Measure & Integration
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Lecture – 30 A
Properties of Lebesgue Measure on R^2

In the last two lectures, we have been looking at the properties of the Lebesgue measure on this space are to with respect to Borel measurable sub sets of R 2 and Lebesgue measurable subsets of R 2. We will continue studying these properties of Lebesgue measure on the space R 2 it bit more today.

If you recall we had looked in the previous lecture how does the Lebesgue measurable sets and the Borel measurable sets behave with respect to the group operation and the topological in i subsets of the plane. So, we showed in a pointed aspect of the Lebesgue measure namely up to a constant multiple Lebesgue measure is the only translation in variant measure on the space of on the set of all Lebesgue measurable and of course, in particular with the Borel measurable subsets of the plane. So, when we making using of that property today, but will begin with looking at some more transformation on the plane with respect to which Lebesgue measure can change and how does it change.

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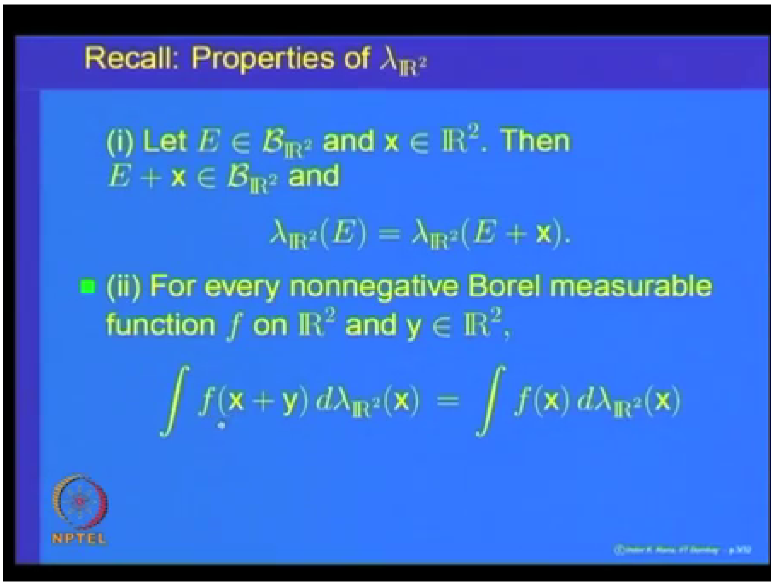

So, let us begin looking at today's topic is going to be Lebesgue measure and its properties further properties. So, let us just recall what we had shown in the previous lecture was that if E is a Borel measurable subset of R 2 then its translate E plus x is also a Borel measurable subset in R 2. And the Lebesgue measure of the translated set is same as the Lebesgue measure of the original set very much similar to the properties on the real line. And we also showed that for a nonnegative measurable Borel measurable function on R 2, if we look at the integral of the translated function that means, f of x plus y d lambda x is same as integral of f x with respect to x.

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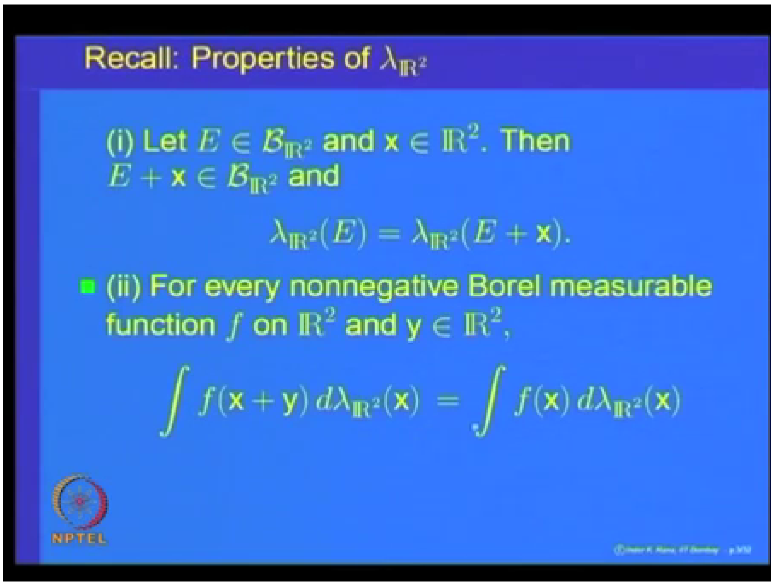

We also showed that this is also equal to actually the integral of f of minus x. So, under reflections and translations the integrals do not change.

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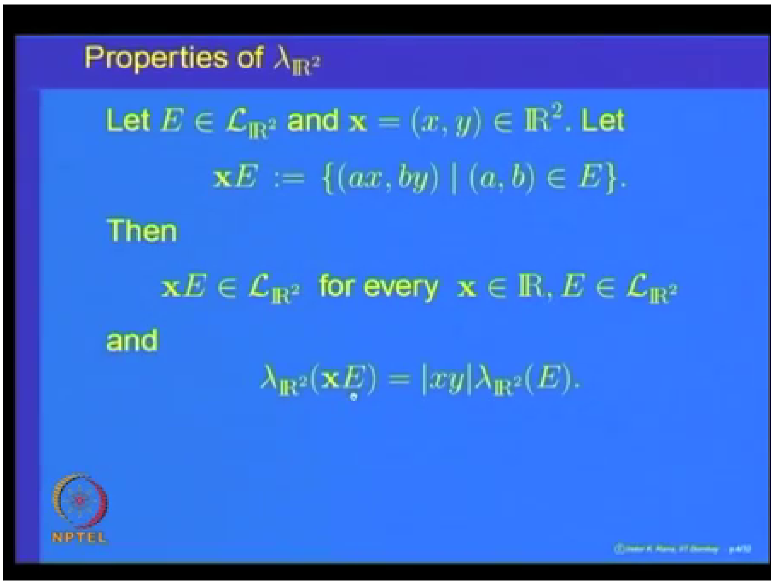

So, today we will start looking at another transformation on the plane namely for any vector x with components x and y in R 2, let us see how does this change if you multiply every element of a set with this vector. So, let us define the vector x dot E to be equal to a x comma by for every element a, b in E; that means, the component the coordinate x coordinate is multiplied by x and the y coordinate is multiplied by y for every element a, b in E. So, the first coordinate changes by a, so the first coordinate a changes by x or it goes to ax, and the second coordinate which is b equal to b times y.

