Measure & Integration
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Lecture – 28 A
Fubini's Theorems

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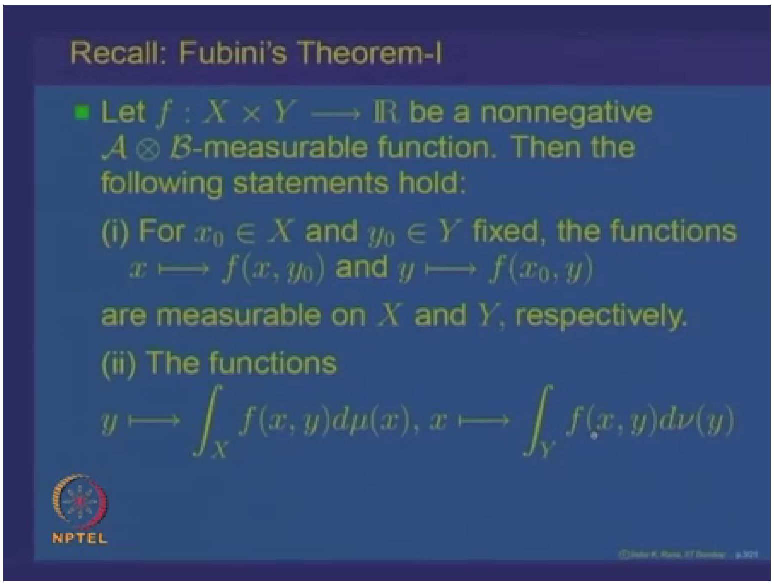

In the past few lectures, we have been looking at measure and integration on product basis. So, let us just we will continue doing that in this lecture also. So, we will be studying what are called Fubini's theorems we have proved some versions of Fubini's theorems in the previous lectures. So, let us recall what are those theorems that we have proved.

So, we have first proved what is called Fubini's theorem 1, which said that suppose f is a function defined on the product space X cross Y taking values in the real line and suppose this function f is nonnegative, and it is measurable with respect to the product sigma algebra A times B. So, f is a nonnegative measurable function. Then we claims the following statements hold namely if we fixed one of the variables for this function say x naught belonging to x or fixed the other variable y naught belonging to y then with respect to the other variable this functions become nonnegative measurable with respect to the corresponding sigma algebras.

So, for example, if x 0 is in X is fixed then the function y going to f of x 0, y is a function of the variable y on the space y. So, it becomes measurable with respect to the sigma algebra b. And similarly, for every fixed x 0 the function for every fixed y 0 in y to the function x going to f of x, y 0 is a measurable function on x with respect to the sigma algebra a. So these two, so for every one of the variable fixed in the other variable it becomes a nonnegative measurable function. So, once it is nonnegative measurable, you can integrate it out.

So, look at the integral of f x, y D mu x the variable y is fixed. So, we are integrating with respect to x, so the integral depends on y. So, this gives us a function y going to integral of x of f x, y with respect to the variable x. And similarly, if you integrate out the other variable y, then you get the function of x namely x going to integral over y of f x, y dv y. So, the second claim is that these functions are again nonnegative measurable functions of y and x respectively.

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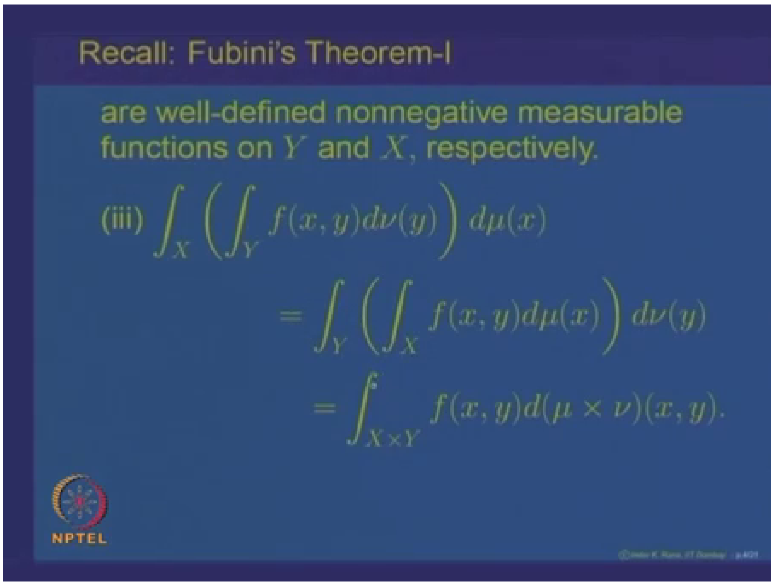

And once these are nonnegative measurable functions, you can integrate out the other variable now. So, integrate the with respect to the variable X, so you get one of the iterated integrals namely integral with respect to X and then integral with respect to Y of f x, y D mu y is equal to the other iterated integral namely first integrate with respect to X and then with respect to Y. These two iterated integrals are equal and the claim is this is equal to the integral of the given nonnegative function f x, y with respect to the product sigma algebra.

