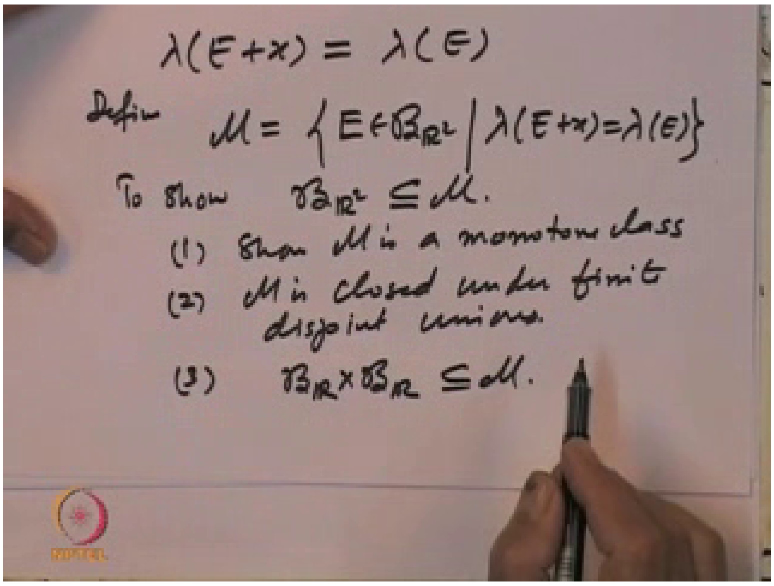
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 29B
Lebesgue Measure and Integral on R^2

(Refer Slide Time: 00:16)



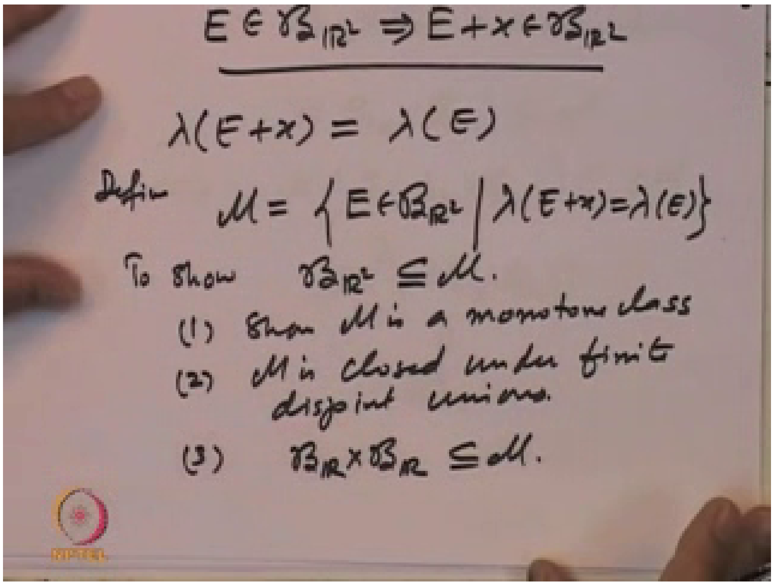
So to show the other thing, to prove that lambda of E plus X is same as lambda of E, everything is in R 2. To show this once again let us define M to be the collection of all those subsets, E M to be the collection of all those subsets, E belonging to B R 2, for which this property is true lambda of E plus X is equal to lambda of e. So, we want to show that B R 2 is inside m, because M is already a subset of B R 2. So, that will prove that M is equal to B R 2, and hence this property will hold for all subsets of B R 2.

Now to show this the technique is the monotone class theorem. So, one show M is a monotone class, two - M is closed under finite disjoint unions, and third, the rectangles B R cross B R rectangles are inside M. So, once this three effects are proved will be through as follows, because these rectangles are inside it, and if this measure monotone, this is a monotone class. So, the idea is that step three will imply that the monotone class generated by B R cross, B R is also a inside M.

And this class is also closed under finite disjoint union. So, this collection, the sets which are inside M will also be closed under finite disjoint unions. So, that will prove. So, it is the monotone class closed under finite disjoint unions. So, that will imply that. So, B R, the rectangles are inside it. So, the algebra generated by the monotone class generated by finite disjoint unions also will be inside it, because this is inside. So, this and is closed under finite disjoint unions.

So; that means, the algebra generated by rectangles will also be inside it, but M is a monotone class, which is closed under finite disjoint unions. So, that must be a sigma. So, it is a monotone class generated by an algebra, is also a sigma algebra. The sigma algebra generated will come inside it. So, hence we will have everything is equal.