So, the claim we want to prove is that for every Lebesgue measurable set E, this multiplied set x dot E is also a Lebesgue measurable set. And the Lebesgue measure of this transformed set is equal to absolute value of x y, where x is x component of the vector x; and y is the second component of the vector x. So, absolute value of x, y times Lebesgue measure of the set E.

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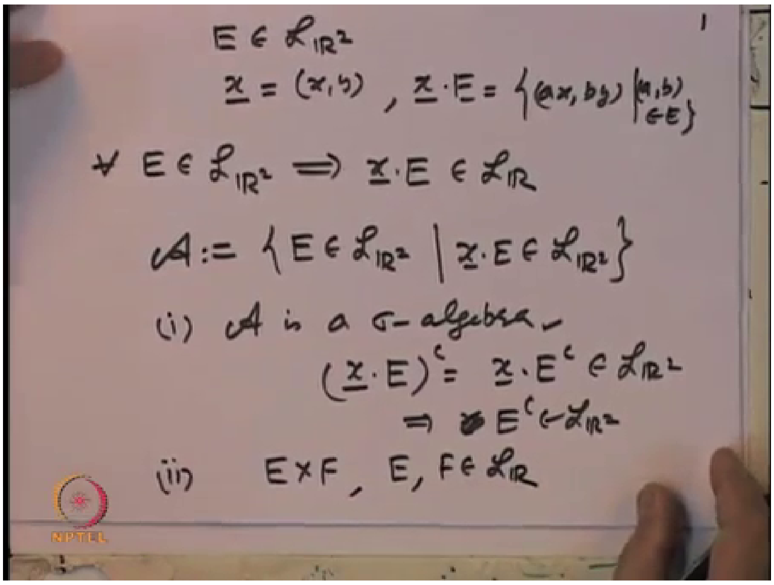

So, let us see how do we prove this property. So, we are given a subset E which is Lebesgue measurable and we were given a vector x with components x comma y, and we look at the vector x dot E which is all elements of the type ax comma by where a and b is an element in E. So, we want to check that for every subset E which is Lebesgue measurable in the plane implies that the vector x dot E is also Lebesgue measurable, so that is first part of the claim. The proof of this is on the similar lines as for the translation namely will collect together all the sets for which this property is true and show that rectangles come inside and show that this class for which this is true form the sigma algebra, so everything will be inside. So, that is the sigma algebra technique essentially we are going to use.

So, let look at that form the collection a to be equal to all sub sets E belonging to L of R 2 such that such that x dot E belongs to L of R 2. So, let us look at that. So, look us observe. So, first observation is that this collection A is a sigma algebra, this collection A is a sigma algebra that is easy to check I will just do not write I will just orally discuss this. So, let us take if a set E belongs to it then the observation is that we want to show that compliment also belongs to it, so that is easy because x dot E compliment is nothing but, so x dot E compliment is same as x dot E compliment. So, if E belongs to this collection a then x dot E belongs to this collection, so its complement belongs to the Lebesgue measurable sets and the compliment is equal to x dot E compliment and that implies. So, this belongs to L of R 2 will imply that x implies that E compliment belongs to L of R 2.

And similar computation will show that this is also A is closed under countable union, so that will prove it is sigma algebra. Let us take the second property namely if I take a rectangle, so if I take a set E cross F, if I take a set E cross F, where E and F both belong to L of R then what happens to x dot e. So, let us compute that.

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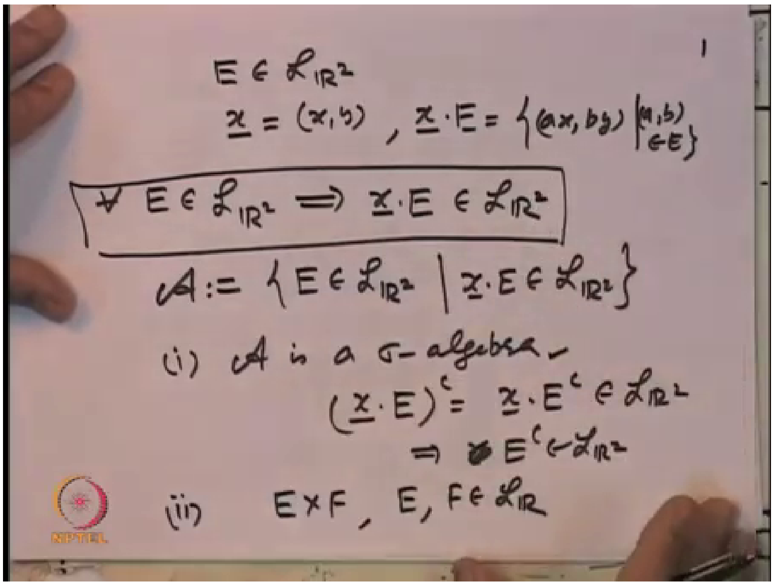

So, in that case the vector x dot E cross F will look like, so that is the x component multiplied with E cross the y component multiplied by F. So, it is xe cross yf. And now if x is a Lebesgue measurable set, we know that x dot E is a Lebesgue measurable set and y times f also is a Lebesgue measurable set in the real line. So, this means this belongs to L of R, this is a set in L of R 2, so that means, if I take a rectangle sets of the type E cross F where E is in L of R and F is in L of r then the rectangle satisfies the property of being in the class A. So, that means, the L R cross L R is inside L of R 2.

