Measure & Integration
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Lecture – 29 A
Lebesgue Measure and Integral on R^2

Ah in the previous lectures we had defined what is called the product measure on product space, in this lecture we will specialised that construction on the set R 2. So, which is of Cartesian product of real line with itself and the sigma algebra bring that of either borel sets or Lebesgue measureable sets and the measure bring the Lebesgue measure. So, the topic for today's discursion is going to be Lebesgue measure and integral on the space R 2. So, let us just recall.

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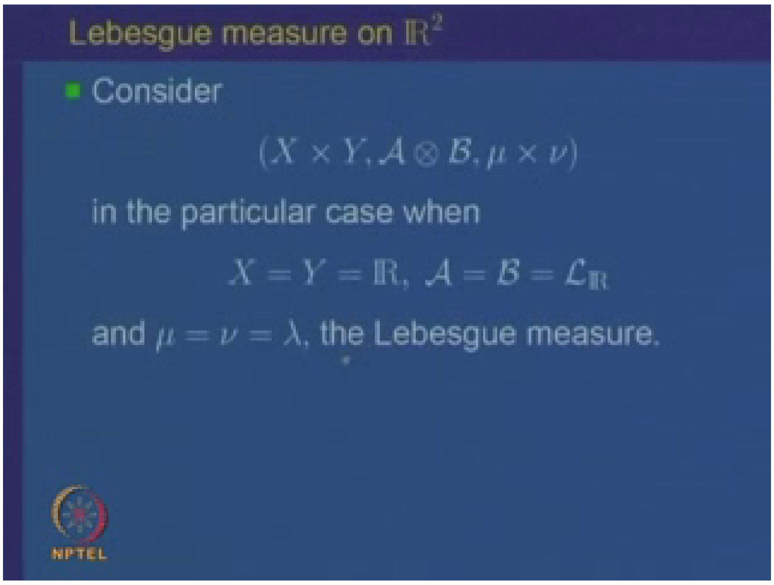

So, we had defined the product measure space, given measure space is x a and mu and y b and nu, we will define the product sigma algebra a times b on the product space x cross y and the product measure mu cross nu.

So, today we will start looking at the particular case when x is equal to y equal to the real line and the sigma algebra a is same as the sigma algebra b is same as the sigma algebra of Lebesgue measureable sets on the real line and mu is same as nu with a same as the Lebesgue measure. So, we are looking at a copy of the real line, the sigma algebra of Lebesgue measurable sets and lambda the Lebesgue measure and taking its product with itself. So, that will give arise to the product measure space R 2 the Lebesgue measureable sets times, the Lebesgue measurable sets the sigma algebra and the product measure lambda cross lambda and if you recall we had mentioned that Even if the original measure spaces are complete the product measure space need not be complete.

So, this product measure space R 2 lambda R cross lambda R and lambda cross lambda is not complete. So, we can always complete it and there is a completion is denoted by R 2 Lebesgue measurable subsets of R 2 and lambda of R 2 so this is called the Lebesgue measure space.

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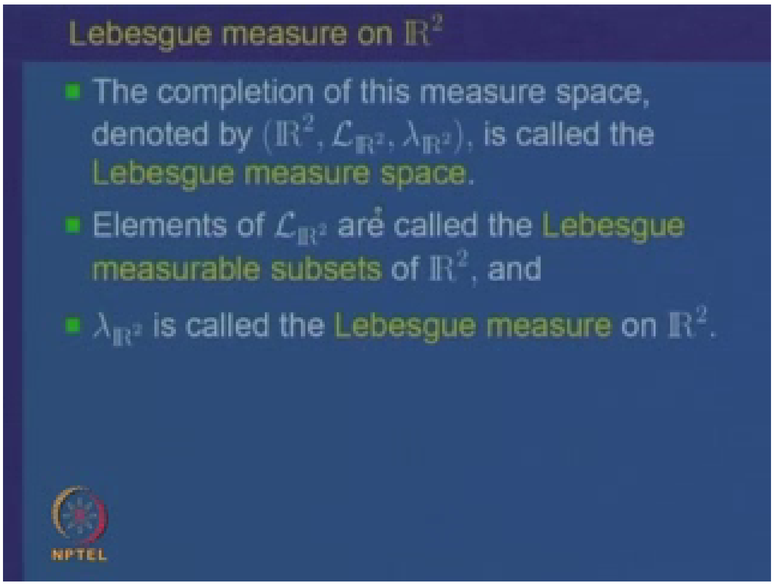

So, Lebesgue measure space is obtained from the sigma algebra Lebesgue measureable sets times Lebesgue measurable sets completed with respect to the product Lebesgue measure on R 2. So, this is normally called the product called the Lebesgue measurable, measure space on R 2 and.

