remember that was also the case when we looked at the algebra generated by a collection of subsets the C only; when C was a semi algebra we were able to describe, what is the algebra generated by the semi algebra? We are not able to give a general description of the algebra generated by a class C.

Similarly, it is not possible to give a description of the sigma algebra generated by a collection of subsets of a set X, but these are the collection; these are the collection of sets such are the collection of sets which are going to play a role in our subject later on. So, we have to study them carefully in detail let us look at some more properties of such kind of objects.

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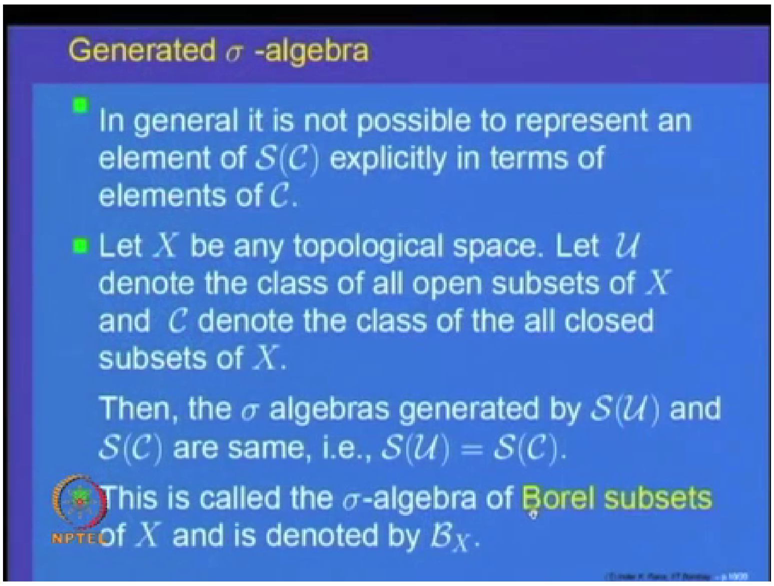

Let us look at something called a topological space. I hope some you are aware of, what is a topological space? A topological space consists of a set X and a collection of subsets of x which is called tau is called a topology.

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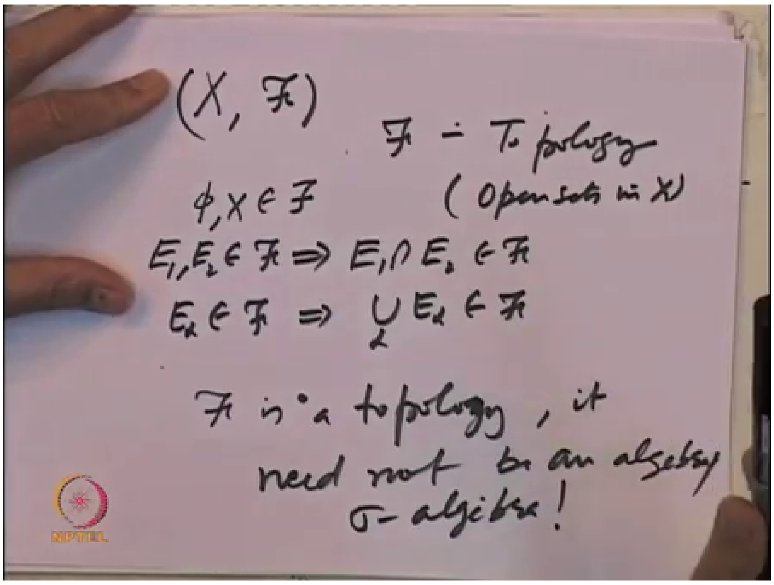

It is a collection of open sets in x. And this is a collection which has some properties namely the empty set the whole space belong to it and if two sets E 1 and E 2 belong to F then that implies E 1 intersection E 2 belongs to F and if E alpha is a collection of sets in F that implies union of E alphas belong to F. So, a topology is a collection of subsets of a set X such that the empty set belongs to it is closed under finite intersections and closed under arbitrary unions. And such collection of sets are called open sets.

So; obviously, if tau if is a topology, it need not be an algebra or a sigma algebra and the reason is; obviously, because this collection need not be closed under compliments which is required for a algebra or a sigma algebra. So, there it lacks (Refer Time: 08:53) in a topological space, the sets whose compliments are open are called close sets. So, if you want the complement also to be in that collection; then those are sets which are both open and close and they are not many examples of such things.