So, just now we observed that this is a sigma algebra, so that will imply that L of R the product L of R is also in L of R 2. So, all the elements in the sigma algebra in the product sigma algebra L R cross L R are inside L of R 2. And now L of R 2 again is the completion of this space, so that is easy to show that if I take a null set then x dot E is also is a null set. So, also if E is subset of R 2 and lambda R 2 of star of E is 0, then it is easy to check by leave it as an exercise that x dot E is again a null set then x dot E lambda star R 2 is again a null. So, lambda outer measure of that is again measure 0, so that imply that also so implies that such sets.

So, lambda sets for which. So, the sets for which the Lebesgue measurable is 0, Lebesgue outer measure is 0 are also in that class, so that will imply this along with the earlier fact. So, this fact and this fact together imply that the A is equal to L of R 2. So, how this is basically the sigma algebra technique which is used to prove that the class for every to prove the fact that for every set E, this property is true. So, x dot E belongs to L of R 2 whenever E belongs to L of R 2. And next we want to check that the Lebesgue measure also it preserved, so that again is a proof which is similar to the earlier proof.

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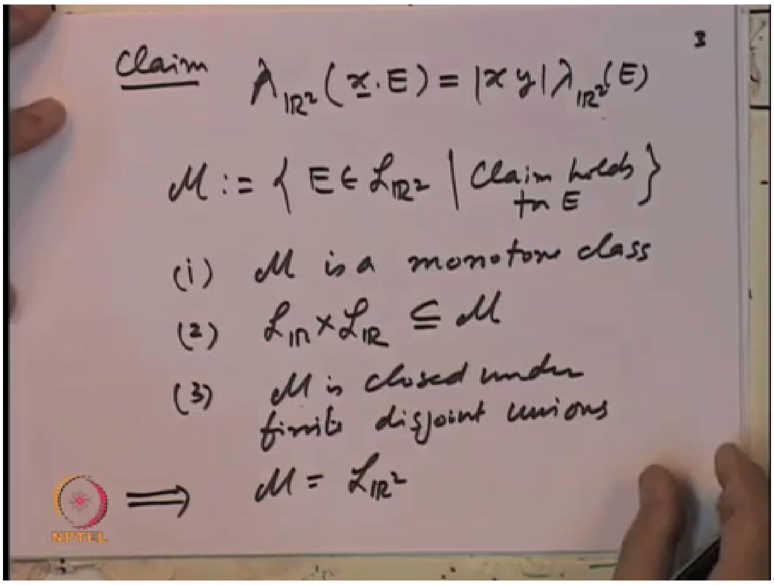

So, let us those who want to check the claim the second part of the claim is that the Lebesgue measure of the set x dot E is equal to x y product Lebesgue measure of the set E. So, this is what we want to check. So, let us observe the following. Basically, we are going to apply the monotone class technique here as in the case of the translation. So, let us define M to be class of all sets E belonging to L of R 2 such that this required claim. So, let us put that as a star. So, this property claim holds for the set E. So, technique is to show that one that this class M is a monotone class. So, that is one step. Two, show it is a monotone class two shows that the sets L R class L R rectangles are inside this class M; and third this M is closed under finite disjoint unions is closed under finite disjoint unions.

Once these three steps are proved, this imply that M is equal to L of R 2 essentially. So, the idea is the following because this rectangles are inside M, so second steps is rectangular inside M and first one says is the monotone class. So, the monotone class generated by these rectangles will be inside M. Now, third property says that this is also closed under finite disjoint unions. So, once it is a monotone class, and it is closed under finite disjoint unions that will imply that the semi algebra generated by the algebra generated by L R cross L R also is inside m. So, the monotone class generated by L R by L R will be in. So, let me just write these steps that why this will imply this.

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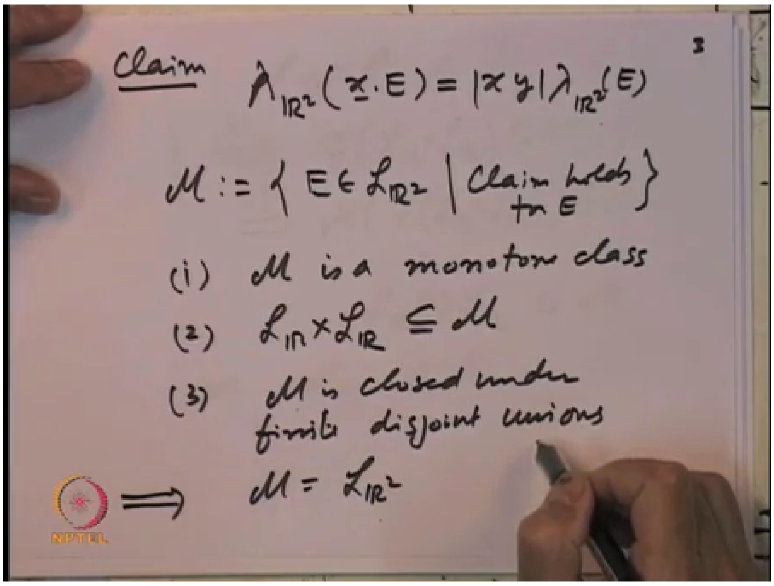

So, basically the reason is the following. So, here is a reason why this will happen. So, L R cross L R the rectangles inside M will imply by a step three finite disjoint unions that the algebra generated by these rectangles will also be inside the class M because it is closed because algebra generated by a semi algebra is nothing, but finite disjoints unions. And now this will imply by 1, that M is a monotone class and this algebra is inside it, so that will imply that the monotone class generated by this algebra, so that is L R cross L R also be inside M. But this our monotone class theorem says that this is nothing but the sigma algebra generated by this class. So, this will imply this is L R the product sigma algebra Lebesgue measurable sets cross Lebesgue measurable sets inside m.

