Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 28 B
Fubini's Theorems

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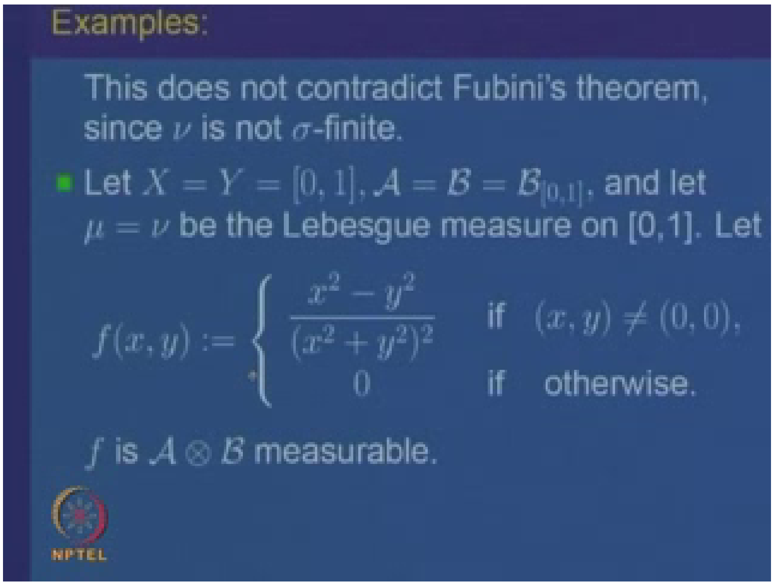

Let us look at another example. Let us look at two sets. Once again the underlying space is same X is equal to Y is equal to 0, 1; and A the sigma algebra on both of them is equal to the oral sigma algebra. And now let us look at measures mu equal to nu to be the Lebesgue measure. So, basically what we are doing is we are taking 0, 1, the Borel sigma algebra and the Lebesgue measure and take a copy of it and take the product of that

Let us define a function of two variables f x, y to be equal to x square minus y square divided by x square plus y square to the power 2 if x and y is not equal to 0 0 and it is equal to 0, if it is 0, otherwise. So, first of all we want to claim that this function f of x, y is a measurable function on the product sigma algebra. So, for that one has to look at the basic properties of functions of two variables one can show that this function is continuous everywhere except at 0. So, it is a almost everywhere continuous function of two variables. And hence it is going to be measurable function with respect to the product sigma algebra that is a Borel sigma algebra on the square 0, 1 cross 0, 1.

So, saying that f is a measurable function requires a proof and the proof I am giving you the hint the hint is as follows. Look at this function this function as a function of two variables. So, let me just write down the step, so that you are able to verify yourself later on.

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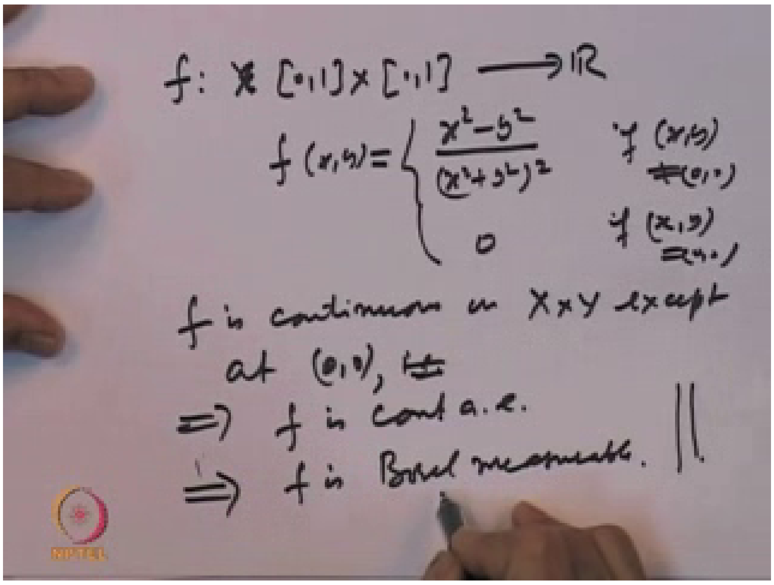

Let us look at the function f on X cross Y, so that is 0, 1 cross 0, 1 to R. So, this is a function which we are a saying is f of x, y, x square minus y square divided by x square plus y square if x, y not equal to 0 0; and it is equal to 0, if x, y is equal to 0, 0. So, the claim is this function f is continuous on X cross Y that is 0, 1 cross 0, 1 except at the point 0 0. So, at 0, 0, if you can you can see that if I take y to be equal to 0 that is x square divided by x to the power 4, so it will looks like 1 over x square and as I suppose is 0 that is going to blow up. So, it is not continuous at the point 0, 0.