So, let us look at a topological space X F; F need not be a topology on X and this need not be a this need not be a sigma algebra. So, the question is can we generate can we generate the sigma algebra given by this topology and the answer is yes. So, let us look at the collection of all open sets in this topological space, and let us look at the collection of all close subsets in the topological space. So, you can generate the sigma algebra given by all open sets and we can also generate a sigma algebra of by the all close subsets of the topological space.

So, the question arises is there any relation between these two sigma algebras. And let us recall a set is closed; if and only if its complement is open. So, using this fact we will prove that the collection of all the sigma algebra generated by open sets is equal to the sigma algebra generated by close sets.

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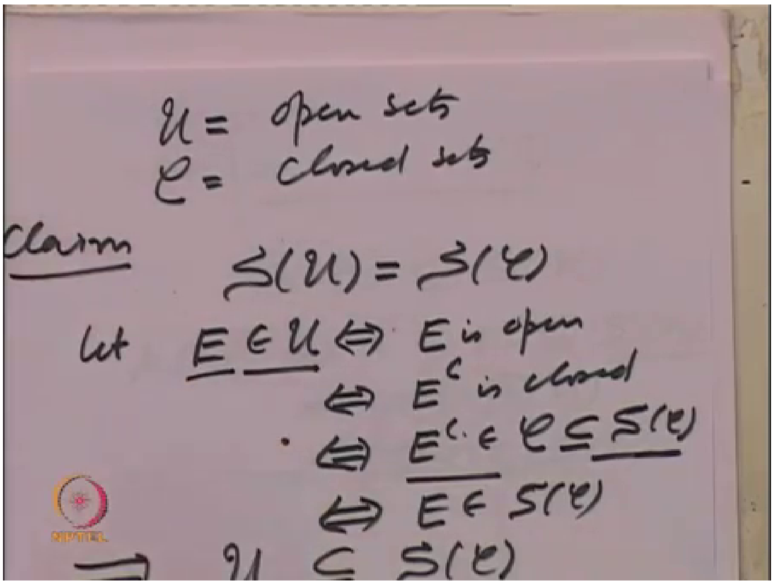

And this proof so u is open sets, C is close sets, and the claim is the sigma algebra generated by open sets is same as the sigma algebra generated by in the close sets. And to prove this: we will follow a technique which is going to be use again and again. So, let us observe. So, let us take a set E which belongs to u; that means, that is same as saying that E is open. And that implies which is same as equivalent to saying E compliment is closed; and that implies that E compliment is in the collection C which is inside S of C.

So, what we have shown is if E is an open set then E compliment belongs to S of C, and as a consequence of this E belongs to S of C, because S of C is a sigma algebra. So, this implies; that all open sets are inside the sigma algebra generated by are close sets. Now this is a sigma algebra which includes open sets. So, this sigma algebra must include the smallest sigma algebra including containing u; that means, once u is inside the sigma algebra S of C the sigma algebra generated by it also must come inside S of C by the very definition. So, this implies that S of u the sigma algebra generated by open sets comes inside S of C. So, this a technique which is used very up. So, to prove; S of u is inside S of C, what we have done is? We have shown that u is inside S of C; and hence S of u is inside S of C. So, this technique is going to be; this technique is going to be used very often in our course of lectures, we want to show certain collection of sets has required property. So, we show that that collection of sets includes a set of generators and hence will include the sigma algebra generated by it provided that collection forms a sigma algebra.

So, by this technique we have shown S of u is inside S of C and by same technique we can show that S of C is also inside S of u, because if a set if I take a set a which is in C; that means, A is close; that means, A compliment is open. So, it belongs to u which is in S of u. So, and that will imply that A belongs to S of u, because that is a sigma algebra. So, hence this implies that the sigma algebra generated by C must come inside S of u. So, the collection S of C is same as S of u. So, this is a very important collection of subsets of a topological space and this is given a name, such this collection is called the Borel subsets of the set X. So, the sigma algebra generated by all open sets or by all close subsets of a set X is called the Borel sigma algebra of subsets of the set X.