So, as we had stress that importance of this term lies in the fact that to integrate a function of two variables f x, y, we can fix either of the variables first integrate it out, and then integrate out the other variable. So, you can in do integration with respect to one variable at a time. So, this was the called Fubini's theorem-1 for functions which are nonnegative measurable f x, y.

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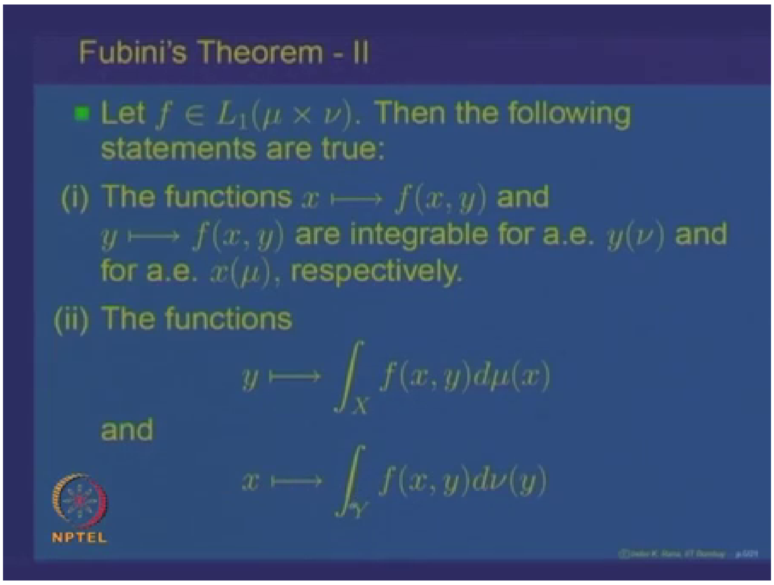

Then we extended this theorem to functions which are integrable, so that we called as Fubini's theorem-2 and that stated that let f be a integrable function on the product sigma algebra product space. So, f is in L 1 of mu cross nu then the following statements are hold, namely, if you fix one of the variables say x or y then as a function of the other variable these functions are integrable, but not for almost all variable fixed for almost all points whichever are fixed. For example, if we are fixing y, then it says that for almost all y, the function x going to f of x, y is an integrable function; and similarly for every almost all x fixed, y going to f of x, y is a integrable function.

So, when you fix one variable at a time for almost all such fixings the function of other variable are integrable. So, you can integrate out. So, you get function y going to integral over X of f x, y D mu x; and similarly the other function is x going to the integral over Y with respect to y of f x, y D nu y. So, these two functions are again so the claim is at these two functions are again integrable with respect to the corresponding measures nu and mu. So, you can integrate them out.

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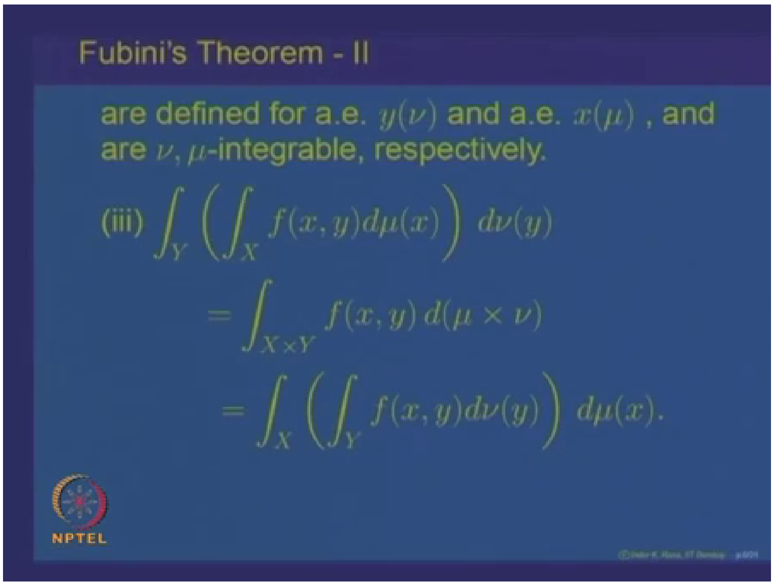

And what you get is that the iterated integrals once again in this case are equal and equal to the integral of the product of the function with respect to the product measure. So, basically what we are saying is that the two iterated integrals are equal to the integral of the function with respect to the product measure, whenever f is a integrable function. So, these two theorems, we had proved; and we want to give a one more version of this theorem.