But that does not an matter so; that means, f is continuous almost everywhere because one point does not matter. So, implies f is continuous almost everywhere. So, and this continuous almost everywhere implies f is Borel measurable, because continuous means inverse images of open sets are open and hence they will be Borel sets. And to show that Borel set is Borel you apply that sigma algebra technique. So, this we have shown that every continuous function is continuous almost everywhere is the Borel measurable. So, using in this step, you will be using the sigma algebra technique. So, look at all sets for inverse images are Borel open sets are inside and so on. So, this I will prove that this is a measurable function.

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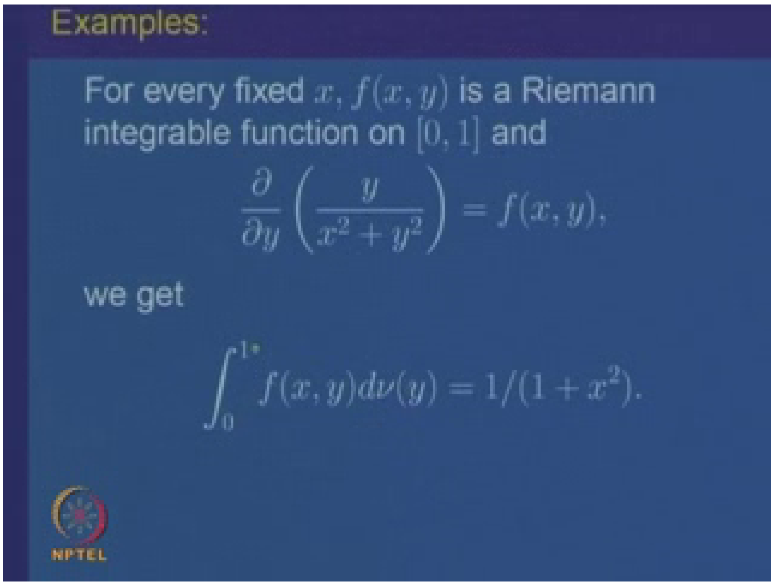

Now, let us compute the iterated integrals of this function with respect to that measures separately. So, let us first observe that for every fixed x. So, let us look at for every fixed x if I fix x any point then this is a function which looks like minus y square divided by x square plus y square whole square, so that is function of y for every x fix continuous almost everywhere except at the point 0. So, it is going to be a Riemann integrable function. So, it is a function for every fixed x, it is a function which is Riemann integrable function of y. And if you look at the integrant, it is the derivative of the function y divided by x square plus y square.

So, partial derivative of the function y divided by x square plus y square is equal to f of x, y. So, I know the anti derivative of this function for every fix x. So, I can integrate it out with respect to the variable y. So, when I integrate f x, y with respect to the variable y, because it is a Lebesgue measure you are integrating and for Riemann integrable function Lebesgue integrally same as the Riemann integral. So, that will give me that the integral 0 to 1 of f x, y d nu y is equal to 1 over because y is 1, so that will give me integral of this is equal to 1 over 1 plus x square.

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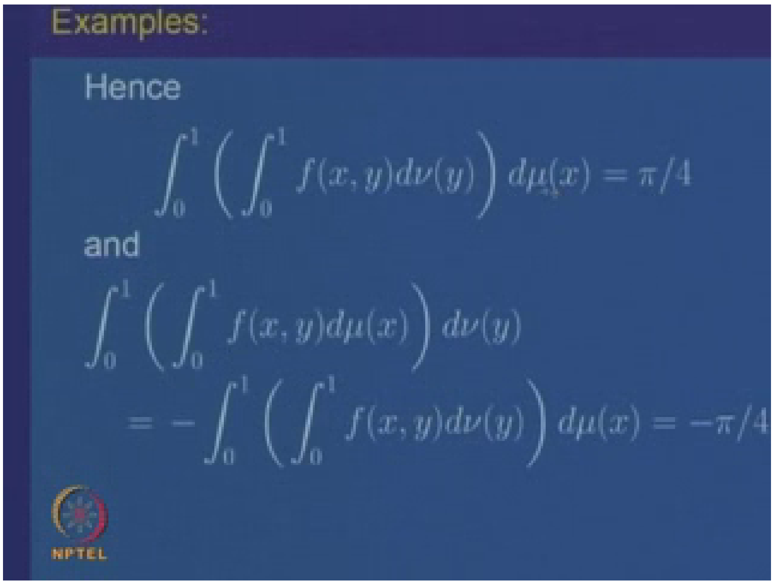

