Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 27 B
Integration on Product Spaces

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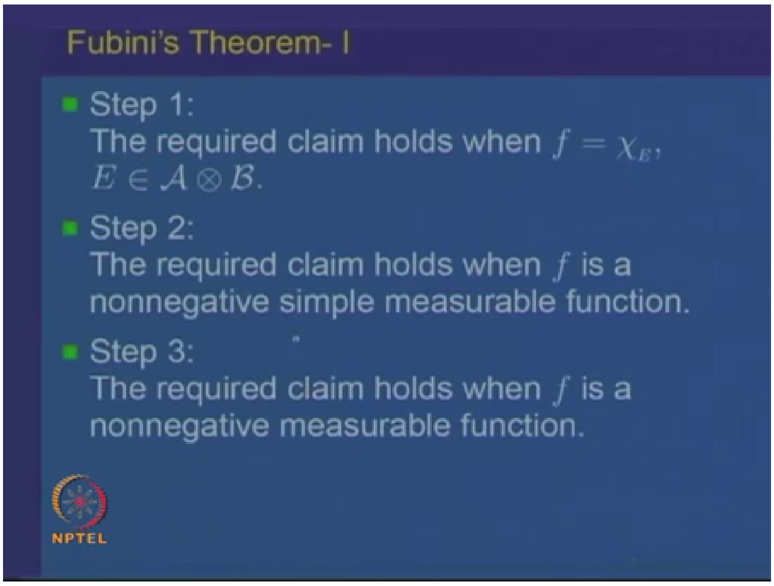

Let us just go through this proof through the slides once again. So, that we have a clear idea what we are doing. So, the step one the required claim holds when f is a indicator function of E that is the previous theorem that we have proved. And step 2, the required claim holds when f is a nonnegative simple measurable function. So, from step 1 to step 2, one goes via the fact that integrals are linear operations; and then one goes to step three that the required claim holds when f is a non-negative measurable function, so that requires applications of basically applications of monotone convergence theorem

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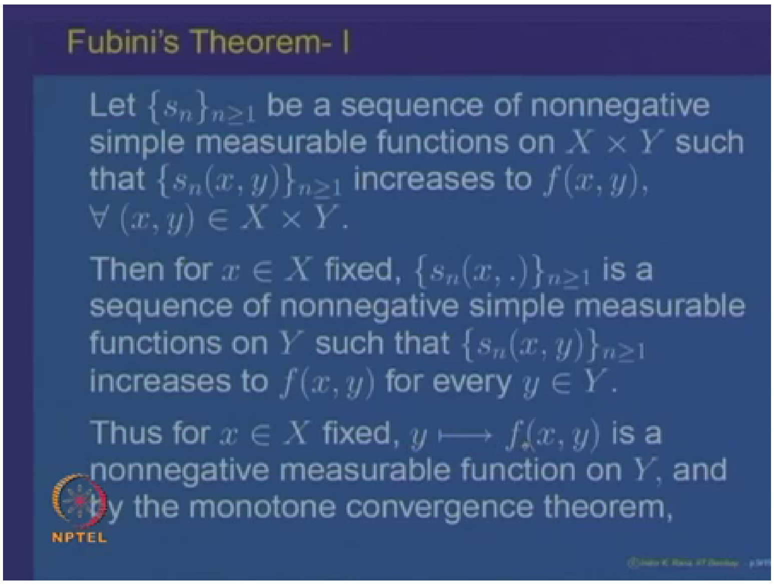

So, step 3 is the crucial one where lot of applications are of monotonic convergence theorem are used. So, let us just go through that again. So, let s n be a sequence of nonnegative simple measurable functions such that s n increases to f, so that is by the fact that f is a nonnegative measurable function. So, now let us fixed X, so then the sequence s n x. X is fixed, so in the variable y is a sequence of nonnegative simple measurable functions on Y. And it increases to the function f of x y for X fixed. So, point wise s n x dot as a s x fixed s n x as a function of Y increases to the function f x as a function of y. So, an application so this is increasing, so an application of monotone convergence, and monotone convergence theorem is not required here.