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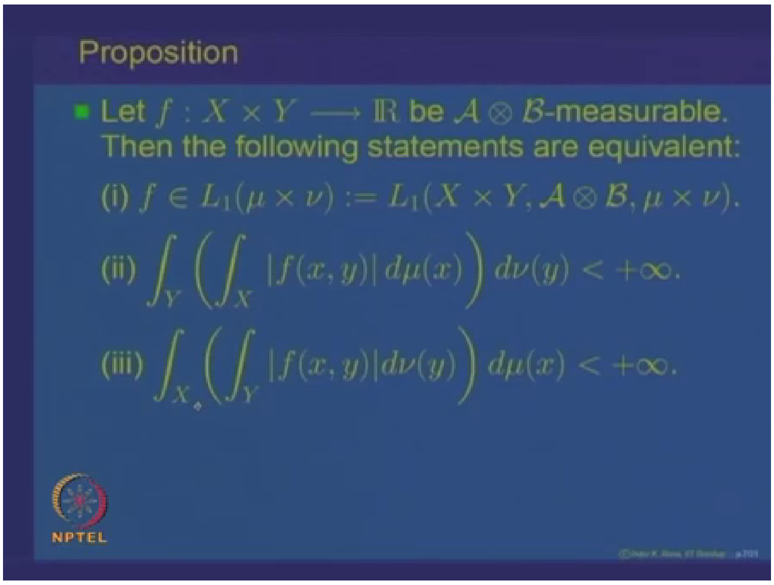

So, to do that, we need to prove a proposition, what integrable functions on the product space. So, let us take a function f on the product space f x, y on the product space A cross B then the claim is that the following statements are equivalent. One the function is L 1 function is as product as a function of two variables the function is integrable, so that is f belongs to value one. Secondly, if you look at the absolute value of the function f and look at it iterated integral with respect to x or y. So, look at the absolute value of the function f there is a nonnegative measurable function. So, look at its iterated integral first integral with respect to x and then with respect to y that is finite, so that is a second condition. And the third is you are interchange this. So, look at the other iterated integral it first integrate with respect to y and then with respect to x. So, then which is the second iterated integral that is finite. And fourthly, so these three conditions may want to show these three conditions are equivalent.

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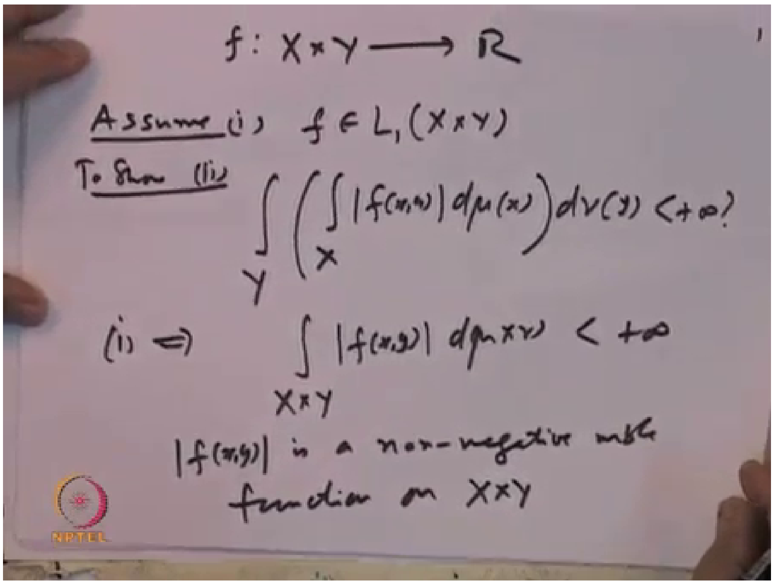

So, let us prove them. So, let us first assume so f is a function given on the product space X cross Y to R. So, let us assume one, assume one that is namely saying that f belongs to L 1 of X cross Y. We want to show two namely that the iterated integral of mod f x, y D mu of x and then integrate that with respect to y, so that it is D nu y is finite. So, this is what we want to show. Now, since f is integrable, so the condition that f is integrable. So, one implies that integral of the function mod of f x, y with respect to the product sigma algebra product measure mu cross nu is finite or the product space right. And now let us are note that mod f x, y is a nonnegative measurable function on X cross Y. So, it is a nonnegative measurable function on X cross Y. So, Fubini's theorem one is applicable.