(Refer Slide Time: 03:42)

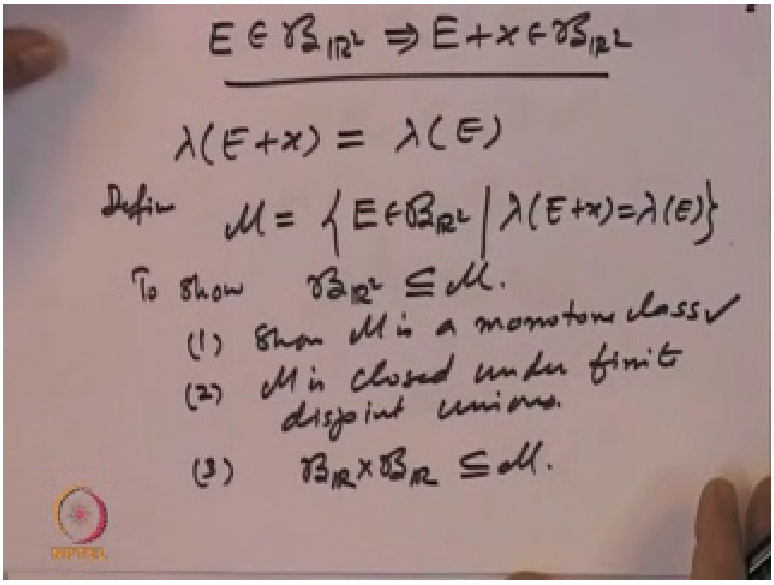


So, the idea of the proof is; that means, one should prove these three things, because after these three things are proved. So, well what will the proof imply. So, see. So, 3 will imply 3 plus 2., this is a semi algebra, because B R cross B R a semi algebra. It is inside M, and M is closed under finite disjoint, unions will imply that the algebra generated by. So, f of B R cross B R will be inside M. So, the algebra generated by this come inside m, but now implies by 1 M is a monotone class. So, it includes this algebra.

So, the monotone class generated by this algebra is also inside M, but the monotone class generated by an algebra is same as the sigma algebra. So, this is same as the sigma algebra generated by this algebra B R cross B R, and that is equal to the Borel sigma algebra of R 2. So, that is the line of argument that will prove that B R 2. So, this is the line of argument, which will prove that B R 2 is a subset of M. So, we have to verify these three things; namely M is a monotone class, M is closed under finite disjoint unions, and rectangles are a Borel rectangles are inside M.

So, that to show that let us look at the first one, that M is a monotone class. So, to show that M is a monotone class, let us look at the proof.

(Refer Slide Time: 05:25)

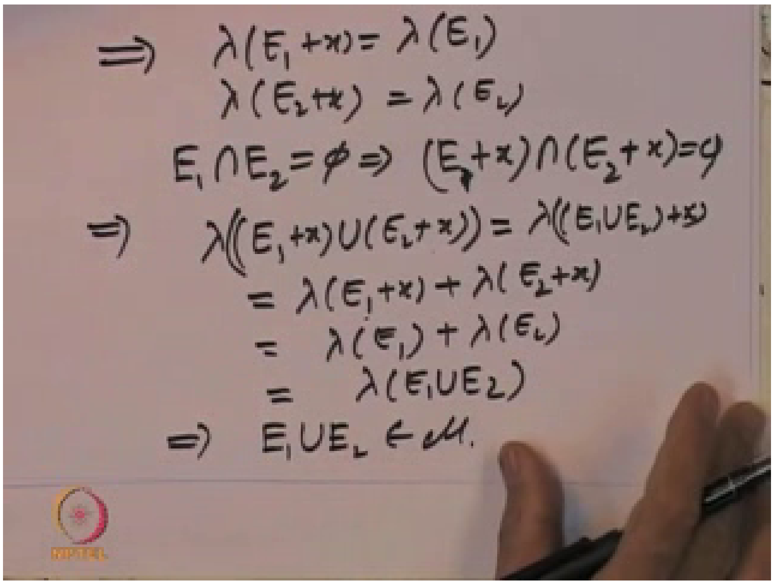


So, proof of one. So, let u look at a sequence E n, which is increasing, increasing to E, in increasing E and let us say E ns belong to M. So, that will imply that lambda of E n plus X is equal to lambda of E n for every M. Now if E n is increasing then E n plus. So, X is also increasing and lambda being a measure this converges to lambda of E plus X, and by the same thing.

This converges to a lambda of e. So, that says lambda of E plus X is equal to lambda of E. So, ens increase to E, then that will imply that these two are equal. So, E belongs to m, and similarly A for A degrees in sequence also similar property, when E ns are decreasing to E and lambda of say, then lambda of E 1 is finite. Then the intersection, the E which is a intersection will also belong to M. So, that will prove the fact that E is M is a. So, that will prove the fact that M is a monotone class. So, that is.

Now, let us show that M is closed under finite disjoint unions. So, for that, to show that M is closed finite disjoint unions.