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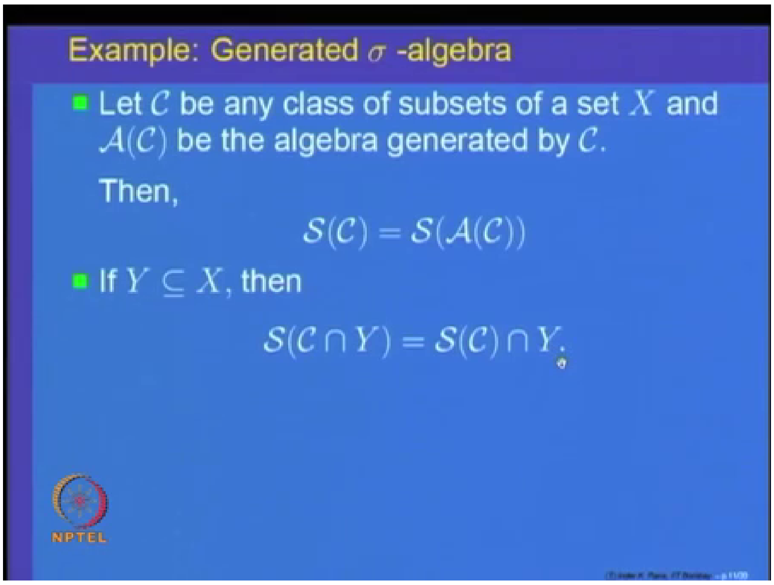

So, let us observe a few more things. So, we recall; we started with a collection C of subsets of a set X. And now we can generate; we said that given a collection of subsets of a set X, we can generate a algebra out of it; which is something algebra has a property which the class a may not C may not have. And now given this collection of C we can also generate a sigma algebra out of it; and we can also generate the sigma algebra by the algebra generated by that class right. So, the question comes what is the relation between these three things.

So, the observation we are going to prove is that given a collection C, you can directly generate the sigma algebra by this collection or you can generate first the algebra and then the sigma algebra by that algebra both processes are same. So, let us give a proof of this obvious fact; because such kind of proofs are going to be useful.

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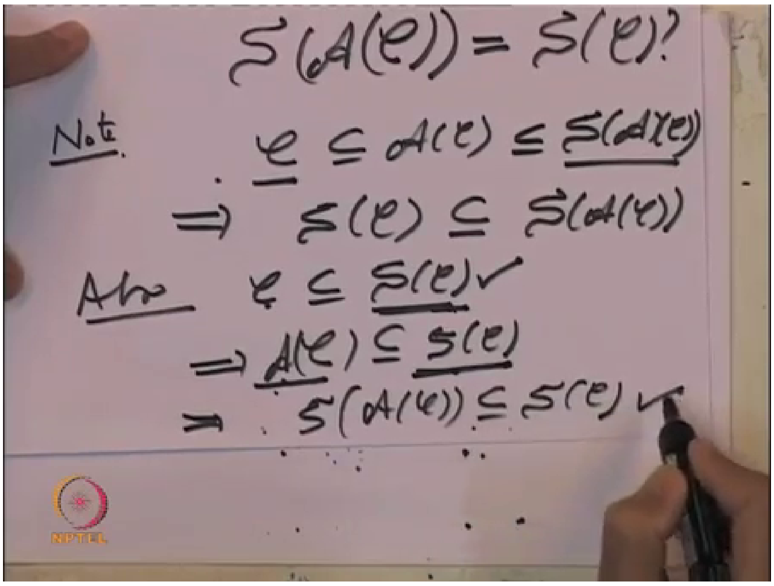

So, let us look at the proof; that if C is any collection of subsets of a set X, and I take the collection C I generate the algebra by this collection and then generate the sigma algebra by this collection that is same as the sigma algebra generated by C. So, this is what we want to prove. So, let us observe. So, note; C is contained in A of C. So, and which is contained in S of A of C by very definition. So, that implies C is the collection which is contained in the sigma algebra so; that means, the sigma algebra generated by a C is contained in the sigma algebra generated by the algebra generated by C. So, that proves one way to prove the other way round what we have to show. So, let us observe that C is contained in S of C also; C is contained in S of C and S of C is sigma algebra. So, hence it is also an algebra;

So, this implies that C; A of C must be inside S of C. So, here we have use the fact every sigma algebra is also an algebra. So, this is an algebra including C. So, the smallest one must be inside it. And now this collection is inside S of C this is a sigma algebra so; that means, the smallest one. So, the sigma algebra generated by A of C must come inside of S of C. So, that proves other way round inequality and this is already proved. So, these two together imply that the sigma algebra generated by any collection C, is same as you can first generate the algebra and then generate the sigma algebra it does not matter both are same. And the proof illustrates the use of this technique again and again if C is inside something and that something is algebra that algebra generated comes inside and so on. So, these are going to be techniques which are going to be use again and again in our course of a lectures. So, this is one observation that the sigma algebra generated by any collection C is the also the sigma algebra generated by the algebra generated by that collection.