The set saying the sigma algebra lambda of R 2 are called Lebesgue measurable sets in R 2 and the measure lambda of R 2 defined on this completed space is called the Lebesgue measure on R 2. So, whenever one refers to the Lebesgue measure space it is the complete measure space obtained by a completing the product measure on the product sigma algebra.

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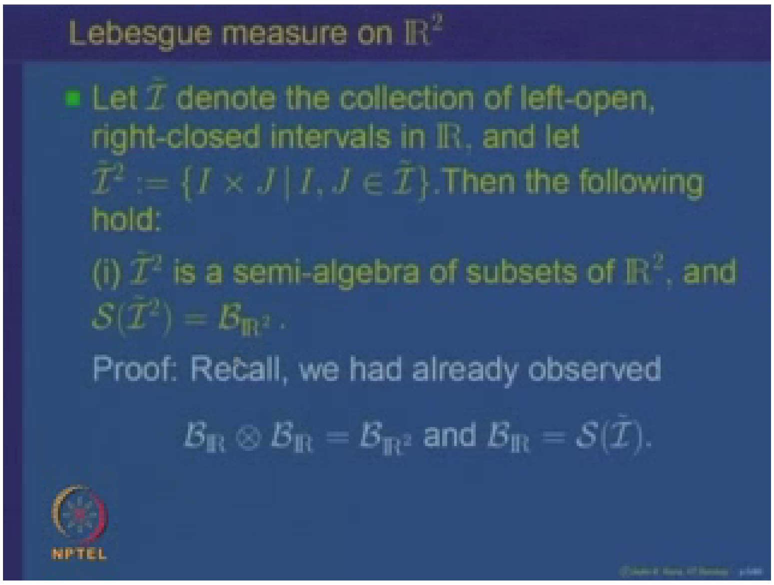

So, today we will start looking at properties of Lebesgue measurable sets and Lebesgue measure. So, let us denote by I tilde as we are done for the real line the collection of all left open, right closed intervals in real line. So, let us look at the rectangles obtained by such intervals. So, that we denoted by I tilde 2 upper superscript 2.

As I cross J rectangles was the sides are left open, right close intervals, then we claim that this I cross J is a semi algebra of subsets of R 2 and the sigma algebra generated by this is equal to the borel sigma algebra of R 2. So, to prove this we already know that I tilde the left open, right closed intervals form a a semi algebra of subsets of real line and we have already shown that if you take rectangle consisting of elements of the semi algebra then itself form a semi algebra namely the product of semi algebras is always a semi algebra.

So, that general construction will tell that the space, the set of all rectangles with left open, right closed intervals is a in the semi algebra. To show that this sigma algebra generated by the rectangles I tilde 2 is the borel sigma algebra, we observe few things first of all if you recall we have shown so let me just recall effect that we have shown in the beginning of defining product sigma algebras.

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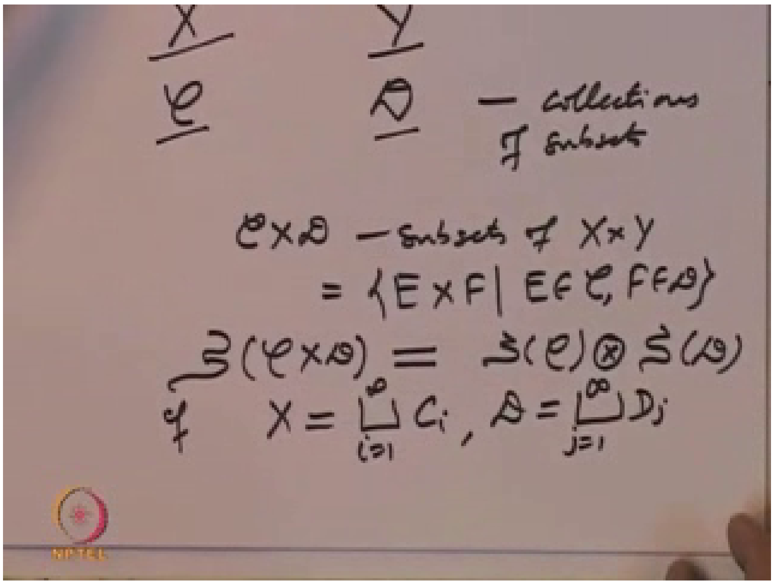

Namely, if we take a set X and take a set Y and here we are get a collection of subset c and we are going to collection of subsets. So, these are collections of subsets, then we can form c cross d that is a collection of subsets of X cross Y right.

So, this is equal to all sets of the type E cross f where E belongs to c and f belongs to d and now one can generate a sigma algebra out of this collection c cross d, on the other hand we can generate a sigma algebra by the collection c, we can also generate a sigma algebra by the collection d of subsets of y and take, take the sigma algebra generated by the rectangles of these types. So, let us call it as s of c cross s of d then the claim is that these 2 are equal not always whenever.