(Refer Slide Time: 07:13)

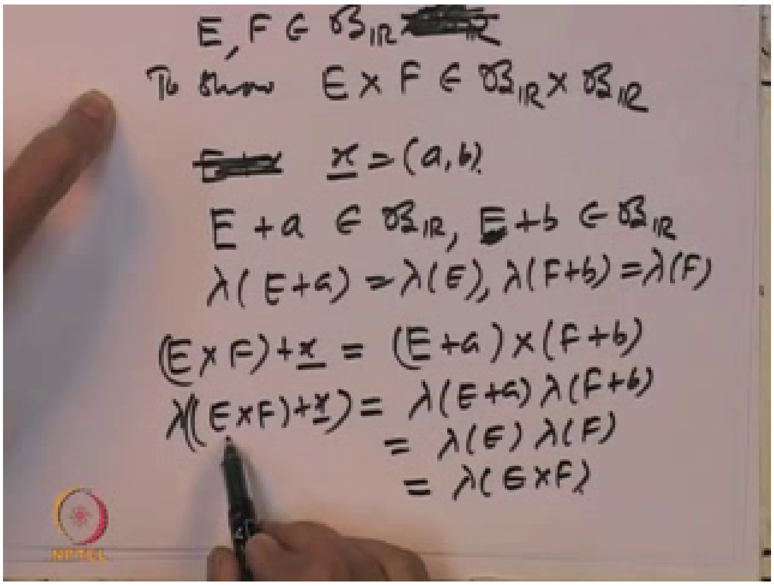


Let us take, let E 1 and E 2 belong to M E 1 intersection E 2 equal to empty set. Now E 1 and E 2 belong to an. So, this fact implies lambda of E 1 plus X is equal to lambda of E 1. And similarly lambda of E 2 plus X is also equal to lambda of E 2. Now E 1 and E 2 disjoint implies. There is the sets E 2 E 1, the translates of E 1 and translate of E 2 are also disjoint.

So, that is a simple thing to observe. So, that will imply that lambda of E 1 plus X union of E 2 plus X, because these sets are disjoint. So, the Lebesgue measure of the union of in R 2 is same as the Lebesgue measure in R 2 of E 1 a plus X plus lambda of E 2 plus X, but E 1 and E 2 belong to M. So, this is equal to lambda of E 1 plus lambda of E 2, and that is equal to E 1 and E 2 are disjoint. So, it is lambda of E 1 union of E 2. So, what we are shown is, if E 1 and E 2 belong to M in their disjoint, then lambda of E 1 union E 2 is same as lambda of E 1 X plus union of E 2 X, but a simple observation we will tell you that, this is also, is same as lambda of E 1 union E 2 plus x. So, whether you take translates first, and then take the union; that is same as taking union and the translates. So, that will imply. So, this will imply that E 1 union E 2 also belongs to M. So, whenever E 1 and E 2 are disjoint, their union also belongs to it.

So, that proves the second fact namely M is closed under finite disjoint unions. Finally, we prove the third fact; namely the rectangles are inside m, that again is a straightforward simple effect to prove. So, to prove that let us. So, observe. So, prove the third thing, let us observe the following namely.

(Refer Slide Time: 10:13)



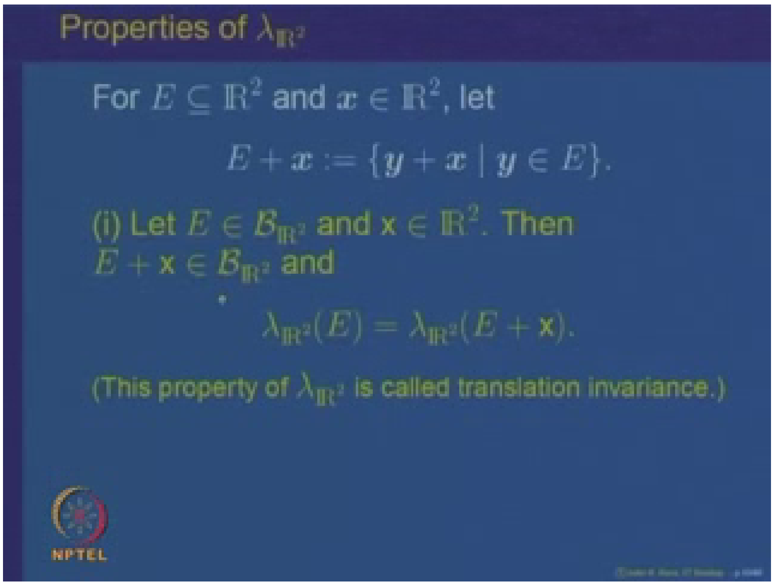
So, let us take E f belonging to B R cross B R to show E cross f E and f, both belong to B R. We want to show that the cross product belongs to B R cross br. So, that is what we want to show. So, to show that, and let us observe E and f belong to B R. So, we know that whenever E is in f. So, E plus x. So, let us take a vector, let us take a vector X which is equal to A coma B.

Then what is E plus, then we know that E plus a belongs to B R, and also E plus b belongs to B R, because E and f are subsets in br. So, the translates belong and lambda of E plus A is same as lambda of E, and that was a set f and lambda of f plus B is same, as lambda of f right. So, now, look at the set E cross f translated by X, X is A B. So, what is that? So, that is equal to E plus a cross product with f plus b.

So, the Lebesgue measure of the set. Sorry E cross f plus X will be equal to, this is rectangle. So, a Lebesgue measure of E plus a into Lebesgue measure of f plus b, but that is equal to Lebesgue measure of E into Lebesgue measure of f, because Lebesgue measure on the real line is translation in variant. So, that is equal to Lebesgue measure of E cross f. So, what we are shown is that, if E cross f is a rectangle, Borel rectangle. Then translate of the Borel rectangle has the same measure as the rectangle itself.