And now once again one shows that this is also true for null sets. So, it will hold for completion. So, that will imply that L of R 2 which is the completion of this is also inside

M and that inside m, so that will prove the required result. So, these steps that this collection M is a monotone class and includes rectangles and is closed under finite disjoint unions, this proof is similar to that of the proof when we had translation of set E by vector. So, I will says that you try this as a exercise yourself and on the same lines as the earlier proof, because it is the repetition of the same idea again and again, so it is better to get used to it by doing it yourself.

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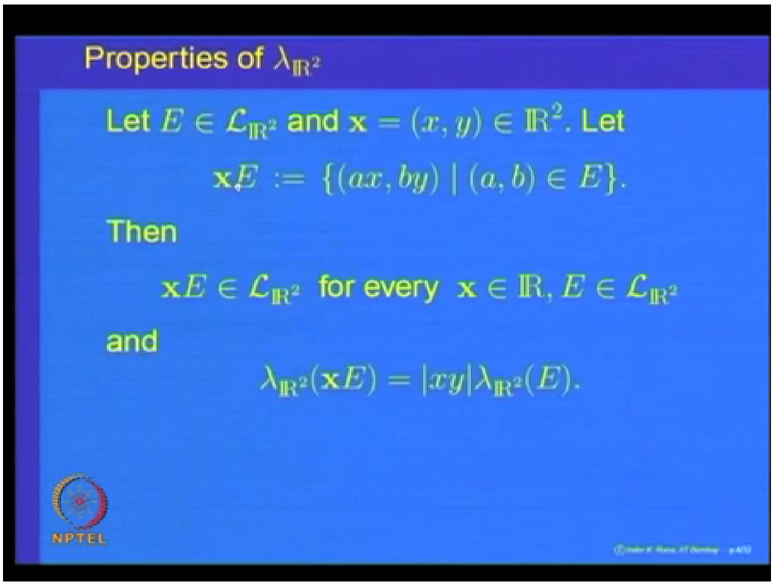

So, to prove that so that will prove the required claim. Namely, so this property for the Lebesgue measurable will be proved, namely if E is a Lebesgue measurable set then the product x times E is also a Lebesgue measurable set and its Lebesgue measure is equal to the Lebesgue measure of the set E multiplied by absolute value of the components of x that were x comma y. So, this property helps something about multiplication, and now we will like to rewrite this property in a slightly different way.

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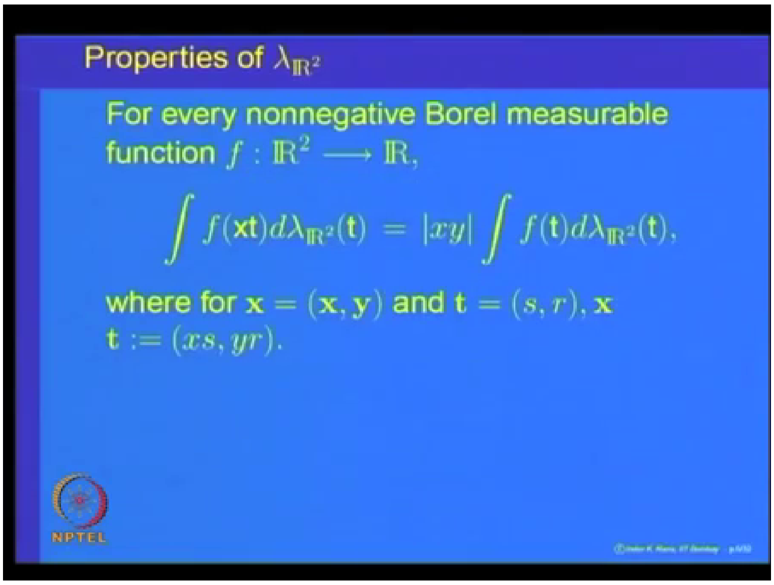

So, before that let me just state the corresponding result for integrals namely that the is f is a nonnegative measurable function on R 2, then integral of x times t this multiplication as we find now is equal to the absolute value of x, y times the integral of function f. So, that is how if I multiply the function f which by vector x then the integral changes by the value mod of x minus y. And the proof of this is once again an application of sigma algebra monotone class simple function technique I am sorry, and I would like to leave it as an exercise once again. So, copy the proof, try to copy the proof or the translation x plus 3 saying that the integral is invariant and here the multiplication comes.

So, the steps are essentially first take f to be the indicator function of set measurable set and that is just now we prove that result. And once it is true for indicator functions, this being equality involving integrations. So, it will hold for finite linear combinations, that means this will be true when f is a nonnegative simple measurable function. And for a general nonnegative measurable function, one takes limit of nonnegative simple measurable function and essentially applies monotone convergence theorem to get that this result is also true. So, the standard simple function technique will give you a proof of this. So, I will says that you have a look at that proof yourself.