So, this is a limit of increasing sequence of measurable functions so that says that the function y going to f of x, y is a nonnegative measurable function because this function is a limit of measurable function. So, first fact to being used is that limits of measurable functions is a measurable function. And now we can also apply monotone convergence theorem to conclude that the integral the iterated integral of s n must come converge to the iterated integral of f with respect to y.

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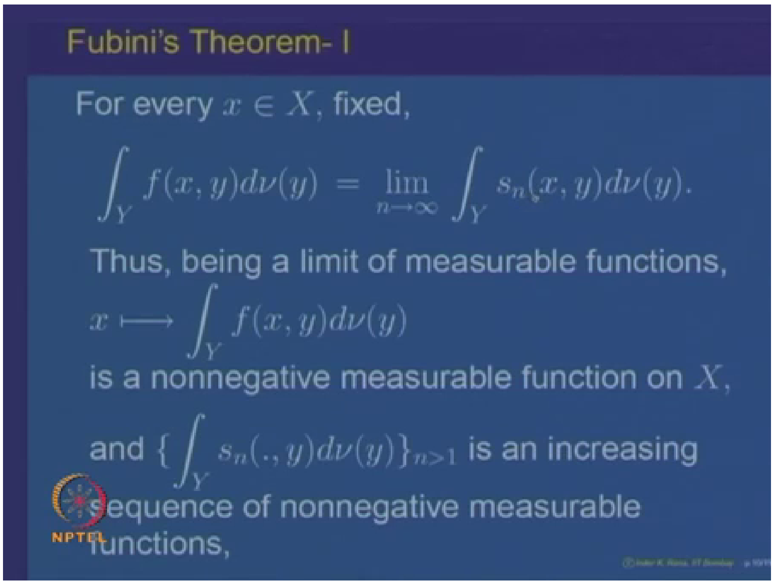

So, that is the first application of monotone convergence theorem that for the nonnegative measurable function f for the variable x fixed, it is a integral with respect to the variable y is well defined, because this is a nonnegative measurable function and it is equal to limit with respect to n, the iterated integral of the nonnegative simple, a measurable functions s n with respect to y. And now this result also says that this equality also says that this the right hand side treated as a function of x, so that means that convergence to this integral right. And by the fact that the required result holds for nonnegative simple measurable functions, this function integral of s n with respect to y is a measurable function of x. So, here we are using the step 2.

So, this is a sequence of measurable functions converging to a function so that means, this integral must be a measurable function. So, again limits of measurable functions are measurable, so that gives you that x going to integral over y f x y dy. So, the iterated integral of f with respect to y is a measurable function with respect to x; and it is nonnegative. And once again this is a nonnegative function, and it is a limit of these integral limits of this measurable functions. So, another monotone convergence theorem application gives that integral of the iterated integral of the integral of s n with respect to y, its integral with respect to x must come to the corresponding integral of f with respect to x.

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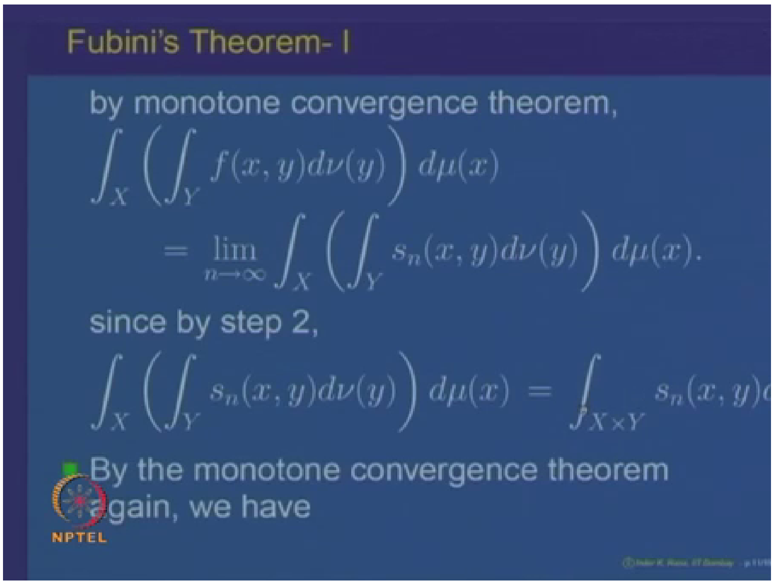