So, we showed that these 2 are equal if x can be represented as a union of sets partition 1 to infinity and d can be written as a union of some sets djs in the collection d of 1 to infinity. So, whenever this x can be represented at a disjoint union of sets from c and y can be represented as a disjoint union of elements of d then whether you take the rectangles first and generate the sigma algebra or generate the sigma algebras and then take rectangles and generate the sigma algebra both will be equal to same. So, this result we had proved in the beginning of the topic.

So, as a consequence of this we obtained.

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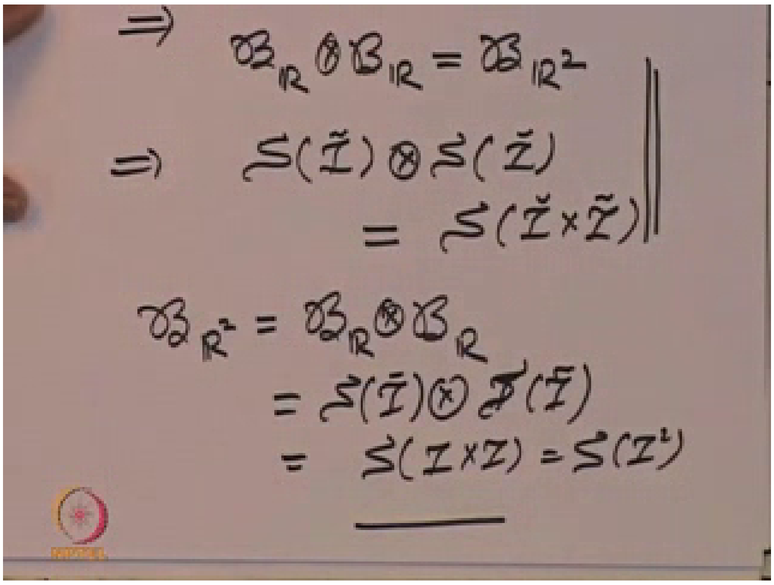

So, this implied 1 observation that if you take the borel sigma algebra cross the borel sigma algebra of R that is equal to borel sigma algebra of the space R 2. So, that is one observation because the real line can be represented as a countable union of say an open sets or intervals and similarly the same argument also implies that if I take the sigma algebra generated by this left open, right closed intervals cross the sigma algebra generated by left open, right closed intervals and then look at the product sigma algebra. Then that will be same as the product sigma algebra of left open, right closed intervals cross left open, right closed intervals because the whole real line can be written as a countable union of left open, right close intervals.

So, these 2 facts follow from our earlier construction. So, we will keep that in mind and now what you want to show is that the borel sigma algebra of R 2. So, that we know it is borel sigma algebra of real line times the product borel sigma algebra of real line and borel sigma algebra we know from our construction of real numbers that is the same as a sigma algebra generated by left open, right closed intervals.

So, left open borel sigma algebra is generated by the sigma algebra of left open, right closed intervals. So, and this just now we observed is a sigma algebra generated by I cross I. So, that is same as the sigma algebra generated by I 2. So, that we are completes the proof of the fact that the sigma algebra generated by rectangles which are left open right closed intervals is same as the borel sigma algebra of R 2. So, that is one observation. So, that is very much similar to the result in the real line where the left open, right closed intervals generated the sigma algebra borel subsets the same result is too if we replace intervals by rectangles which are left open, right close ok.

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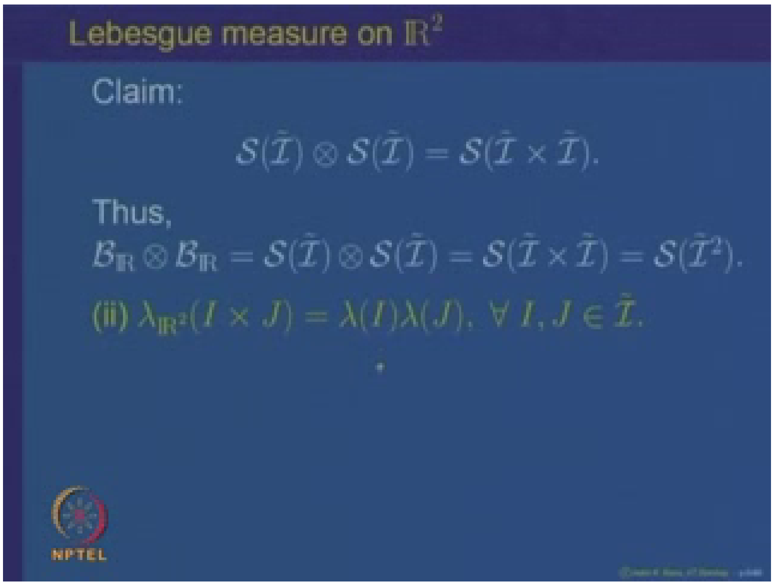

So, that is the proof we are just now said. So, SI is equal to SI cross I.