So, that will, that proves the third thing; namely that the Borel sets cross, the Borel sets is inside M. So, all the three facts are proved, and that will imply that B R 2 is a subset of m, and hence for all. So, that is what we have shown is that.

(Refer Slide Time: 12:47)

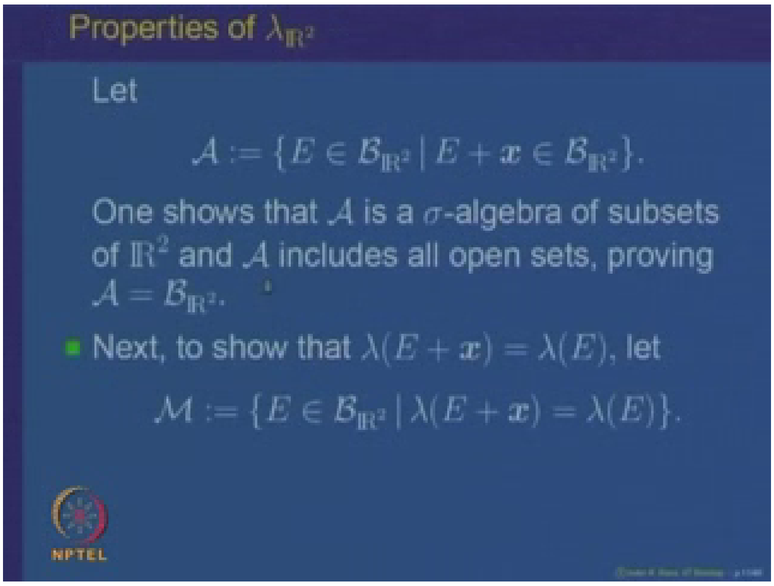


So, we have, what we have shown is that the Lebesgue measure is a measure on the plane, which has the property that Lebesgue measure of every. For every a Borel set E it translated, is also translation is also a Borel set, and the Lebesgue measure of the translated set is equal to Lebesgue measure of the original set.

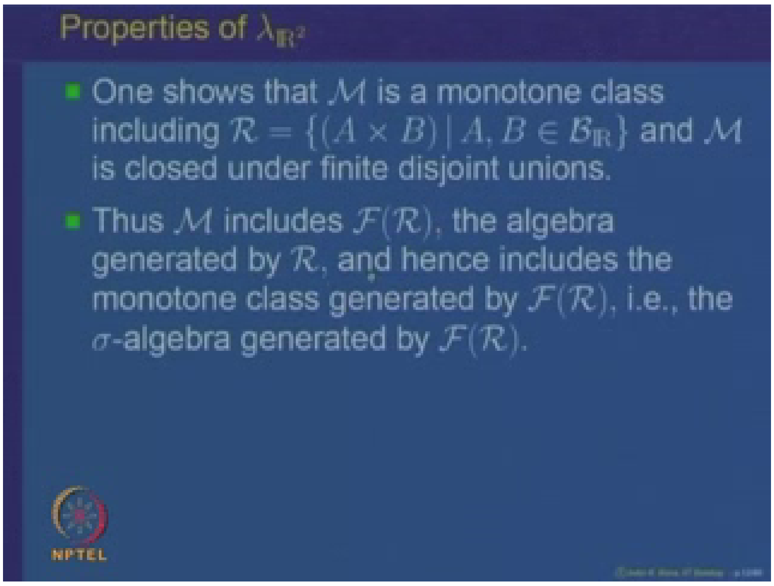
So, this is called the translation invariance properties of the Lebesgue measure on the plane. So, as in the case of real line. The real line we will showed that the Lebesgue measure on the line is a translation invariant measure and. So, similarly we have shown that the product of that Lebesgue measure taken on R 2 is also a translation invariant measure. Of course, the natural question arises on the real line we are shown, that essentially Lebesgue measure is the only translation invariant measure, and we will show for this Lebesgue measure on the plane also, is essentially a unique, is the unique translation invariant measure. Unique in the sense that a scalar multiple is again translation invariant anyway. So, up to a multiplication by a scalar.

We will show that the Lebesgue measure in the plane is a unique translation invariant measure on the Borel algebra, so, but before that let us prove a property about the integrals of functions on the plane. So, the next property we want to analyse is the following; namely. So, this proofs we have already gone through the sigma algebra monotone class technique.