So, here we are applying monotone convergence theorem that the integral over x of the integral of f with respect to y must come must be limit of the corresponding integrals with respect to the nonnegative simple functions. And now come back and for nonnegative simple functions, we know the result is true. So, this iterated integral must be equal to the double integral, so that says so this is equal to the double integral. And now s n is a sequence of nonnegative measurable functions on the product space, so this again by either you can say application of monotone convergence theorem what just by the definition this limit must be. So, this is equal to this and the limit of that must be equal to the integral of the function f over X cross Y, so that says the corresponding result holds.

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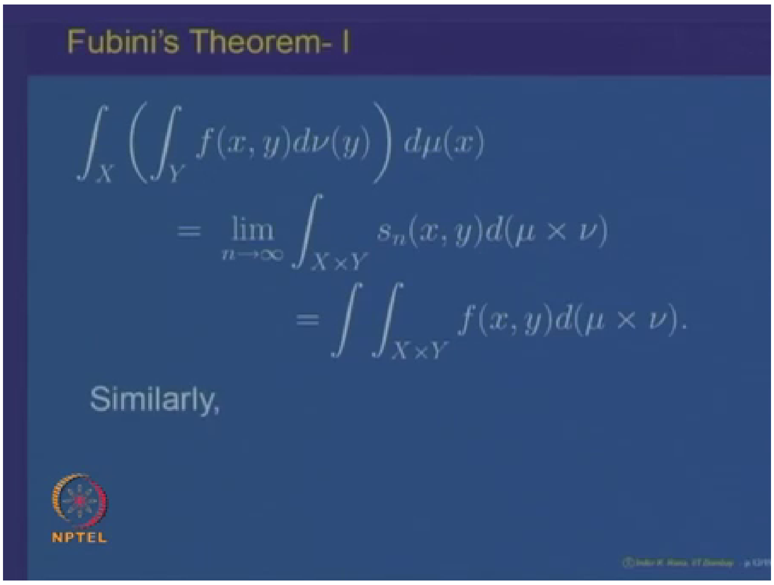

So, this iterated integral of f is equal to the double integral of f with respect to mu cross nu.

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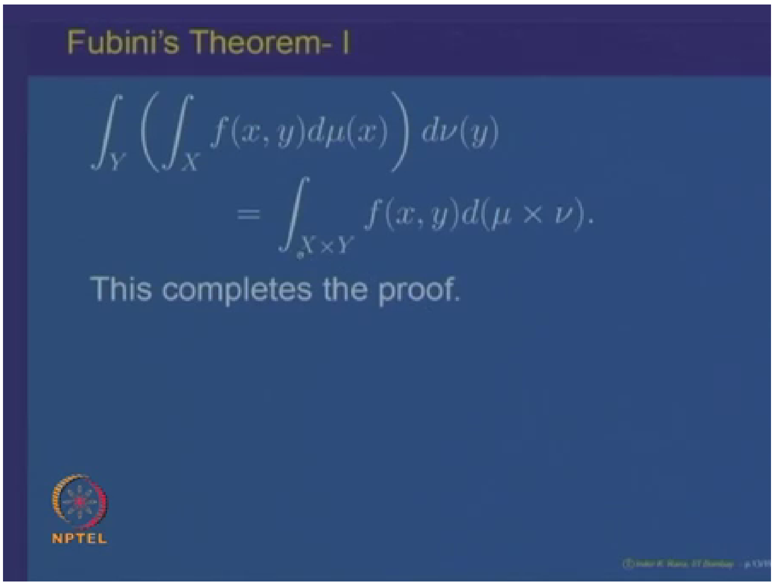