So, borel sigma algebra Br cross Br is the sigma algebra generated by intervals left open, right close cross left open, right closed intervals measure same as the rectangles. So, let us look at the the next property that the Lebesgue measure that we are defined for a rectangle is lambda of I cross lambda of J it is same as lambda of I into lambda J. So, that is obvious because we obtained the product measure extension of the measure on the rectangles. So, what we are saying is the labla Lebesgue measure on R 2 is the natural extension of the notion of area in the plane.

So, this is properties obvious built in the definition of the product measure.

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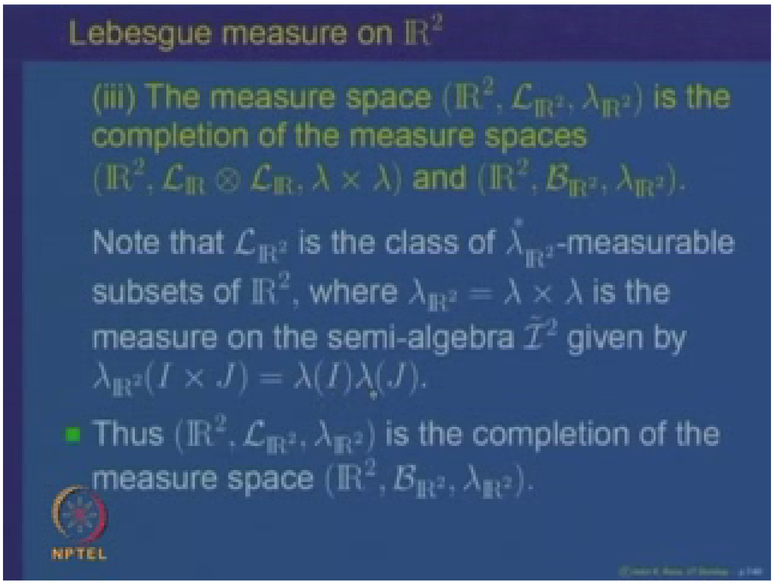

And the third observation is. So, recall we just now said that the Lebesgue measure space R 2 Lebesgue measures subsets of R 2 and Lebesgue measure the Lebesgue measurable subsets. So, this space which is the space Lebesgue measure space, on one hand we defended as the completion of the Lebesgue measurable sets cross Lebesgue measurable sets and this is also the completion of the measure space of real line with borel subsets of R 2 and that is once again by the effect that the borel subsets of R 2 are inside this and the borel sets subsets of R 2 and the Lebesgue measurable sets they defer only by sets of measure 0.

So, that is also the completion. So, one way of looking at this look at look at the Lebesgue measureable subsets R 2 be that being the completion. So, it is the is the class of all out of Lebesgue measurable subsets in R 2 with respect to the product measure and on the semi algebra I 2 the of rectangles it is given by the product so this; obviously, the completion of the measures space R 2.

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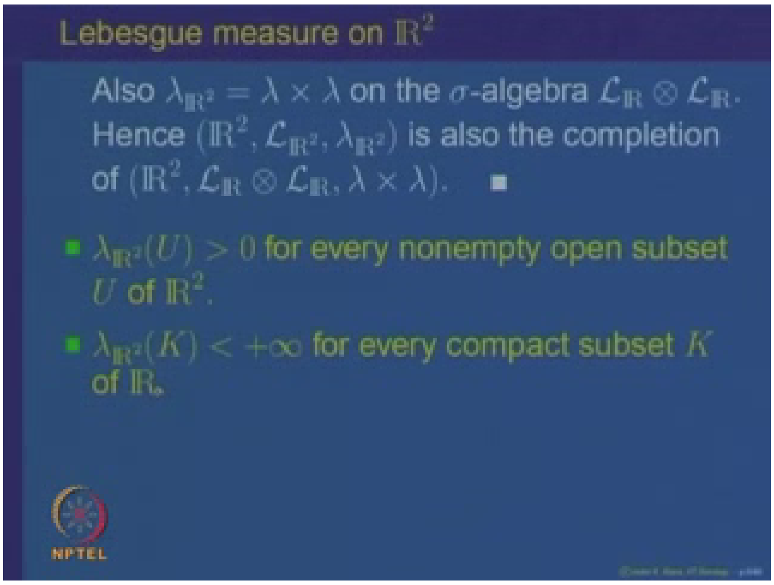

So, these are obvious fact so so we should keep in mind which are very much similar to that of the real line, the place a role later on given to look at null sets in R 2.