And now that we want to integrate with respect to x. So, integrate this with respect to x and that we know how to integrate this function 1 over 1 plus x square by substitutions putting trigonometric substitutions. So, we leave it for you to verify that the integral of 1 over 1 plus x square between 0 to 1 is equal to pi by 4. So, one of the iterated integrals this equal to pi by 4, but a simple observation in the function tells me that if I change x to y. So, let us look at the function if I change x to y interchange x and y then you get a negative sin outside because there is a negatives x square minus y square.

So, I do not have to compute the other iterated integral by using this property then if I interchange x and y that gives me a negative sign. So, if I want to integrate first with respect to x and then with respect to y, it will answer will be same as the earlier one with the negative sign. So, the other integral is going to be minus pi by 4. So, for this function of two variables, we have got two iterated internals, one of them equal to pi by 4 and the other is minus pi by 4, and both the measures here are sigma finite.

So, the question is what is going wrong. So, we have got two sigma finite measure spaces clear the same measure space 0, 1 Borel sigma algebra and Lebesgue measure. And on this we may got a function f x, y equal to x square minus y square divided by x square plus y square when not point is not 0, 0; and this function has got iterated integrals which are different. So, this not contradict Fubini’s theorem. The answer is no this is simply because though this you iterated integrals are different because the function is not integrable on the product space. So, we cannot apply Fubini's theorem-2 to it. So, let us verify that with respect to the product sigma algebra, the function is not integrable and that is a very simple observation.

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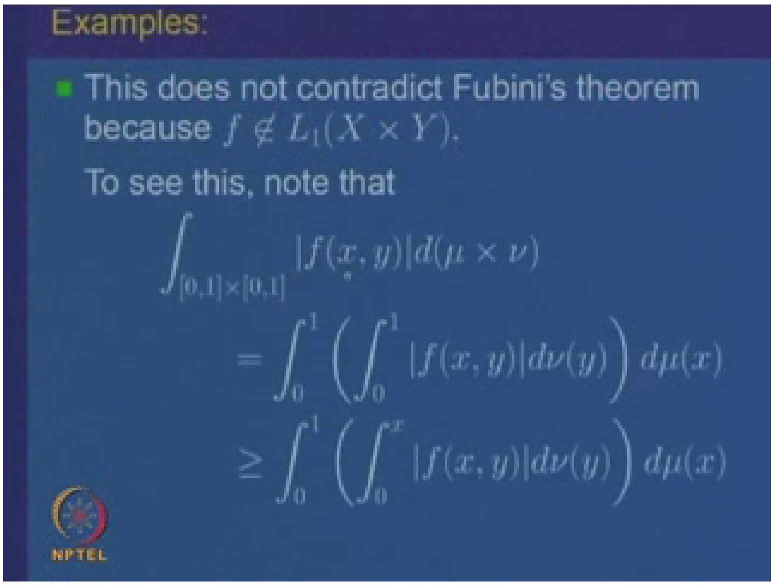

So, let us look at that. So, look at the function for two variables. So, let us look at mod of f x, y. So, this mod of f x, y respect to two variables this is a nonnegative function. So, by Fubini's theorem-1, we know that this is equal to the iterated integral of mod f x, y with respect to same variable y, so that we can because the integrant is nonnegative and x is in the inner integral the x is fix between 0 and 1. So, we can write the iterated integral from 0 to x. So, it will become bigger than or equal to. So, the integral of f should value of f x, y with respect to the product sigma product measure is bigger than or equal to integral 0 to 1 integral 0 to x of mod f x, y d nu y, but mod of that is nothing but 1 over x.