And similarly the other thing can be proved you can interchange the variables x and y. So, the result is true. So, this is a result which is called Fubini's theorem one which says. So, this is the first Fubini's theorem if it says that for a nonnegative measurable function on the product space if you want to integrate find its integral with respect to the product measure, you can do it by integrating one variable at a time. Either you can fix x integrate out with respect to y, and then integrate with respect to x or interchange, choice is yours you can first integrate with respect to x and then with respect to y. So, the two iterated integrals for a function of two variables is equal to the double integral for nonnegative measurable functions. So, this is called was first Fubini's theorem which helps one to integrate functions of two variables. So, next we want to show that this result also holds for functions which are integrable. So, we want to prove that for a integrable function the corresponding result holds.

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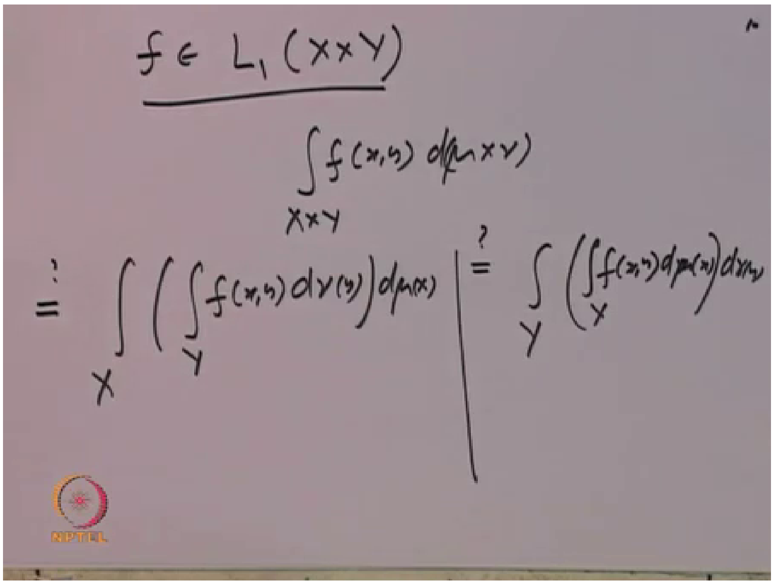

So, let us look at the proof of that. So, let us look at. So, let us take a function f which is L 1 on X cross Y, it is integrable with respect to X cross Y. And we want to say that that the integral of f x y over X cross Y d mu cross nu. We want to say that this integral on one hand is equal to you can integrate first x, y with respect to nu, we want to claim this with respect to y over y, and then integrate out that with respect to x, so d mu x. Or one should be able to say that this is also equal to you take the function f x, y integrate out the variable with respect to mu, so x and then integrate out with respect to y d nu of y. We want to say that these two this results hold.

Now for that on obvious if these equations are true hold where f is not necessarily nonnegative so; that means, what first of all the inner integral, for example, the integral of f x y with respect to y must exist. That means, we should be able to say for a function of two variables, which is integrable when I fix the variable x as a function of y that is integrable. So, that is integrable and then that gives us a function of x and then we should be able to say that is integrable with respect to x and finally, these two are equal. And similarly, the other result must hold. So, the theorem which we want to prove is the following namely that is called Fubini's theorem 2.