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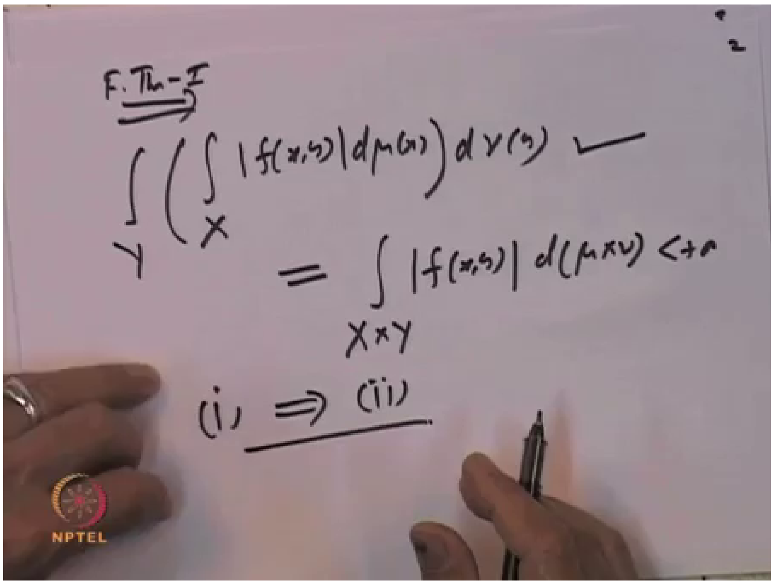

So, by Fubini's theorem-1, so we get implies by Fubini's theorem-1, which was for a nonnegative. So, by Fubini's theorem-1, for nonnegative functions what we get is that the iterated integral of so mod f x, y the iterated integral with respect to x and then with respect to y D mu x D nu y must be equal to the double the integral over the product space X cross Y of mod f x, y D of mu cross nu which is given to be finite, so that implies that this iterated integral is finite by the second condition. So, what we have shown is, so hence, so one implies two. Let us try to show that two implies one. So, let us show that two implies one.

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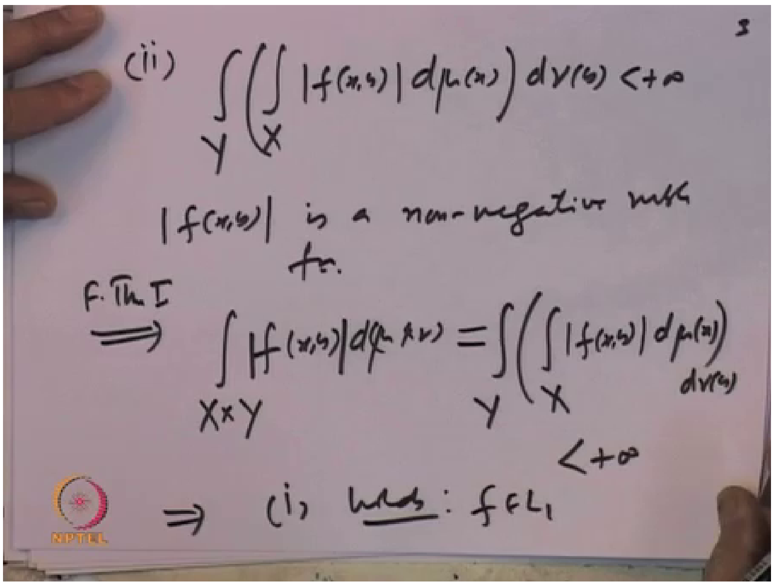

So, what is two? So, the condition two says that integral over X of mod f x, y D mu x that integrated with respect to Y D nu y is finite that is given to be finite. So, once again we observe that mod of f x, y is a nonnegative measurable function. So, by implies by Fubini's theorem-1 that the double integral. So, now, we can revert that, so double integral is that means, integral of the product space of f x, y D mu cross nu must be of absolute value of f x, y must be equal to the iterated integral. So, that is integral over Y integral over X of mod f x, y D of D mu x and D nu of y and that is given to be finite by two. So, implies that one holds namely integral of mod f with respect to the product measure is finite that is that f belongs to L 1.

So, what we have shown is that the condition one is equivalent to condition two. And a similar proof implies that condition two is also equivalent to condition one. So, saying that the function is integrable is equivalent to saying that the either of the iterated integrals of mod f are finite. So, all these three are equivalent conditions. So, these can be put into the statement of the Fubini's theorem. So, combining Fubini's theorem-1, 2, and then this proposition gives us what I call as the combined a Fubini's theorem basically combined Fubini's theorem gives you conditions under which you can say that the iterated integrals of a function of two variable are equal and equal to the integral of the function over the product measure space.