(Refer Slide Time: 14:41)

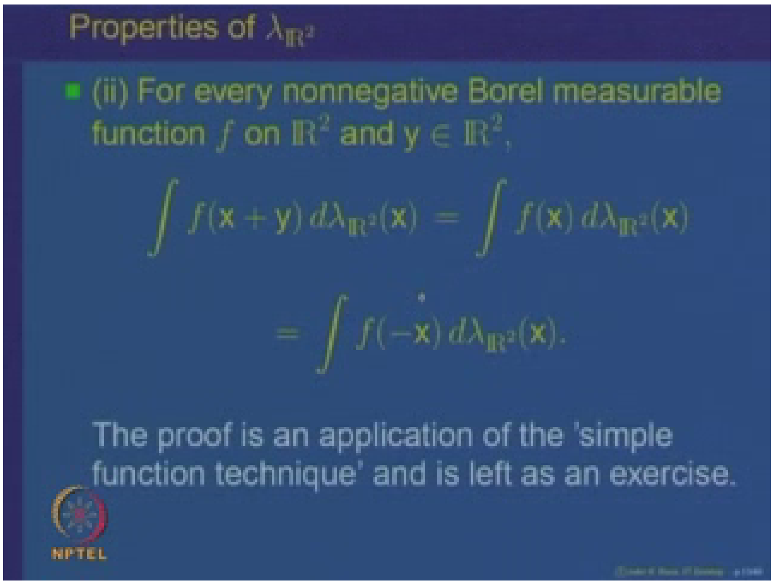


(Refer Slide Time: 14:44)



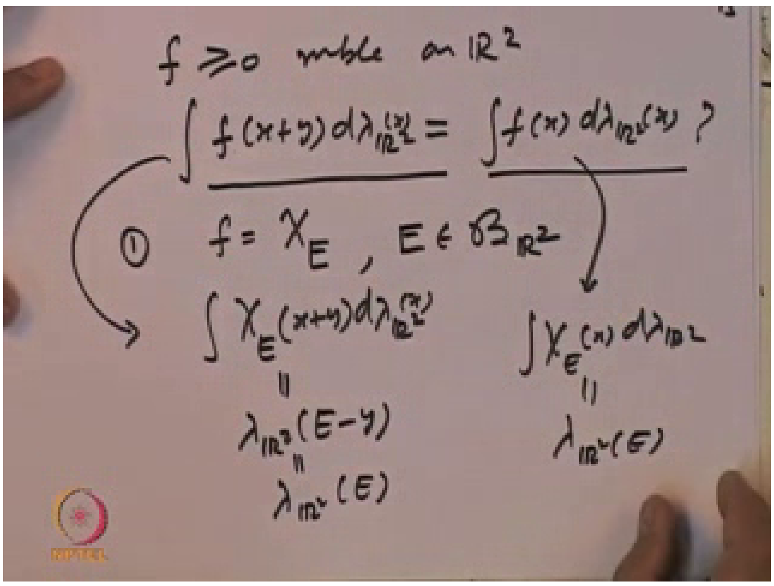
So, that sigma algebra monotone class technique that we have already explained. So, that is just shown here. That shows the M includes f of R, and hence it will include the sigma algebra generated by E time that will proved.

(Refer Slide Time: 15:02)



So, the next property we have wanted to illustrate is the following; namely for every nonnegative Borel measureable function f on R 2, and any vector y in R 2, the integral of the translated function. So, integral of f of X plus y with respect to the Lebesgue measure is same as the integral of the function itself, and it is also same as integral of the negative of the function namely f of minus x; that means, the Lebesgue integral for nonnegative functions is invariant under translation, and this is what is called reflection X goes to minus x. So, a proof of this is basically applications of the simple function technique. So, let me just illustrate 1 or 2 steps of this proof that this is true.

(Refer Slide Time: 16:08)

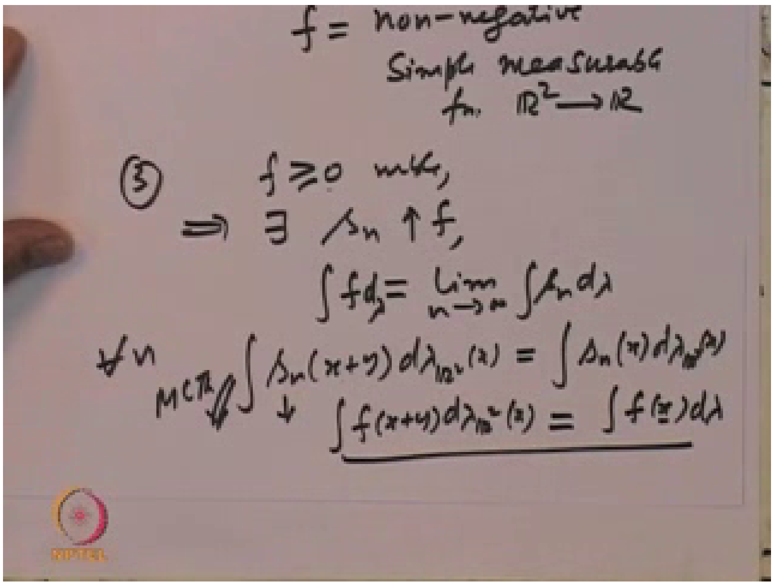


So, let us look at the first one. So, let us try to, let us prove that if f is a nonnegative measurable function on R 2 then we want to prove that the integral of f of X plus Y d lambda R 2. So, this is our R 2 is equal to integral of f of X d lambda R 2 of X.