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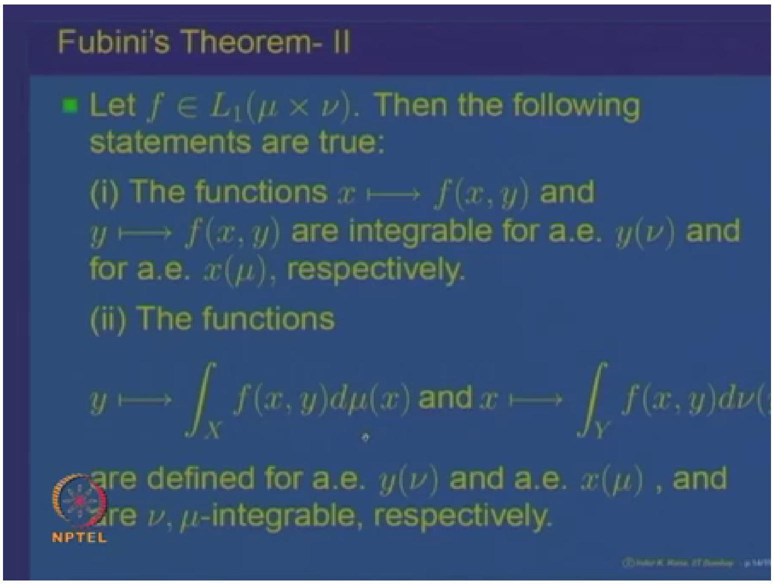

So, namely if f is a integrable function f is a integrable, so we want to prove the following that if f is integrable function then the following statements are true namely one that for the function of two variables. If I fix either of the variable then with respect to other variable its integrable not for all, but we are able to say that the function x going to f x, y and y going to f x, y for the other variable are integrable for almost all y and for almost all x.

So, for almost all fixing of coordinate, the other variable it becomes a function which is integrable with respect to the other one, so that is one. And then secondly, once these are integrable, you can integrate out. So, it says that the function y going to integral of f over x with respect to mu; and similarly the integral of f with respect to Y, these two are defined almost everywhere. And of course, they are defined almost everywhere and are integrable.

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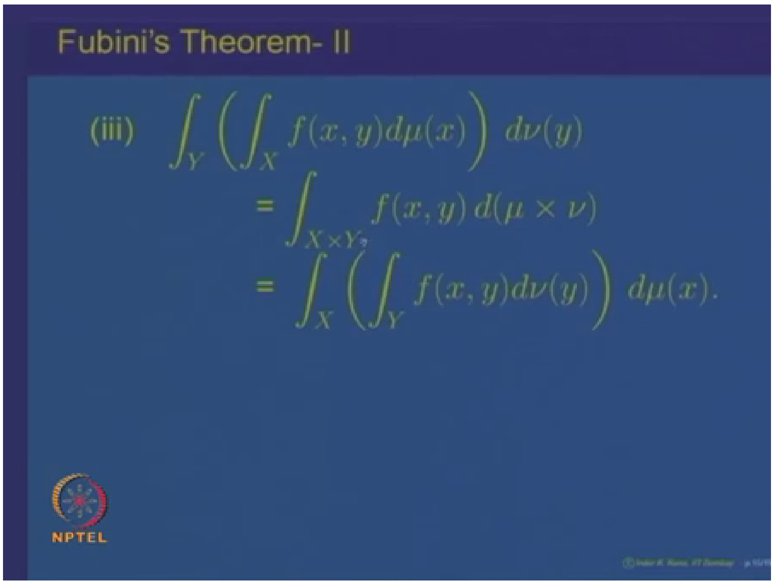

And hence the third step says they are integrable and indeed the integrals the iterated integrals are equal to the double integral. So, we would like to prove this theorem. So, to prove this let us proceed as follows. So, we are given that the function f belongs to L 1 of X cross Y.