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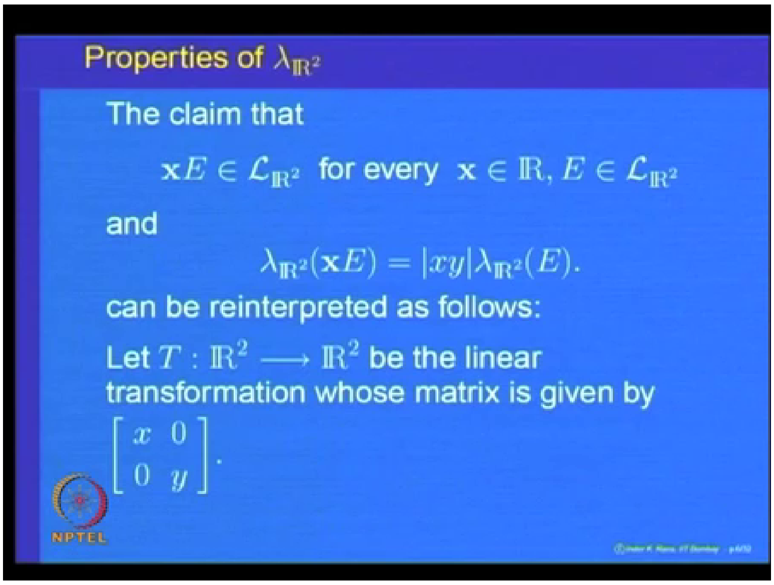

So, let us go to some more properties of Lebesgue measurable. So, just now we proved that for a Lebesgue measurable set E and a vector x in R 2, so they should be R 2. If you multiply. So, lambda R 2 of x E is equal to the absolute value of the product of the coordinates of the vector x with which you are multiplying into the Lebesgue measurable of E. One can reinterpret this result as follows, namely let us look at a transformation t from R 2 to r a linear transformation whose matrix is given by the vector by the diagonal by the diagonal matrix x 0 0 y. So, here is some knowledge of linear algebra is required. So, let me state a few things about some facts about linear algebra that we are going to use. So, here is some facts about linear algebra that we are going to use.

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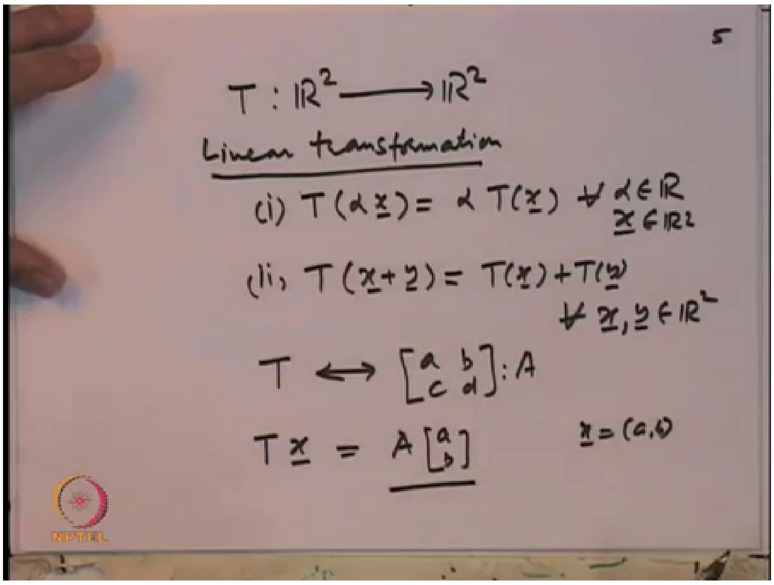

So, the first thing is that a linear transformation is in linear algebra. So, let us look at I will be looking at linear transformations on R 2 only. So, let us look at linear transformations on R 2. So, what is a linear transformation a linear transformation a linear transformation is a function T from R 2 to R 2 which has the following property. The T of if I take alpha times a vector x, it is alpha times T of x for every alpha belonging to R and the vector x belonging to R 2 that is one property. And the second that for any two vectors x and y; if I look at image of T of x plus y that is equal to T of x plus t of y for every x, y belonging to R 2. So, such a map is called a linear transformation and any such linear transformation T is also given by a matrix two by two matrix. So, let us call it as a, b, c and d.

So, what is the meaning of this, that means, that T apply to a vector x is same as the if you call this as the matrix A that is a matrix A applied to the component of x. So, let us call it as a, b, T applied to a, b where x is equal to a comma b. So, this is very centered thing in linear algebra that linear transformation described by matrix multiplication where A is called the matrix corresponding to the linear transformation and that is obtained by a basis of R 2. So, I will not go into details, so that is what essentially will require.

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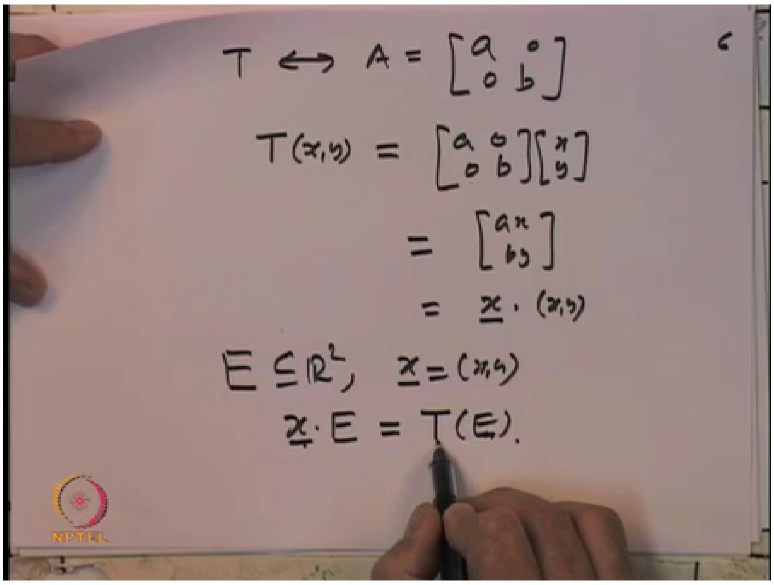

So, let us look at a special linear transformation namely T which comes from the matrix which is a diagonal matrix. So, a b 0 0. So, what is the effect of the linear transformation T, T applied to a vector with components x, y that is same as the matrix a 0 0 b applied to the column vector x, y and the matrix multiplication says it is just ax and by ax plus zero y plus 0 x and by. So, that is same as in our notation that is the vector x multiplied with the vector with components x y.