So, basically the sets which are going to be of importance are going to be the Lebesgue measurable sets are, Lebesgue measurable sets cross Lebesgue measurable sets in R 2 or borel subsets in R 2. Here is another useful fact about Lebesgue measure in R 2 which connects it with topologically in I sets namely the Lebesgue measure of R 2 of a and it opens nonempty open set is always bigger than 0. So, that follows from the fact that if U is contained in R 2 is open and u is not equal to empty set then so here is the set u.

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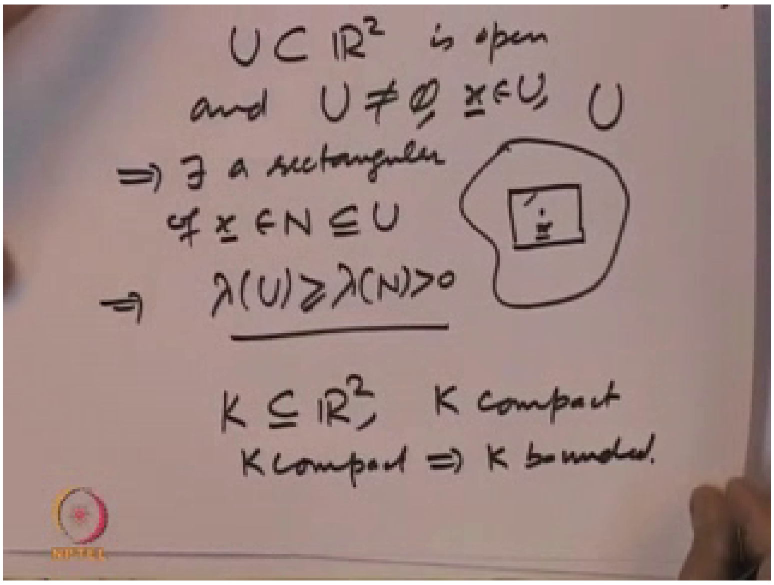

So, there is always a rectangle left open, right closed rectangle inside it so it is a rectangular neighbourhood.

So, implies they are exists a rectangular so with nonempty so there is a point x belonging to U. So, there is a there is a rectangular neighbourhood of x. So, let us call that neighbourhood as n. So, this is the rectangle n which is contained an U, but the Lebesgue measure of so; that means, the Lebesgue measure of u will be bigger than Lebesgue measure of N which is always going to be bigger than 0 because it is a na is a nonempty neighbourhood. So, for every nonempty you have open set the Lebesgue measure is always, Lebesgue measure is always positive if here a set is nonempty and

The Secondly, important thing is (Refer Time: 14:09) you will take a set k which is compact subset of R 2. So, let us look at a compact subset of R 2. So, k is contained in R 2 and k compact by the set is compact that implies it must be bounded. So, k compact implies k bounded and; that means, so saying the set is bounded implies that.

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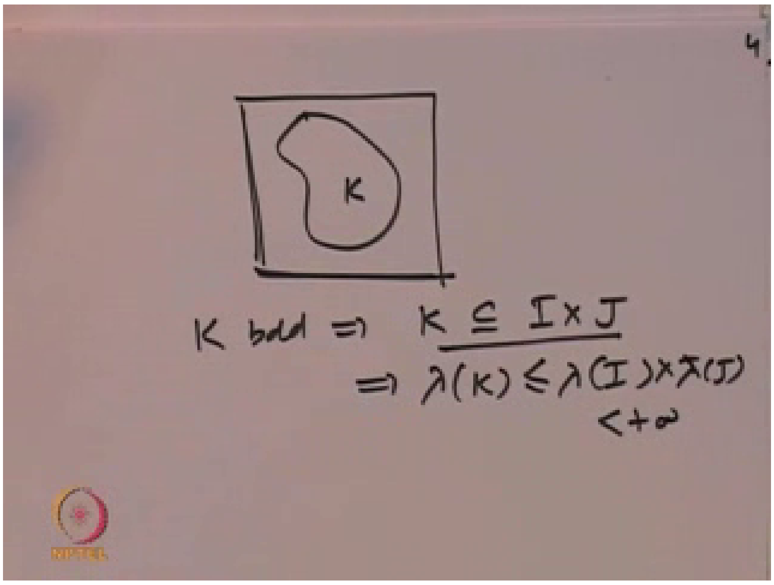

So, this is a set k which is compact so; that means, it is bounded. So, it must be inside a a rectangle. So, k bounded implies so k bounded implies k is inside some I cross J with I lambda of I. So, implies lambda R 2 of k you blows the lambda of I cross lambda of J which is finite.