So, this is what we want to prove. So, the simple function technique as we recall is the following first step. Let us take f to be the indicator function of a set E, where E is a Borel subset of R 2. So, in that case the left hand side. So, this left hand side is integral of the indicator function of E X plus Y d lambda R 2 which is nothing, but, so the integrating with respect to X. So, that is same as X plus y belonging to E. Means here it is X belonging to E minus Y. So, this is integral of the indicator function of E minus Y. So, it is lambda R 2 of the set E minus y, but that is same by the translation invariant property, it is lambda of the set E. So, and this thing f is the indicator function, indicator function of X d lambda R 2 which is same as lambda R 2 of E.

So, what we are saying is that, as a first step the required claim namely integral of f of X plus Y is integral to integral f holds, whenever f is the indicator function of a set E. Now both sides being a integrals. So, implies.

(Refer Slide Time: 18:10)



So, step one. So, step one implies step two namely required claim holds for f equal to nonnegative simple measurable function R 2 to R. So, this claim recalled, because any nonnegative simple measurable function is a finite linear combination of characteristic functions on the indicator functions. So, for E, each indicator function we have shown this.

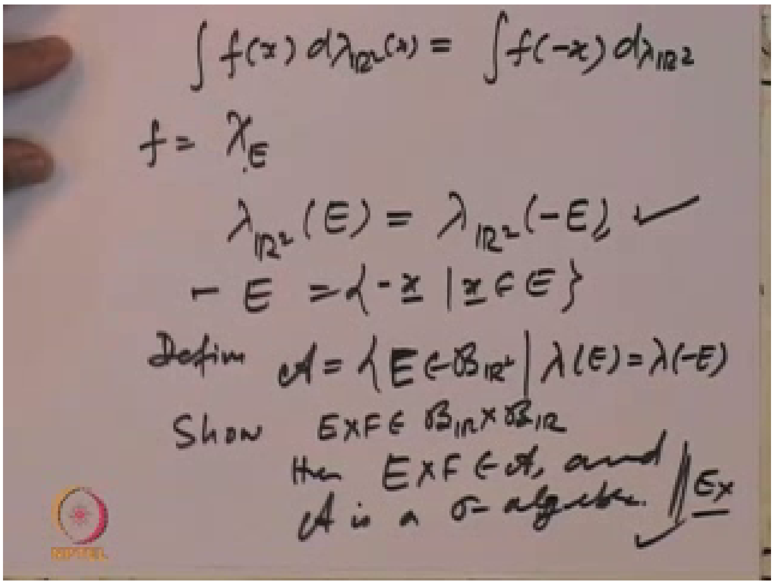
So, that we will imply that the required claim holds for nonnegative simple functions. So, and the third step if f is nonnegative measurable then we know implies they are exists a sequence Sn of nonnegative simple measurable functions, Sn increasing to f and integral of f to be equal to limit n, going to infinity integral of Sn d lambda. So, saying that f is nonnegative measureable, means that f is limit of nonnegative simple measurable functions and.

The integral of f can be defined as the limit of the integrals of nonnegative simple measurable functions, but for nonnegative simple measurable function in each Sn. So, for every n we know that the required claim holds by step two. So, by step two, we know that Sn of X plus y d lambda R 2 of X is equal to integral of Sn of X d lambda R 2 of x. So, that is by step two. Now as Sn is increasing to f. So, clearly the translates this will increase to the translate of the function f.

So, this implies the, in the limit by monotone convergence theorem. So, an application of monotone convergence theorem will say that as n goes to infinity this will converge to integral of f of X plus Y d lambda of R 2 of X right. On the other hand, we know this converge is to integral of f X d lambda of R 2. So, these must be equal. So; that means, for a nonnegative measurable function this required conclusion holds. So, that is how one proves the claim namely f of X plus Y is equal to f of integral of the translate is equal to the integral of the original function a, basically is a, what we call as the simple function technique applied to it.

So, the same argument will im similar argument will show that integral of f of X is same as integral of f of minus X. So, for that one, has to use the fact that the Lebesgue measure of a set E in R 2 is same as the Lebesgue measure of. So, for the step two. So, let me just indicate what we need for step two to show that integral of f of X d lambda R 2 of X is equal to integral of f of minus X d lambda R 2 ok.

(Refer Slide Time: 21:58)



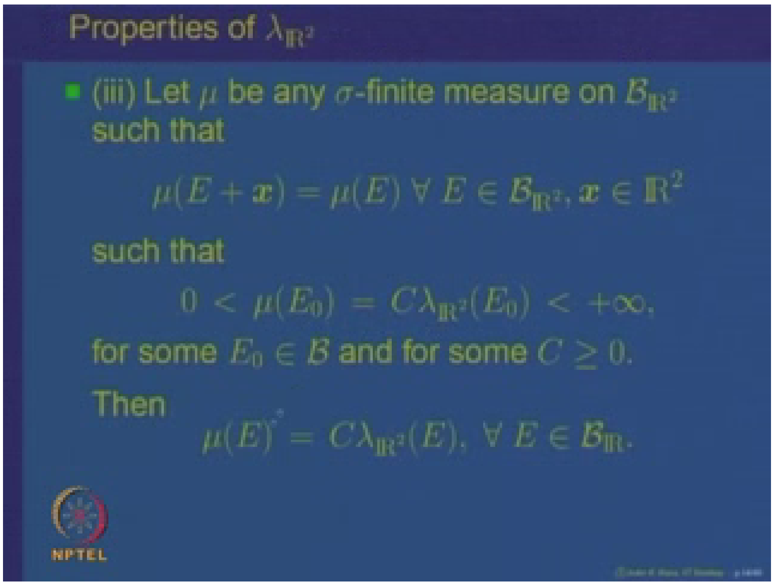
When f is equal to indicator function of the set e; that means, we need the fact that lambda R 2 of a set E is equal to lambda R 2 of minus of E.