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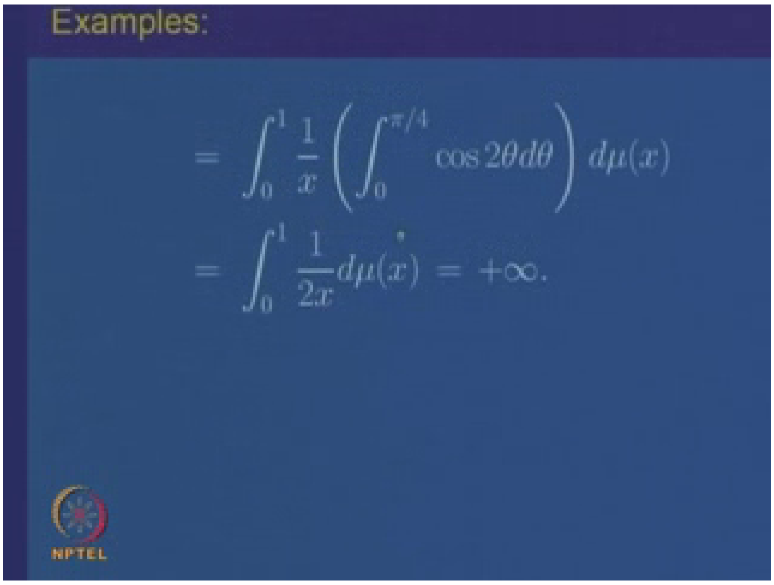

So, let us just look at mod of and 0 to x, here x is fix with respect to y. So, once you compute that inner integral, so that is nothing but 1 over x, 0 to pi by 2 cos 2 theta and those and that can be computed and that comes out to be 1 over 2. So, it is 1 over 2 x d mu x which is equal to plus infinity because 1 over x is not integrable. So, a simple computation shows that this function is not integrable. So, as a consequence, the again the Fubini's theorem is not contradicted because the two integrals are not equal because the function is not integrable.

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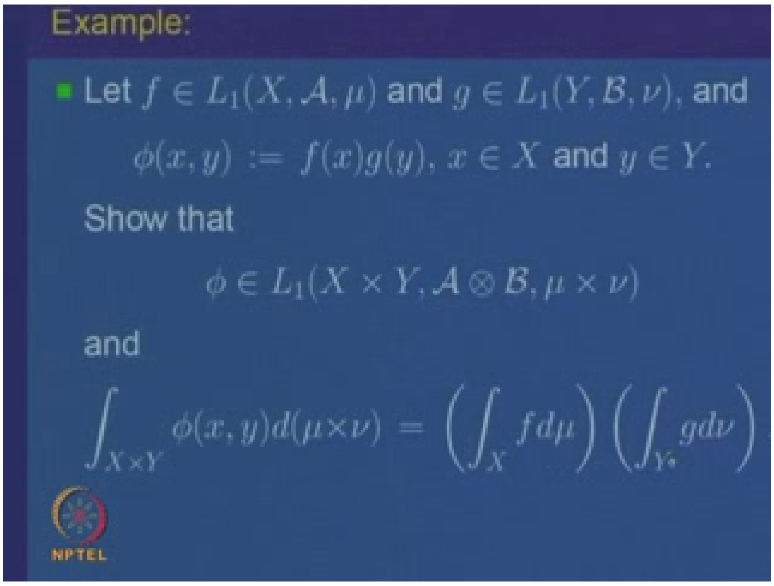

Let us look at an application of Fubini's theorem. We want to prove that if f is a integrable function on X, A mu; and g is another integrable function on Y, B we nu then look at the product of the two functions f x and g y. So, phi x, y is equal to f x into g y the claim is that this function is integrable and its integral is equal to the integral of f into integral of g.

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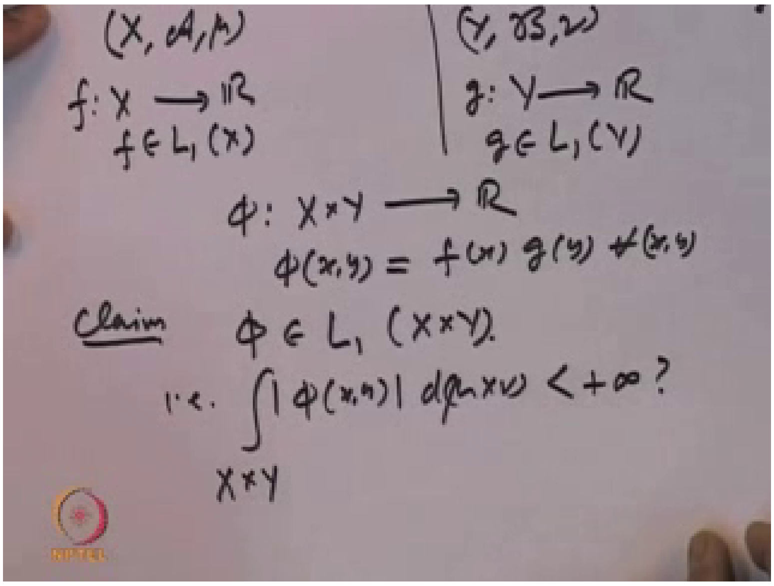