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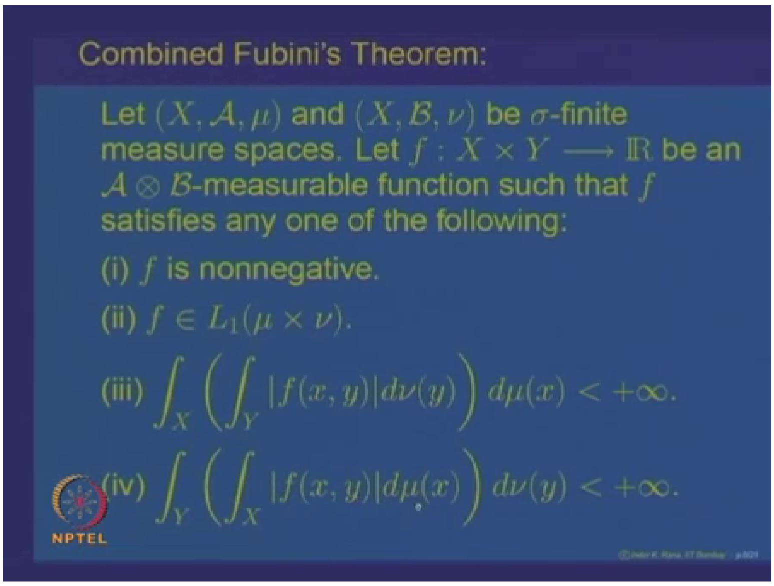

So, let us look at the combined Fubini's theorem, which says let x a mu and y b nu be sigma finite measure spaces might be have been working under sigma finite finiteness because a product measure is defined only for sigma finite measures. So, if two sigma finite measures spaces are given, and you are given a function f on the product set X cross Y, which is measurable with respect to the product sigma algebra A times B then the following conditions any one of the following conditions namely f is nonnegative. And Secondly, f is integrable; and third condition that is the iterated integral of mod f with respect to y and then with respect to x is finite or the other one the iterated integral of mod f with respect to mu and then with respect to nu is finite.

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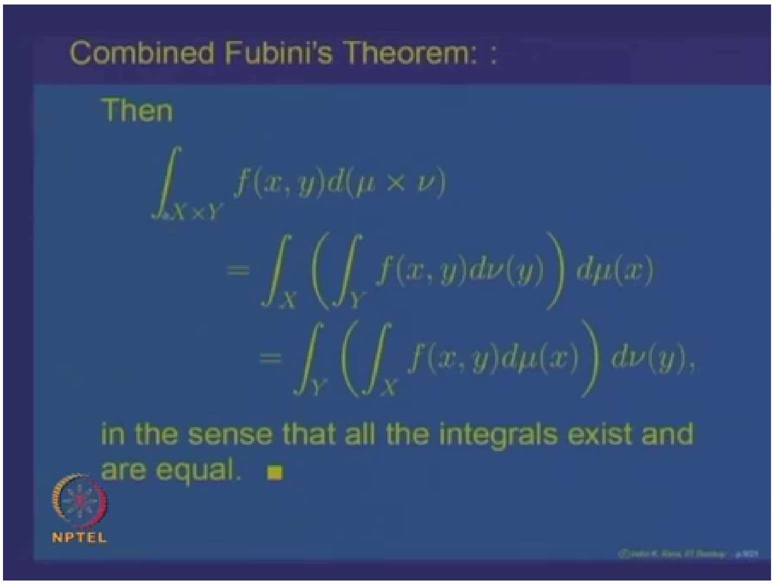

So, if any one of these four condition is satisfied then we can say that the integral of the function f over X cross Y with respect to the product sigma algebra is equal to both of the iterated integrals namely either integrating with respect to y first and then with respect to x or integrating with respect to x first and then with respect to y. So, basically these three integrals are equal in the sense that if either of them exist then all the three will exist and are equal.

So, basically Fubini's theorem says that the two iterated integrals are equal and equal to the product equal to the integral over the with the respect to the product measure. Whenever either of these integrals are define for example, if f is nonnegative then these are all defined and hence they should be equivalent by Fubini's theorem-1; and if f is L 1 then the product then the integral of f x, y over X cross Y that exists. So, this these all three must be equal and other iterated integrals of mod f there equivalent to saying f is L 1, they all will be equal. So, this is what is called combined Fubini's theorem and it is of importance, we will see lot of applications of this soon.