Let us observe another thing; another way of generating more examples of sigma algebras. If you take any collection C and take a subset Y of it; then that gives us C intersection Y is the collection of subsets of Y, which are inside C right; and then we want to show that, the sigma algebra generated by this. So, we are restricting C to Y and then generating the sigma algebra by it and the claim is it is same as the sigma algebra generated by C first and then restricting it to Y.

So, let us give a proof of this fact. So, let us start with; so X is any set and C is a collection of subsets of P of X. So, and Y is a subset of X.

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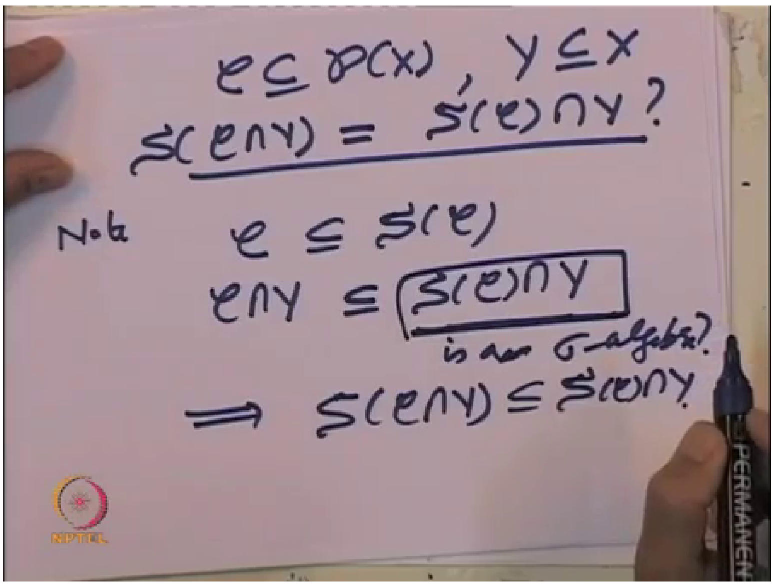

So, we want to show that the sigma algebra generated by C intersection Y is equal to the sigma algebra generated by C restricted to Y. So, this is what we want to prove. So, let us observe; so note C is contained in S of C right by the very definition C is inside. So, if I look at C intersection Y. So, look at the intersection of these sets that is going to be inside S of C intersection Y right. And if I can show that this is a sigma algebra. if I can show this is a sigma algebra, then I would have that S of C intersection Y will be inside S of C intersection Y. So, one should try to show that this is a sigma algebra. So, let us try to show that that is a sigma algebra.

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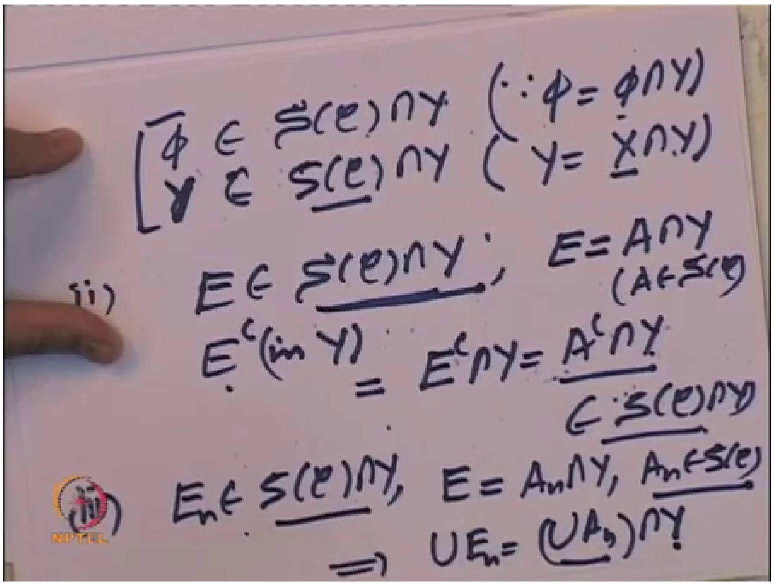

So, does empty set belong to S of C intersection Y? Yes, because empty set can be written as empty set intersection Y; and empty set belongs to S of C. Similarly the whole space so, what is the whole space? The whole space here is Y because S of C intersection Y. So, claim is that this belongs to S of C intersection Y that is true; because Y can be written as a X intersection Y and X is in the sigma algebra S of C. So, both this things belong.