So, in our notation when we had this matrix, so we had this for any set E in subset in R 2 and we had a vector x with components x y. So, saying that we are multiplying x with E the property that we studied just now is same as looking at the image of T image of the set T under this linear transformation E, so that is what the interpretation is. So, we can say that this multiplication by the vector x is transforming the set E by the map linear transformation T which is the map from R 2 to R 2.

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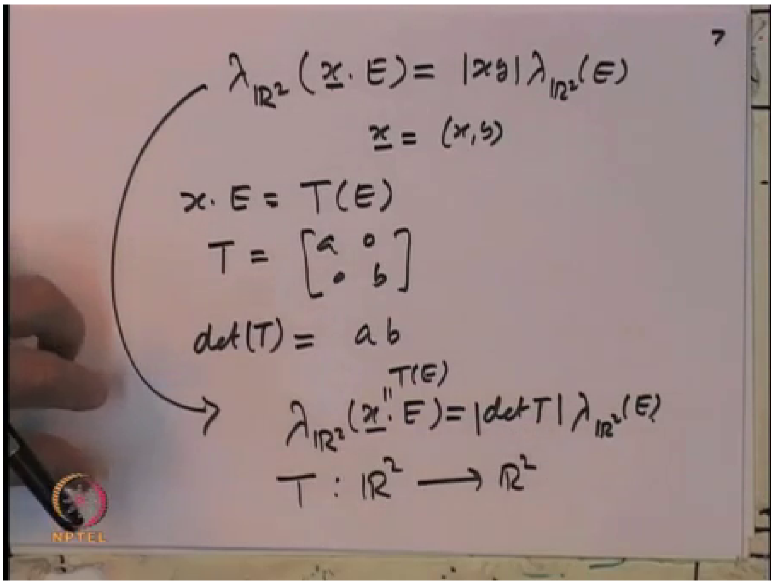

And our results says that the result we proved just now says that the Lebesgue measurable of R 2 of the set x dot E is equal to the absolute value of x, y times the Lebesgue measurable measure of the E, where the vector x is equal to with components x and y. So, we said x dot E is the image of E under the linear transformation T, and also this T is given by the diagonal matrix a 0 0 b. So, for this diagonal matrix there is a notion of what is called determinant of this transformation T. So, what is the determinant, determinant for two by two, this is cross multiplying and subtracting the value. So, it is a times b for this determinant.

So, absolute value of x, y, when you multiplying by x. So, here our vector is with component x and y. So, this result can be interpreted as lambda R 2 of x dot E is equal to this absolute value of x, y is nothing but the determinant of T times lambda R 2 of E. So, our result that under multiplication that is how the value changes of the Lebesgue measure can be interpreted in terms of the linear transformations that if I take the linear transformation T from R 2 to R 2, which gives this multiplication as interpreted earlier. Then the Lebesgue measure of the translated set, so this is T of E Lebesgue measure of the translated of the transform set is determinant of T times the Lebesgue measure of the original set. So, this is the property interpreting in terms of maps.

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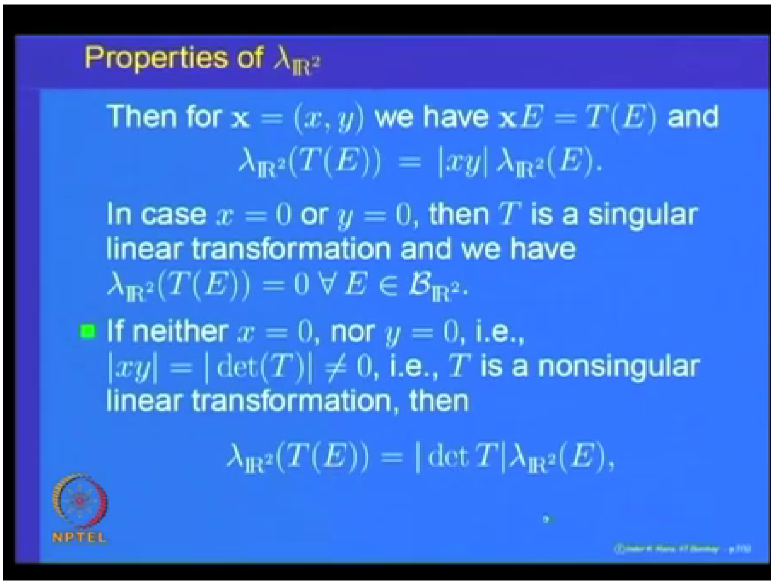

So, let us also observed something called T is singular in that case determinant of T is 0. And basically saying that if x or y are 0 then this both sides are 0. So, if neither x nor y is 0 then determinant is not 0 and T is non-singular. So, let us just look at this facts bit more seriously.