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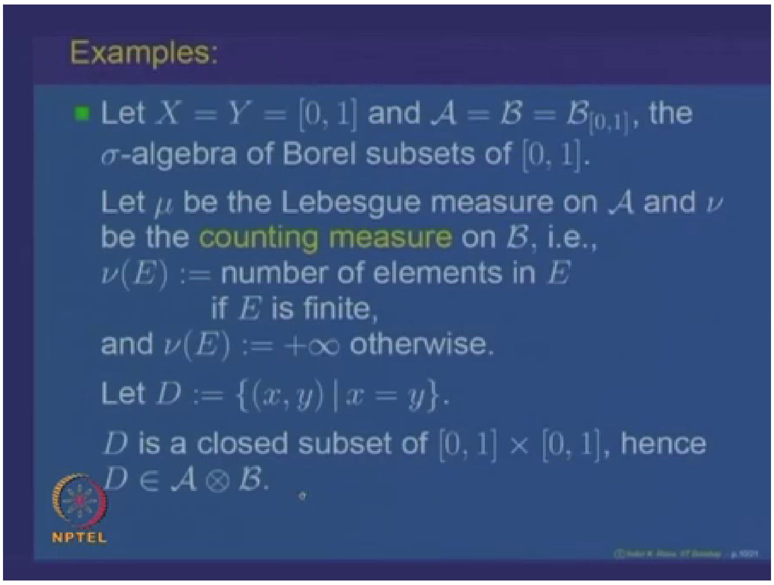

So, let us look at some examples. So, I want to stress this point namely that in the statement of the theorem, we have assume the condition that mu and nu are sigma finite measures. So, these conditions are going to be important. So, let us look at some examples to illustrate this. So, let us look at the space when x is equal to y is equal to 0, 1. So, my underlining spaces x same as y as the interval 0, 1; and the two sigma algebras sigma algebra A is same as a sigma algebra B is equal to Borel sigma algebra, the sigma algebra Borel subsets of 0, 1. So, let us define the measure mu to be the lebesgue measure on A and the measure nu on when treated as y 0, 1 is treated as the other space we define it to be the counting measure. So, what is the counting measure counting measure is the number of elements in the set E, if the set E is finite otherwise it is equal to plus infinity.

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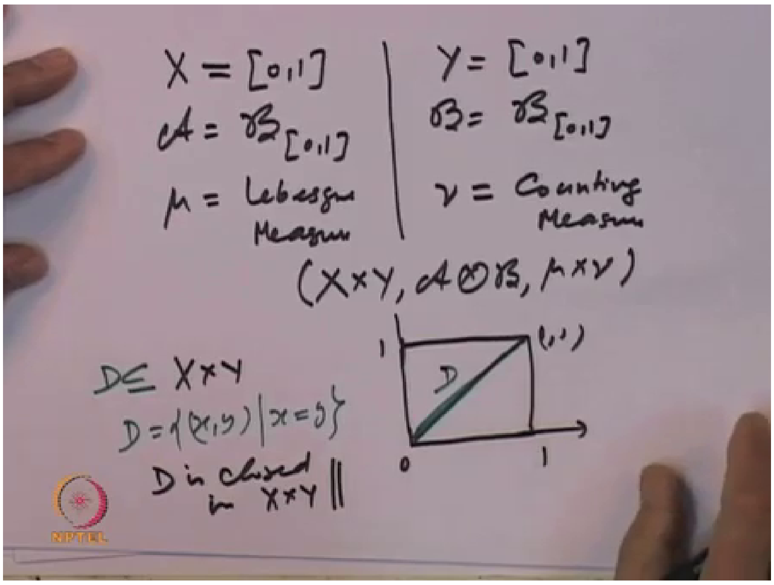

So, we have got two measures spaces. So, let us look at the example that we have gotten. So, we have got X which is equal to 0, 1, the sigma algebra A is the Borel sigma algebra on 0, 1; and mu is Lebesgue measure. On the other side we have got Y, which is again 0, 1 and the sigma algebra B is again the sigma algebra of Borel subset of 0, 1 and we have got nu which is equal to the counting measure. So, counting measure as you define it is if the set E is finite then it is number of elements whether it is equal to plus infinity. So, now let us look at the product space X cross Y A times B and mu cross nu. So, let us look at the product space.

So, let us look at the set. So, we are going to look at the set D which is the set x is equal to y. So, this is you will note that X cross Y this is a subset of X cross Y. So, the claim is that this is a closed subset of X cross Y, and hence it is an element in A cross B. So, what is our space X cross Y, X cross Y is if you picturized, it is just 0, 1 cross 0, 1. So, that is the our space this is 1 1 and what is the D, we are looking at the set D, which is contained in this the set D which is contained in this. So, that is D is equal to x comma y x is equal to y. So, that is a this line is my D. So, the claim is that this D is a Borel set. So, one-way of looking at it is the D is closed in X cross Y. What is the meaning of that this set is closed. So, one way of showing that this is a closed subset of a 0, 1 is to show that it contains all its limit points that is one way of showing it.