What is minus of E. So, minus of E. So, minus of E is the set minus the vector X, here X belongs to E now. So, the proof that, this shows once again, one has to go to the sigma algebra technique. So, consider define a to be the collection of all those sets E belonging to B R 2, where for which you can say that lambda of E is equal to lambda of minus E. So, look at all these collection of the sets. So, claim. So, one will show the rectangles are inside it; that means, if i take sets E cross f belonging to B R cross B R, then E cross f belongs to A and A is a sigma algebra.

So, once again if these two steps are approved that will prove that this claim holds for every Borel subset also, and hence for the indicator function of a set E. So, that is a value it as exercise. Once gain is a straight forward verifications. So, do that. So, once that is done. So, that will prove a second equality also. And now in this proof, one more observation we want to make here, is the following. If I replace lambda R 2 by any. See in this proofs of these two things we have not used anywhere. The fact that we are on lambda is especially the Lebesgue measure.

Essentially we use the fact that this measure lambda of R 2 is translation invariant. So, if you replace this measure Lebesgue measure on R 2 by any translation invariant measure, then this result that f of X plus Y is equal to integral of f of X will remain true for lambda of R 2 replace by any translation invariant measure. So, this is an observation you should keep in mind for the future reference.

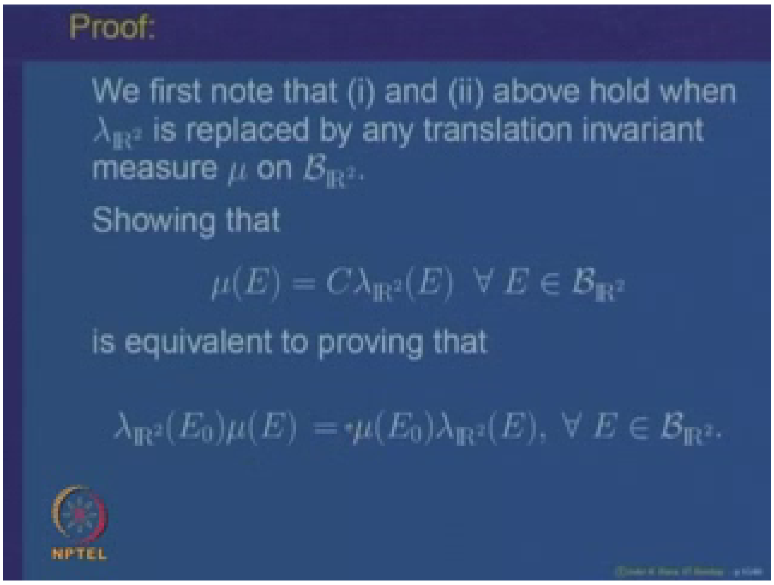
(Refer Slide Time: 25:12)



So, finally, we want to prove the fact that the translation invariance is a unique property for the Lebesgue measure. So, let take any measure mu with this sigma finite on the Borel subsets of R 2, and assume it is translation invariant.

Let us assume that there is some particular set E naught says that the measure of the set E naught is positive, and the measure mu of E naught is c times a constant multiple of Lebesgue measure of the set E naught and it is finite. So, there is a set of finite Lebesgue measure. Finite positive Lebesgue measures says that mu of E naught is a constant c times Lebesgue measure of E naught. For some particular set E naught then the claim is that this property holds for every subset of Borel subset; that means, mu of E is constant multiple of the Lebesgue measure. So, that will prove the uniqueness of the Lebesgue measure with respect to translation invariance. So, let us prove this.

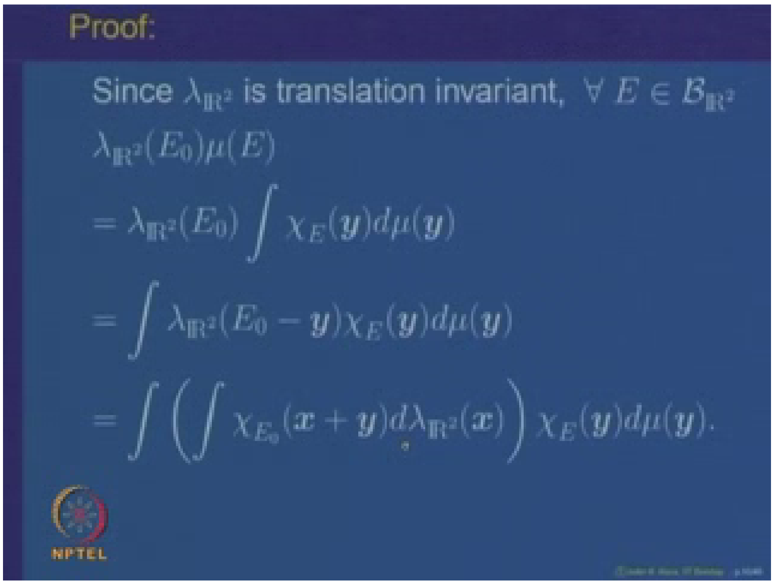
(Refer Slide Time: 26:29)