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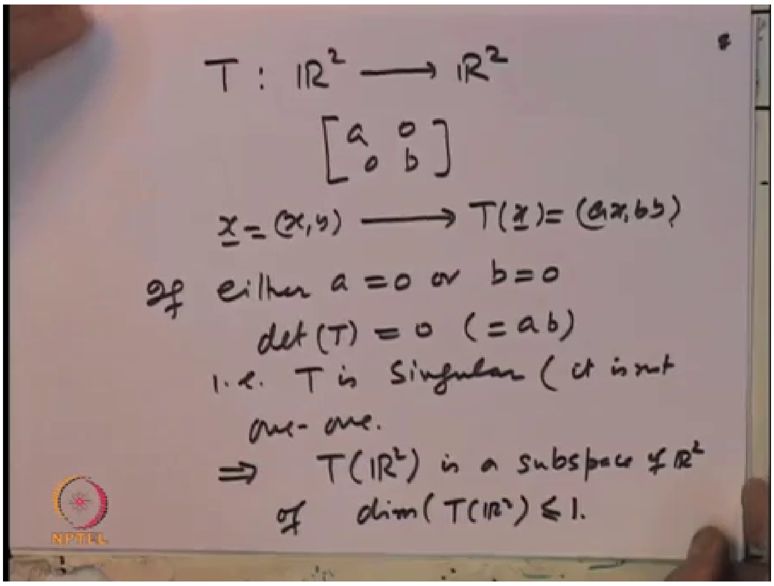

So, let us we have a map T from R 2 to R 2. And this is given by the diagonal matrix a 0 0 b. So, this is the matrix so; that means, any vector x y goes to, so x comma y a vector x goes to T of x which is nothing by a x comma b y. Now, let us observe thing here that if either a is 0, or b is 0, if either a is 0 or b is 0, then determinant of T is equal to 0, because determinant of t is equal to equal to a times b. So, determinant of that means, that is T is singular whenever determinant of a linear transformation is 0, T is singular, that means, it is in terms of functions it is not 1 1, it is not 1 1.

So, T in a linear transformation, which is singular, so it is not 1 1, so that implies the image T R 2 of the whole space is a subspace of R 2. Because under linear transformations the image is always a sub space of dimensions of T R 2 has to be less than or equal it cannot be two because then it will be 1 1. So, because it is not 1 1, so that is less than or equal to 1. So, here I am discussing about of linear algebra because that will be required here.

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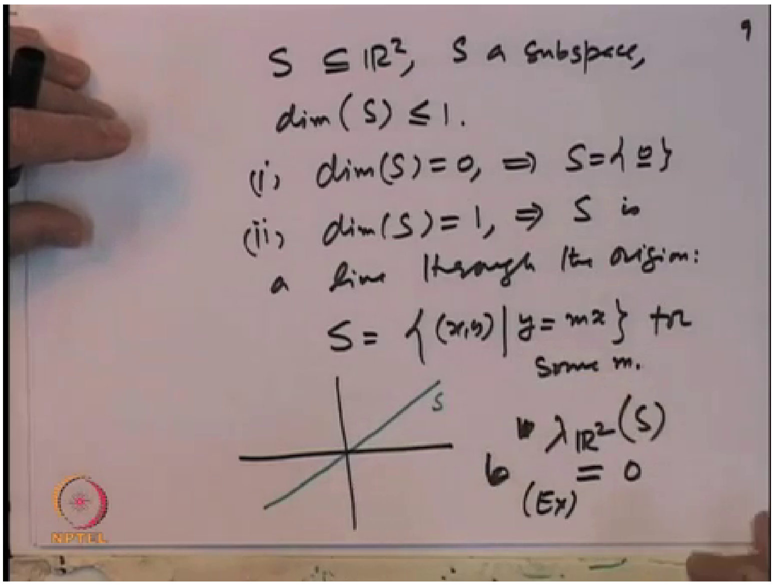

So, the question arises what are subspaces S of R 2, S a subspace and dimension of dimension of S less than or equal to 1. So, for dimension of so one possibility is dimension of S is equal to 0, so that implies that S is just the vector zero vector. Or secondly, dimension of S is equal to one in that will imply that geometrical, S is a line through the origin or mathematically I can write S as all x comma y where it is all x comma y where a subspace S should be look like y is equal to m x for some m. That means in R 2, a subspace has to be nothing but a subspace is just a line through the origin, so subspace S to be a line through the origin.

So, once it is a line through the origin what is going to be the Lebesgue measure of this line. So, let us look at the Lebesgue measure of R 2 of this line S, so obviously, the guess is this is going to be equal to 0. There are various ways of proving this what one can do is to prove that Lebesgue measure of any line is equal to 0, what one can do is try to approximate this line by small rectangles or. So, I think this is the good exercise to leave. So, I leave it as a exercise to check that the Lebesgue measure, saying that the area of the line is equal to 0.

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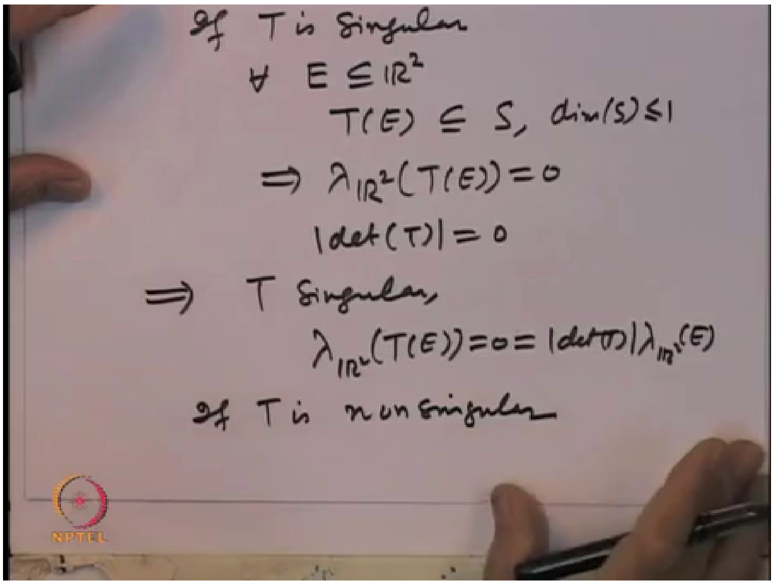

So, let us use these facts. So, if T is singular, then for every subset E contained in R 2, T of E is going to be a subset of dimension one. So, is a subset of S dimension of S less than or equal to 1 implies that the Lebesgue measure of R 2 of this set E is going to be equal to 0. On the other hand, we also know the determinant also of T is also equal to 0, so that implies for singular transformation T singular then the Lebesgue measure of T of E equal to 0 equal to determinant of T which is again 0 time Lebesgue measure of E. So, that property holds when T is singular.