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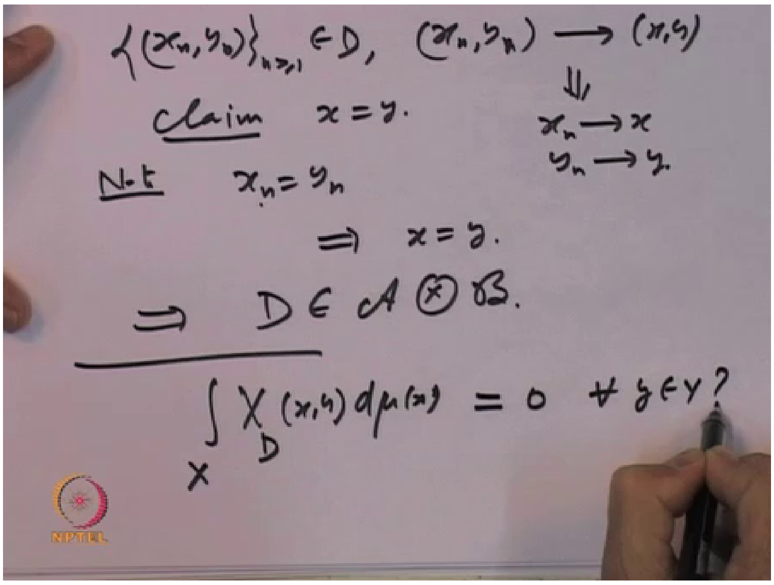

So, if I take a sequence x n comma y n belonging to D, and x n comma y n converges to say x y then claim we should be able to show that x is equal to y, but let us note x n y n belonging to D that implies. So, note that x n is equal to y n, because x n comma y n belongs to D is the diagonal and x n, y n converging to x y. So, this implies. So, this condition implies that x n converges to x n, y n converges to y. So, x n is equal to y n; x n convergence to x n, y n converges to y, so the all that will implies at x is equal to y. So, the set D is a closed subset of 0, 1. So, implying that D belongs to the product sigma algebra A times B, so that is the D is a closed subset of 0, 1.

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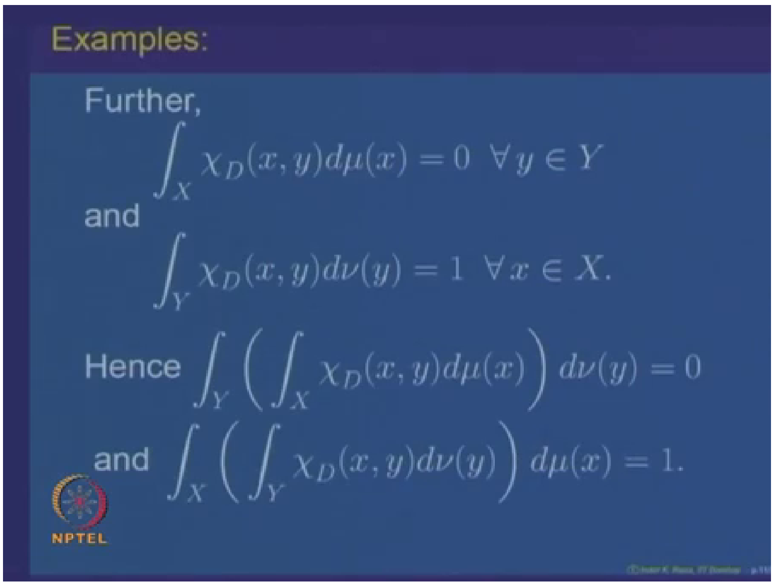

Next we want to compute the iterated integrals of the indicator function of the set D of this diagonal with respect to mu and the claim is that the if I integrate if i fix one of the variable say why and integrate the indicator function of D with respect to x then this is equal to zero for every y. And the other integral when you fix x integrate with respect to y the counting measure that is equal to one for every x. So, that will imply that the two iterated integrals are not equal one of them is equal to 0; and the other one is equal to 1.

So, let us verify this first. So, let us verify the condition that if I take the indicator function of the set D, x y and integrate with respect to mu over x. So, the first claim is that this is equal to 0 for every y belonging to y. So, let us see why is this integrally equal to zero to note that to show that.