So, second thing let us look at a set E belonging to S of C intersection Y. I want to look at the compliment of this, but what is the compliment of this? If this set belongs to S of C intersection Y; that means, this set E must be equal to some element. So, the some A intersection Y where is A; where A belongs to S of C that is by the definition. So, what is E compliment, but E compliment in keep in mind; we are looking at subsets of Y. So, what is E compliment in Y? That is same as E compliment in Y; that means, E intersection Y and that is same as A complement intersection Y.

So, and that again belongs to S of C intersection Y right. So, this collection S of C intersection Y is closed under compliments. And finally let us show it is closed under countable unions. So, if E ns belong to S of C intersection Y. Let us assume E n because it belongs here. So, it will be some A n intersection Y where A n belong to S of C. So, that will imply union A ns is union A n intersection Y and this belongs to S of C. So, is intersection Y. So, it belongs to so, that will implies that union also is an element if E ns belong to this then union also is an element of this.

So, we have proved that; this collection is a semi is a sigma algebra this collection is inside this. So, this will prove one way inequality. We have to prove the other way round inequality namely; so let us try to prove the other way round inequality namely. So, claim that S of C intersection Y is also a subset of S of C intersection Y.

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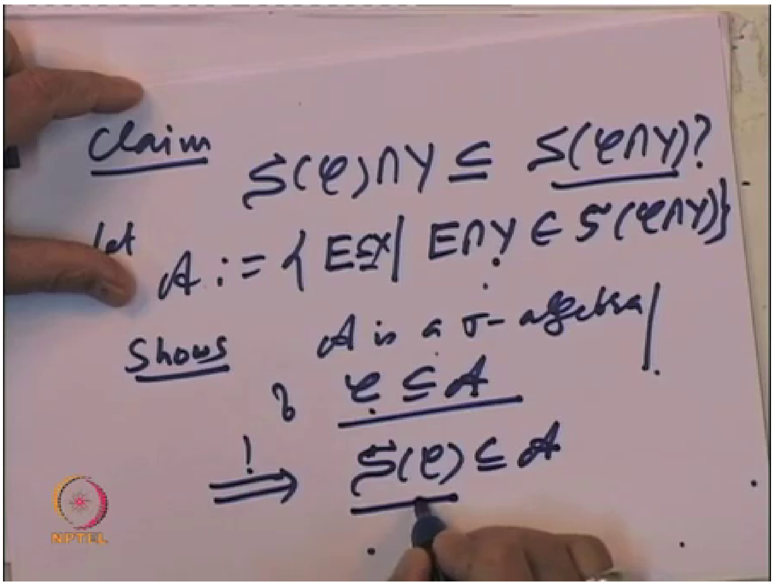

So, this is what is to be proved? And the proof once again, using the similar technique, so let us write A, look at all those subsets E, such that E intersection Y is an element in S of C intersection Y. So, look at all the subsets in x such that the intersection in its Y is inside the sigma algebra. So, one shows A is a sigma algebra and C is inside A; that means, what; if so that this will imply, because C is inside A it is a sigma algebra that will imply that S of C is inside A; that means, for all elements in S of C if I take its intersection that is going to be inside it. So, that will prove this required inequality. So, the claim this claim is equivalent to proving these two things; namely; A is a sigma algebra and C is inside A. So, let us observe why it is a sigma algebra; that is once again by the similar properties namely if E. So, let us show that this is a sigma algebra because if if you look at. So, let us try to show that E is a sigma algebra.