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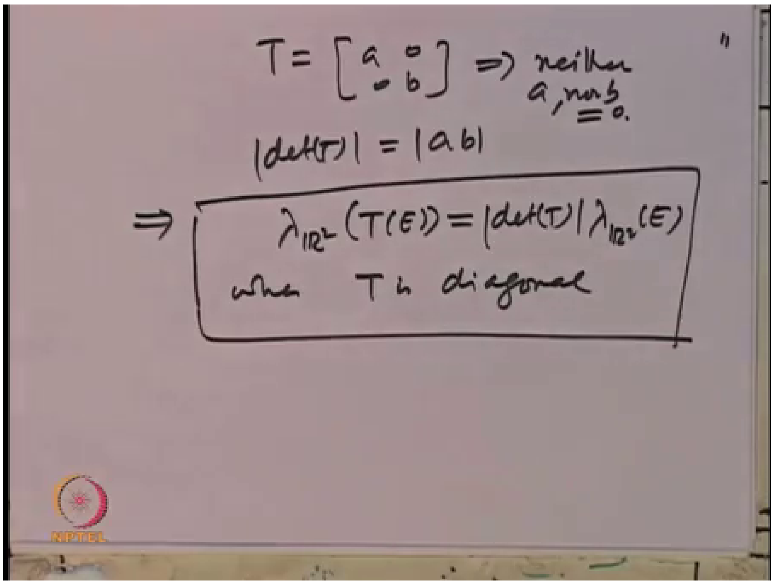

And if it is nonzero, if T is nonsingular, if t is non singular that means where T is given by a 0 0 b that implies that neither a nor b is equal to 0 and determinant of T is equal to a b. So, once again implies that our earlier result implies that lambda of R 2 of T of E is equal to determinant of T times Lebesgue measure of the set E, so when T is diagonal. So, for diagonal transformations we have this is result that if we take a Lebesgue measurable set E and transform it according to a linear transformation, then the transformed set has got Lebesgue measure which is determinant of T absolute value times the original Lebesgue measure of R 2.

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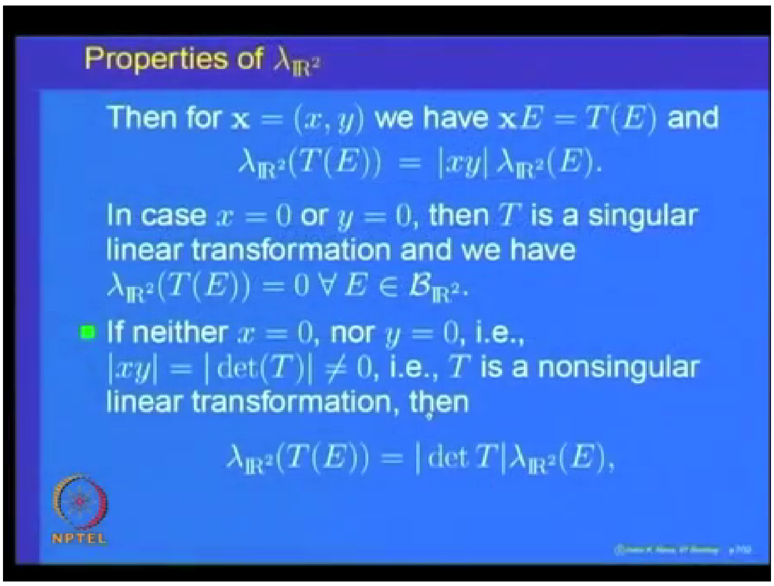

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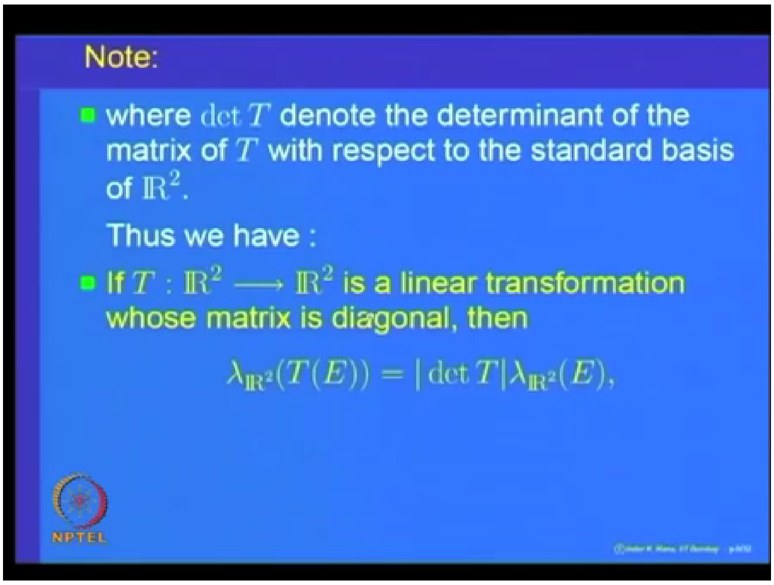

So, this is for non singular determinant. So, the question arise is can we say that this result is true for all linear transformations.