So, as i observed that the integral of the translate of a function is equal to integral of a function remains true, for any translation invariant measure. So, in particular for mu. So, that property will be using. So, now, let us. So, we want to show that mu of E is constant multiple of Lebesgue measure of E for every set E, but C is equal to. So, what is C. Let us just look at C. The C, i can compute from here, C is equal to mu of E 0 divided by lambda of E zero. So, we put that value. So, to show that mu of E is equal to C times lambda R 2 of E. It is equivalent to showing that lambda of E 0.

Lebesgue measure of E 0 into measure of E, this same as measure of E 0 mu of E 0 into Lebesgue measure of E for every subset. So, this equality we should show for every subset E of R 2. So, that we will show it as an application of fubinis theorem. So, let us take the left hand side.

(Refer Slide Time: 27:35)

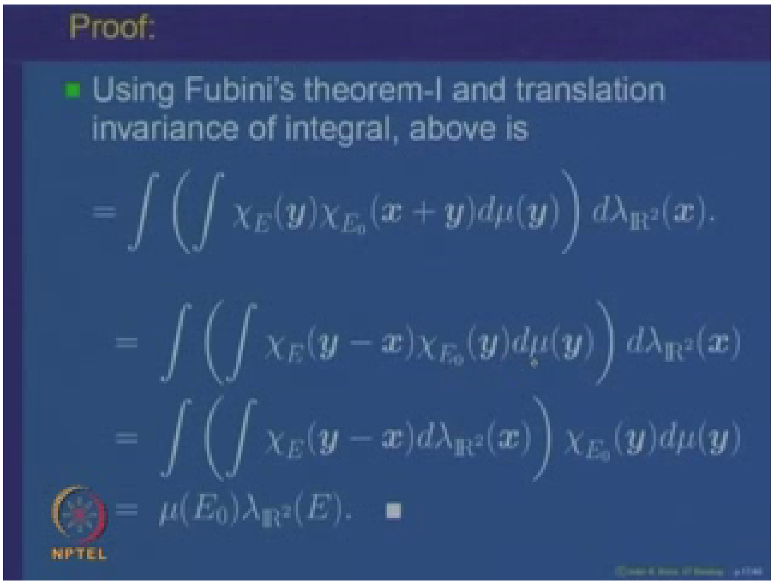


So, lambda of E 0 mu of E is equal to lambda of E 0 and mu of E is integral of the indicator function, with respect to y d mu y. Now take this lambda R 2 in inside, and the use the fact that this is translation invariant. So, lambda R 2 of E 0, this same as lambda R 2 of E 0 minus y, and I put it in under the integral sign. So, the required quantity is equal to integral of lambda Lebesgue measure of E 0 minus y.

Into indicator function of e, and now this Lebesgue measure, I will write it as a integral in the form of integral. So, I get Lebesgue measure of E 0 minus Y is integral of a indicator function of E 0 minus Y same as it. So, it is same as the integral of a indicator function of E 0 of X plus y d lambda R 2. So, here we got double integral, iterated integral and the function involved are nonnegative. So, by fubinis theorem, the first part for nonnegative functions I can inter change the order of integration. So, let us interchange. So, earlier we had inner integral was with respect to lambda and outer with respect to mu.

So, when we interchange mu comes inside and lambda goes outside.

(Refer Slide Time: 28:52)



So, that is the integral, and now once again mu is translation invariant. So; that means, in this integral, if I shift y to y plus X the integral will remain the same A Y minus X, the integral will remain the same. So, let us do a shifting, shift this to Y minus x. So, indicator function of Y minus X indicator function of E 0 X plus Y. So, that becomes Y d mu Y, and now once again we apply fubinis theorem and go back. So, when I apply. So, mu goes out and lambda R 2 comes inside. So, that is a indicator function of E Y minus X lambda of R 2 of X, but that is same as Lebesgue measure of the set E, and this is Lebesgue mu of E naught. So, that is equal to this.

So, twice an application of the fact fubinis theorem for nonnegative functions, and the earlier property gives us the required fact; namely the Lebesgue measure is the translation invariant measure unique translation invariant measure on. So, today we have looked at the properties of Lebesgue measure with respect to the topologically in E sets; namely open sets, compact sets, and with respect to the group operation of translation and the plane. There is another transformation possible; namely you can take a set E, and rotate it not only you can translate, you can also rotate it or magnify a set. So, in next lecture, we will analyse How Lebesgue measure changes with respect to what are called linear transformations in the plane, and which include rotations and magnifications.

Thank you.