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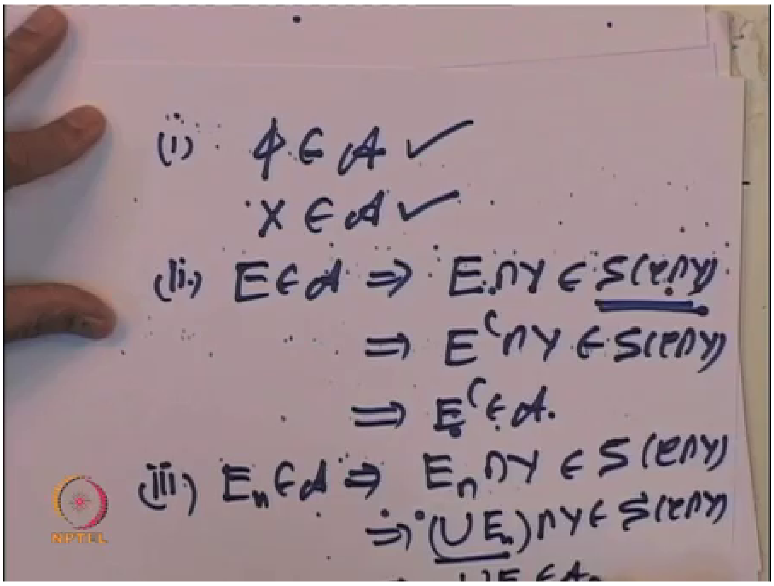

So, one does empty set belong to A? Yes, because empty set intersection Y is empty set which belongs to this, because this is a sigma algebra. So, that is similarly the whole space belongs to A that will be; second property let us take a set E which belongs to A. So, what does that imply? E intersection Y belongs to S of C intersection Y, but that, but this is a sigma algebra.

So, it must be closed under compliments and complements in Y so; that means, E compliment intersection Y also belongs to S of C intersection Y. And that implies that E compliment belongs to A. So, the collection A is close under compliments. And similar arguments will show that is closed under unions also. So, E ns belonging to A will imply right E n intersection Y belongs to the sigma algebra. And hence imply union of E ns intersection Y also belong to the sigma algebra and hence this set union E n belongs to A. So, this proves clearly that that A is a sigma algebra. And clearly C is inside A because E intersection Y in that case will belong to this. So, this will prove the required fact that what do we what trying to show that if I take a set Y inside X and restrict the collection C to subsets of Y and generate the sigma algebra that sigma algebra is same as the first generate the sigma algebra and then restricted to Y.

So, this is another way of generating more sigma algebras out of a given sigma algebra for a given collection right. So, this is a kind of technique which we are going to use later on in our subject for example, x will be the real line we look at Y will be an interval. So, we look at the open sets in the whole space of real line generated the sigma algebra; that is a Borel sigma algebra of subsets of real line. And then we can restrict them to the interval and it will say it is same as looking at the open sets in the interval and generating the sigma algebra out of it right. So, these are various ways of generating sigma algebras.

So, what we had done today is the following; we started with looking at algebras and described a special property of the algebras namely; we said that in an algebra any countable, union can be represented as a countable, disjoint, union. A very important aspect of an algebra. Then we moved on to looking at restricting the algebra to a set. So, we said if E is a subset of the set x and C is a collection, then you can restrict the class C to E and generate the algebra out of it that is same as first generating the algebra and then restricting it to it.

So, that is another way of restricting of algebras. And then we defined, what is called a sigma algebra of subsets of it? It is a collection of sets which is closed under compliments includes empty set and the whole space and is closed under countable unions in unlike algebra; algebra is closed under compliments and finite unions only finite unions. So, every sigma algebra is also an algebra and we gave examples of an a example of a collection of subsets of a set x which is an algebra, but which is not a sigma algebra.

So, sigma algebra is something more stronger than being an algebra. And then we finally, looked at how we one can generate a sigma algebra out of a given collection of sets and that is called the sigma algebra generated. The process is similar to the algebra generated take the intersection of all sigma algebras that include that collection and that will be the sigma algebra generated by it. The sigma algebra generated by the open subsets of a topological space is an important sigma algebra called the sigma algebra Borel subsets of that topological space and that is same as the sigma algebra generated by I all close subsets of a the topological space and that is an important thing. And finally, we proved that you can take any collection restrict it to a set and generate the sigma algebra that is same as generating the sigma algebra first then restricting it to the set. So, nest time I mean in the next lecture we will continue the study of classes of subsets of a set X and we look at some another important class called the monotone class of subsets of the set X.

Thank you.