So, let us show this as a simple application of Fubini's theorem. So, we want to look a function f, so, we have got emerges space X, A, mu of course, sigma finite and f is a function defined on x and f belongs to L 1 of X. And on the other hand, for Y, B, nu we have got a function g Y to R and g belongs to L 1 of Y. So, define phi. So, we are defining a function phi on X cross Y taking values in R and the function defined is phi of x, y is equal to phi of x, y is equal to f x g y for every x, y. So, claim that phi belongs to l one of X cross Y, so that is what we want to a show.

So, let us see how does the proof how will the proof go how will the proof of this theorem goes. So, phi belongs to L 1, so that is we want to show mod phi x, y integral over X cross Y d mu cross nu is finite. So, this is what we want to show. So, to show that it is enough to show, so let us observe to show this enough to show because of Fubini's theorem one or earlier. So, to show this, so this is what we want to show.

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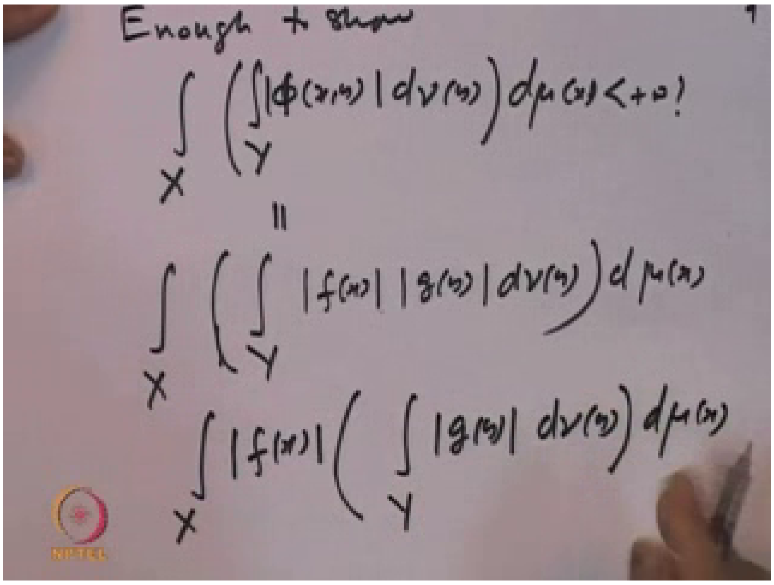

So, enough to show say for example, integral with respect to Y of phi x, y integral with respect to X this is d nu of y d mu of x is finite, so enough to prove this that this is finite, but let us compute what is this quantity. So, this quantity is equal to the inner one integral over y phi x, y is f x g y. So, it is mod of f x mod of g y d nu of y integral over x of d mu x. So, this which we want to show is finite is equal to this. Now, this is independent of integral x is fix. So, this is independent. So, it is integral over x mod of f x inside is integral over y of mod of g y d nu y d mu of x. And now g is integrable. So, this quantity is finite. So, this quantity is finite. So, what is that quantity equal to.

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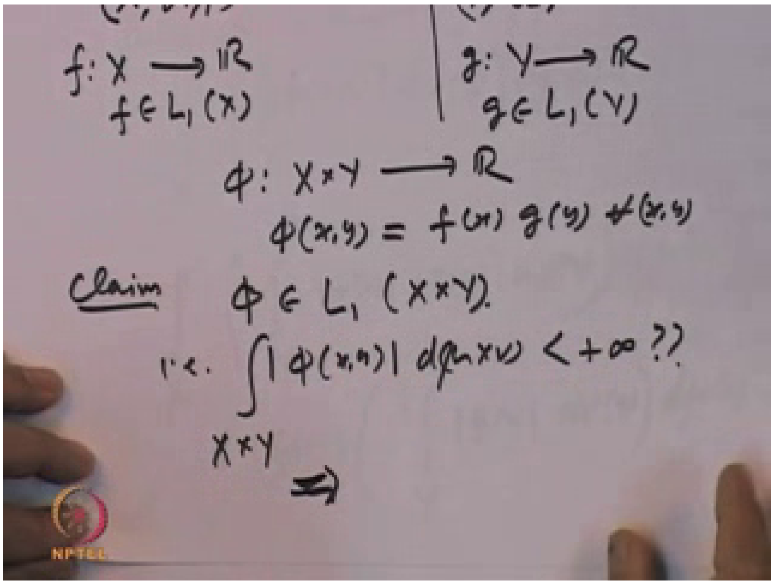

So, let us write what is this quantity equal to. So, this quantity is equal to. So, this is a constant, it is finite. So, it comes out. So, I can write this quantity is equal to integral over x mod f x d mu x integral over y mod g y d nu of y and that is finite because both of them are finite. So, what we have shown is that integral of mod of f x, y the iterated integral is finite, so that just now we observe that is equivalent to saying that the function mod f x, y is less than finite. So, this will imply that phi belongs to L 1. So, phi is a L 1 function. So, we have proved.