So, finite intervals compact implies bounded. So, there is a finite rectangle including it so; that means, it is a finite. So, these are 2 relations about open sets and compact sets the more relations which relate like in the real line.

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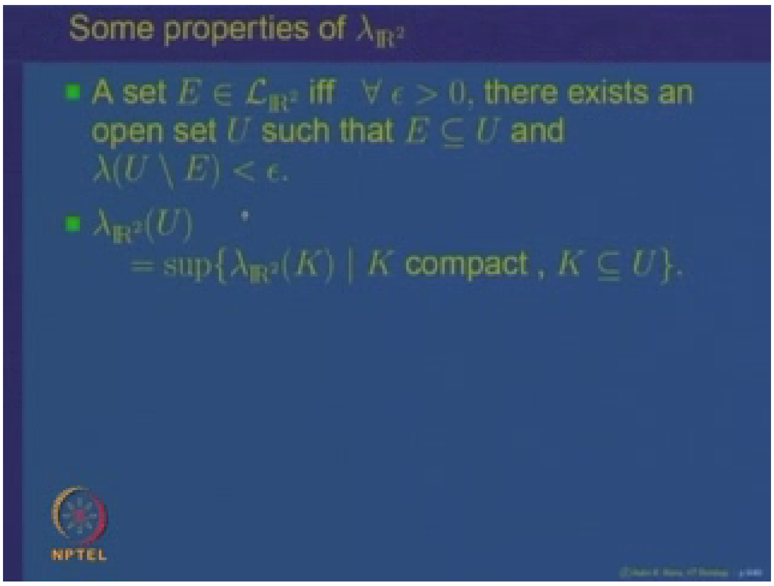

There we can one can prove your result that for example, a set E is Lebesgue measurable iff and only for every epsilon you can find a open set which includes it and the difference I has measure small. So, that is very much similar to the real line and the proof is also very much similar to the real line. So, we will not prove this result interested, if is somebody who is interested should try to copy and achieve the proof of the real line and extend that proof to the case of R 2 and similar. So, the this will give us that another result is that for the Lebesgue measure of R 2 you can approximated by this from inside by compact set so supremum of lambda of R 2 where case compact. So, these results basically all of important. So, these are called regularity conditions for the Lebesgue measure in R 2. So, we will not prove these results just for the sake of knowledge I mention these results here.

So, that later on if you come across we can look at proofs of these results.

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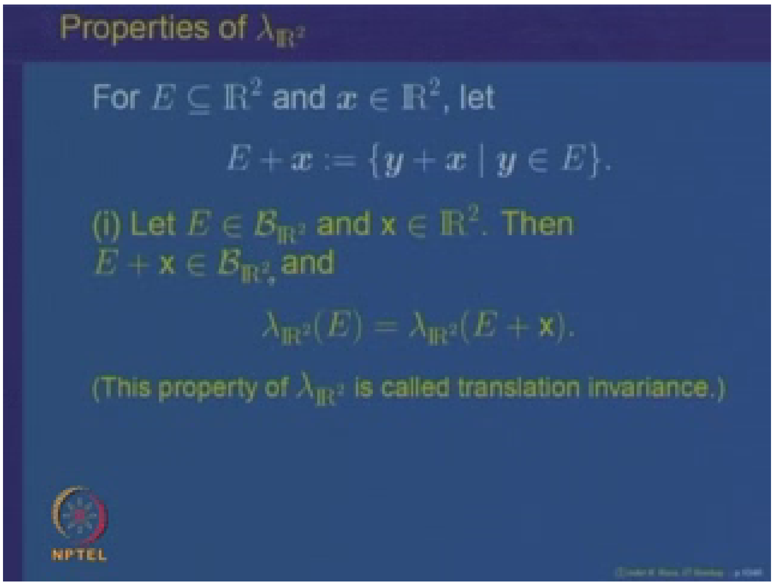

So, the next result we want to look at is how are the Lebesgue measurable sets related with the group structure of the space R 2. So, let us take a subset E of R 2 and let us look at as a point x vector x in R 2. So, we will define the, translate of the set if by x to be as in real line y plus x where y belongs to E. So, I would take the set E and shift every element, every element of E by the vector x. So, that is y plus x. So, the claim the first claim is that if E is a borel set and the point x belongs to R 2 then E plus x also is a borel set that is 1 property and the Lebesgue measure of R 2 of the set E is same as the Lebesgue measure of the set E plus x.

That means the Lebesgue measure is one says it is translation in variant and the class of wall borel subsets of R 2. So, the proof of this set that for every set E,E plus x belongs to B R 2 and the fact that the Lebesgue measure of the translated set is equal to Lebesgue measure of the original set are standard applications of the techniques that we have been using namely the sigma algebra monotone class theorems. So, let me illustrate this once again so that this idea of using the a monotone class convergence theorem monotone class sigma algebra technique settles down in the mind.