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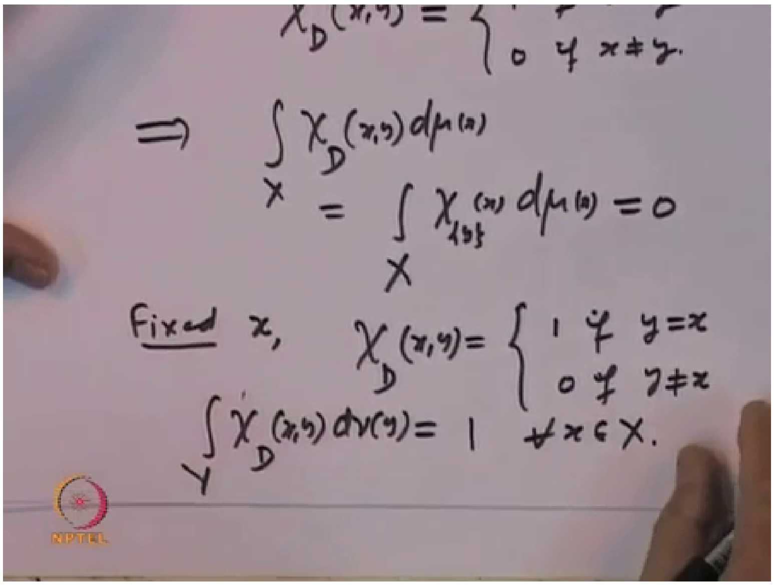

Let us observe that for fixed for fixed y we want to compute chi the indicator function of D at x y. So, what is that equal to? So, this is equal to this is equal to 1, if x comma y belongs to D; that means, x is equal to y; and it is 0 if x is not equal to y. So, this function the indicator function of the indicator function of the diagonal for y fixed takes only two values 1 and 0; and only at one point it takes a value y it takes a value one when x is equal to y, and all other points it is just 0. So, it is a indicator function of a singleton set. And hence so this implies that integral over x chi D x y D mu x is equal to the indicator function of the singleton set namely y x D mu x and that is single point and mu is the Lebesgue measure. So, this is equal to 0.

So, a simple observation that the indicator function of the diagonal x, y is equal to 1, if x, y is fixed. So, if x is equal to y, otherwise it is 0, so that proves that this is equal to 0. And similarly let us compute the other one for fixed x, let us look at the indicator function of x, y equal to. So, what is that equal to? So, that is again equal to 1, if y is equal to x; and in 0, if y is not equal to x that is again the same function. But now let us observe that, so the if I integrate with respect to y of the indicator function of D x y with respect to D nu y, here nu is the counting measure.

So, for one point for one of the one point when y is equal to x, the value is 1. So, the value is one for one point and the measure of the single point is equal to, so that is equal to one for every x belonging to for every x belonging to x. So, that proves the required claim namely the integral of D indicator function of D for every y fixed is 0 and for every x fixed is 1. So, the iterated integral with respect to one of one of the iterated integral is equal to zero while the other iterated integral is equal to 1, so that seems to contradict Fubini's theorem. Because the indicator function is a nonnegative function then the two iterated integrals are not equal, but that is not the case because the measures involved not both of them are sigma, Lebesgue measure is sigma finite, but the counting measure is not sigma finite.

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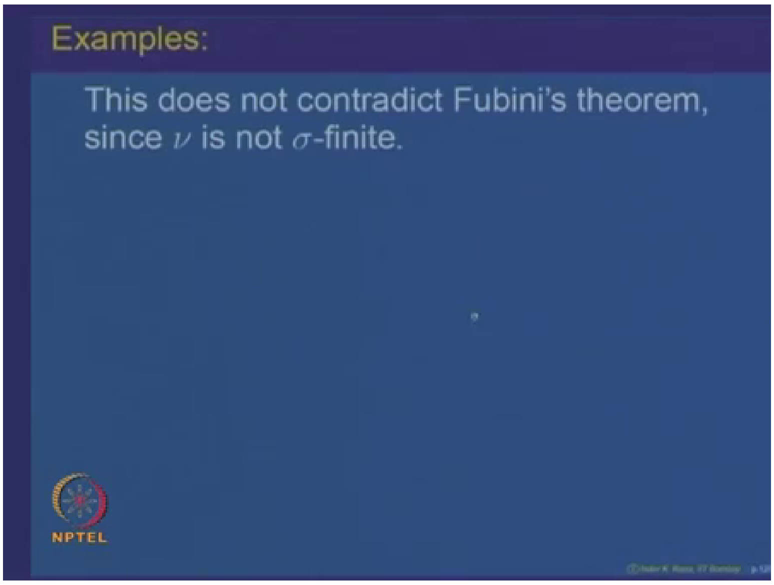

So, this does not contradict Fubini's theorem since the measure counting measure is not sigma finite if the counting measure of the whole interval 0, 1 is infinite and you cannot divided into countable number of pieces each having finite, because 0, 1 is uncountable. So, the counting measure is not sigma finite, so that is why this does not contradict Fubini's theorem.