So, let us just say. So, this last thing that we proved implies that phi is L 1, because of product space X cross Y. Because it is in L 1, so Fubini’s theorem is applicable. So, implies by Fubini’s theorem-2 that the integral often X cross Y of phi x, y d mu cross nu is equal to the iterated integral either one we can write. So, let us a write it over X integral over Y of phi x, y what that is f x g y d nu y and d mu x. And that as now just now we have observed this is independent of the integrant is independent of y. So, this we can take it out. So, this is equal to integral over X f x integral over Y integral over g d nu d mu and that is precisely. So, this integral that we are written is precisely equal to integral over x of f x d mu x.

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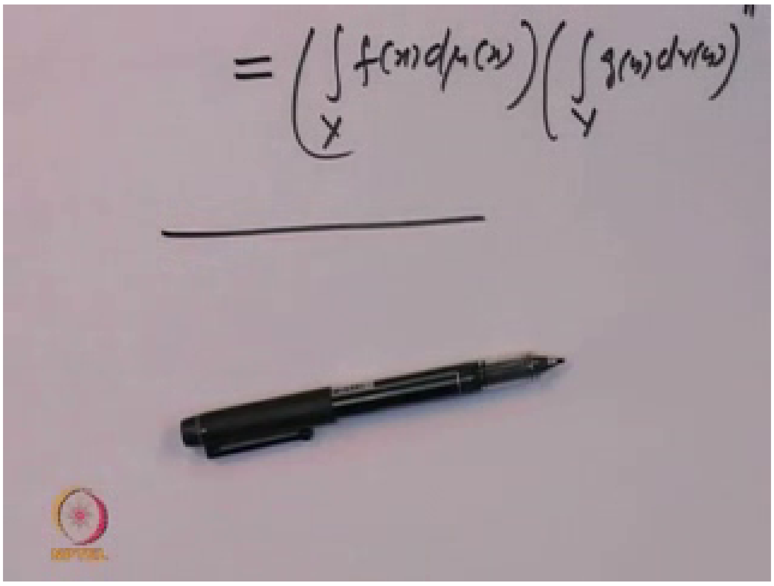

The first one and the second one is integral over y g y d nu y right, so that is how Fubini’s theorem is apply. So, you have seen that two applications of Fubini’s theorem to proof this result namely that if f is L 1. So, if f is a L 1 function then and g is a L 1 function on y then the product is a L 1 function on the product space and integral is equal to the product of the 2. So, this is how Fubini's theorems are going to be applied. So, let me just recollect or revise what we have done till now given product given the product measures X, A, mu, and Y, B, nu which are both sigma finite we constructed the product sigma product sigma algebra A times B the product measure mu cross nu. So, we got the product measure space X cross Y A times B and mu cross nu.

So, for this product measure the first thing we did was how to compute the measure of a set in the product sigma algebra. So, we said we can go a sections, so that give us that the product measure mu cross nu of a set E is same as you look at the x section E x, look at the measure of that nu of E x and integrant with respect to x, or similarly do with respect to y. So, that give us ways of computing the product measure of a set in the product sigma algebra. And that result when interpreted as an integral give us the first Fubini's theorem namely for nonnegative measurable functions, you can fix one variable at a time and integrated out. And then we extended this to functions of two functions which are not necessarily nonnegative, but integrable. So, those give us the Fubini's theorems.

So, in the next lecture, what will do is will now specialize when x is the real line, y is the real line, a is the Borel sigma algebra, b is the Borel sigma algebra or Lebesgue sigma algebras and look at product of the Lebesgue measure on x that is real line and product measure and Lebesgue measure on y that is again real line. So, will look at the Lebesgue measure space on the real line and take its product with itself to come to a notion of a measure Lebesgue measure on the plane, which will extend the notion of area in the plane. So, we will continue this study in our next lecture.

Thank you.