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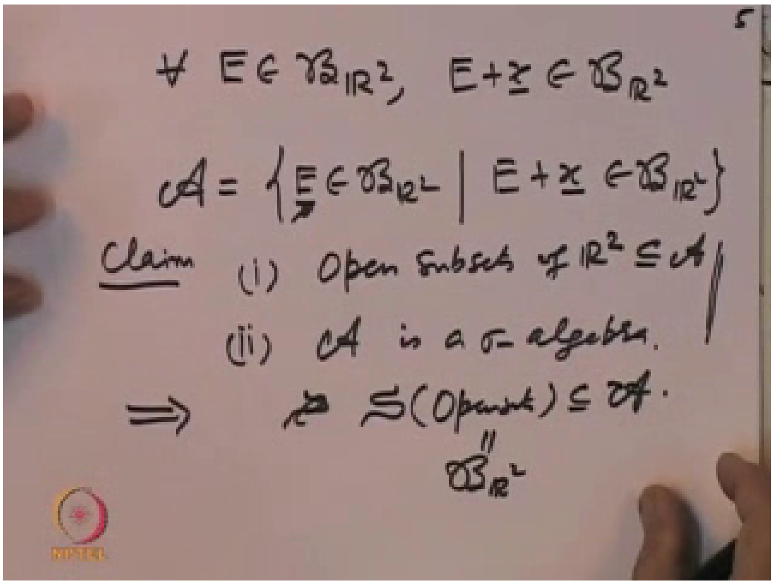

So, we first want to so prove namely that we want to show that for every E a borel subset of R 2, if I look at E plus x that is also a borel subset of R 2. So, the technique is as follows let us collect together all sets a. So, form the collection a of all though subsets E belonging to Br 2 all borel subsets say that the required property is true, E plus x belongs to br 2 so look at all sets saving this property. So, claim so, that is the sigma algebra technique claim, 1 all opens up sets of R 2 are inside this collection. So, we will prove the 2 claims 1 and secondly, that the class a is a sigma algebra.

So, if you prove these 2 sets about the class a then that will imply because it includes open subsets of R 2. So, it will include the smallest sigma algebra generated by. So, these 2 effects will imply, these 2 effects will imply that the sigma that the sigma algebra generated by sigma algebra generated by open sets will be inside, inside the class a and that is equal to the borel sigma algebra. So, that will prove that borel sigma algebra is equal to a. So, let us first show that the open sub sets of R 2 are inside a so let us take a open set.

So, whether to prove the first fact where to show that if a a set.

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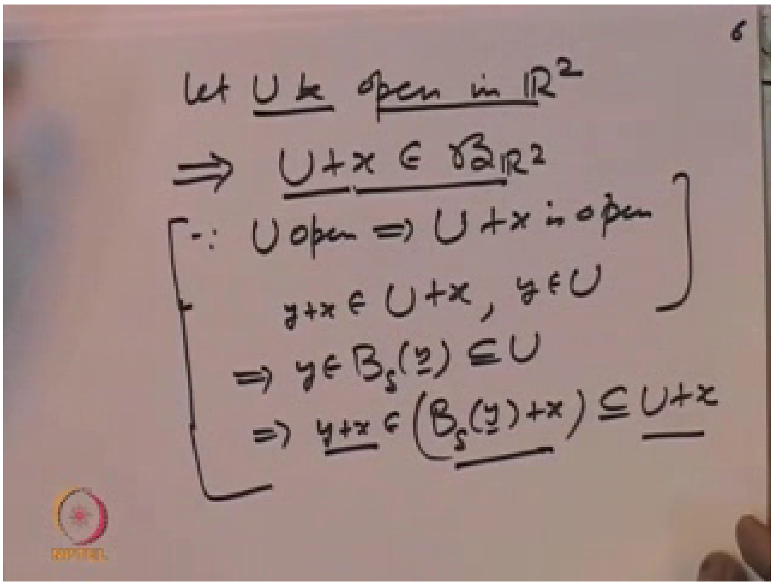

So, to to show the first one so, let u be open in R 2. So, you want to show that this implies u plus x belongs to BR 2 and this follows because this follows because if u is open implies u plus x is open. So, that is a simple effect because how do we will show that u plus x is open the, basically saying that u is open. So, let us take a point y plus x belonging to U plus x if y plus x belongs to U plus x where y belongs to u and u open implies there is a neighbourhood.

So, let us call it as b delta y. So, y belongs to a neighbourhood which is contained in U but then that implies that y plus x belongs to the translation of the neighbourhood that is contained in U plus x so; that means, for every point y plus x there is a neighbourhood where shift it when you shift a ball that remains a ball in the plane so right. So, that is a basic fact we are using, if a translate a neighbourhood that remains a neighbourhood in u plus x. So, that implies that if U is open, if u is open in R 2, then U plus x is also an open set and hence belong to br 2.

So, that proves the first fact namely open subsets belong to a. So, let us now to show that. So, this proves the first fact that opens up sets belong to R 2. So, it to show that a is a sigma algebra that is this very sender technique we have been using at very often if, if a set E belongs to a; that means, E plus x belongs to BR 2 so let us write that. So, if E belongs so if E belongs to.

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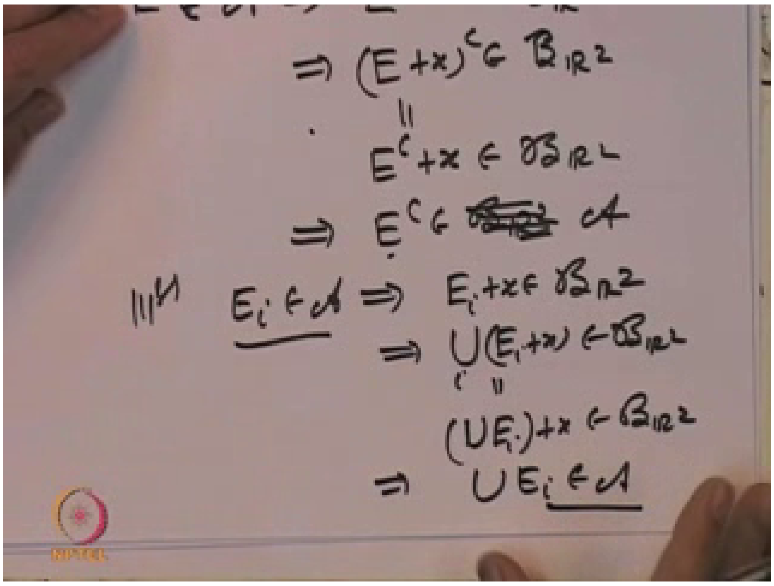

So, if a set E belongs to a collection a. So, that implies E plus x belongs to BR 2 and that implies because BR 2 is a sigma algebra. So, that will imply its complement belongs to BR 2, but this is same as the first taking complement and then (Refer Time: 23:23) taking translate.

So, that belongs to BR 2 so; that means E complement belongs to BR 2. So, if whenever a belongs to collection a gets E complement plus x belongs to BR 2 so; that means, E complement belongs to a. So, E is close in to complements and similarly Eis belonging to a will imply that union of so each Ei plus x belongs to BR 2. So, that will imply the union of Ei plus x belongs to BR 2 union over i, but this is same as union of Eis plus x belongs to BR 2. So, that will imply that the union of Eis union of Eis belong to A so Eis belong they belong to union also belongs to a.

So, that will prove that A is a so a is an algebra, so A is a sigma algebra. So, a is a sigma algebra including open sets. So, it will includes everything. So, that will prove that. So, this is the sigma algebra technique I have been mentioning that sigma algebra technique that we are mentioned says that implies that whenever E belongs to BR 2 implies E plus x belongs to BR 2.

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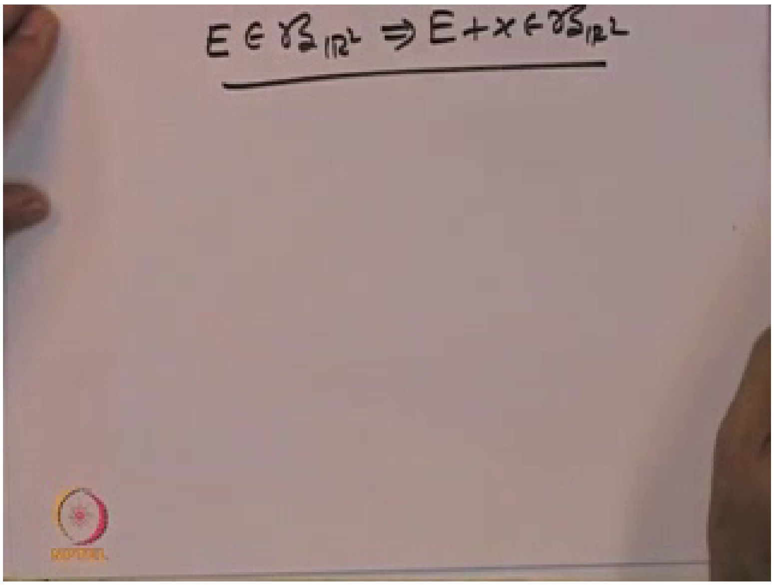

So, that is what we have proved.