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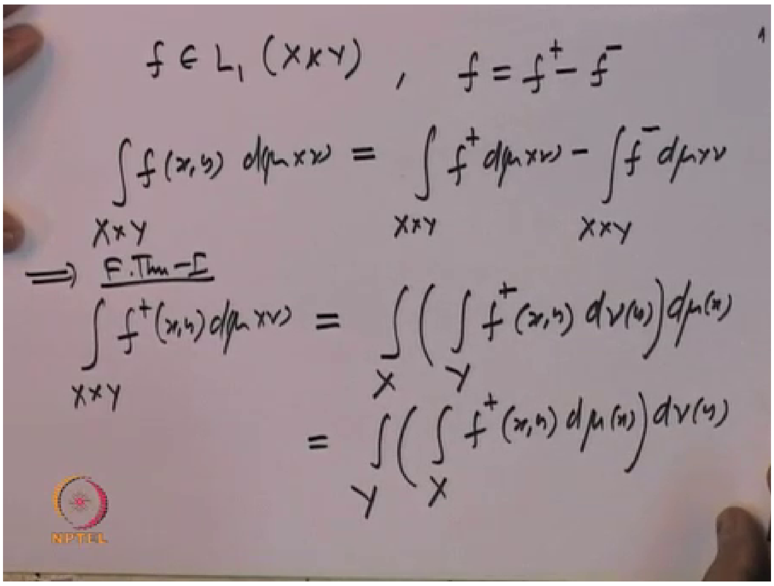

Let us write the positive and the negative parts of the function. So, f is equal to f plus the positive part minus the negative part. And there is the integral of f x, y with respect to the product measure mu cross nu is equal to integral of the double integral of f plus with respect to the product measure minus the integral of the negative part the mu cross nu over X cross Y, so that is a definition of the integral. If f is integrable then the integral of the function is nothing but the integral of the positive part minus the integral of the negative part of the function.

So, now let us look at them separately. So, f plus x, y of course d mu cross nu over X cross Y. So, f plus is the nonnegative function and is a nonnegative measurable function. So, by the result Fubini's theorem-1, I can write this as integral over x integral over y f plus x y integral over f x, y d nu with respect to y and then d mu with respect to x. So, implies by Fubini's theorem-1 that is Fubini's theorem for nonnegative measurable functions that integral of a nonnegative measurable function can be computed by iterated integrals. So, this is and also equal to let us write the other one also, you can interchange integral over X f plus of x, y d mu x and d nu of y. So, this is for f plus we have used the Fubini's theorem-1. And now let us observe f being integrable this quantity is finite.

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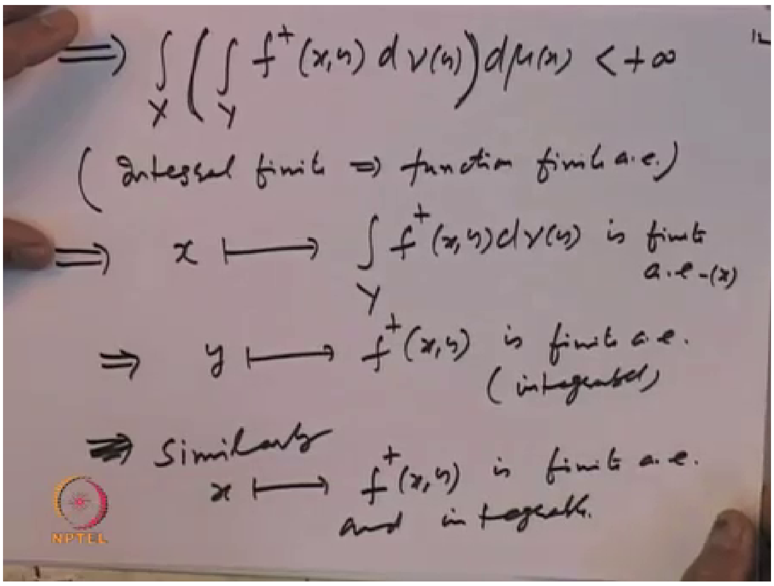

So, all these integrals are finite quantities so that means, so this the all this being finite implies, so for example, the first one implies, so implies because of integrability that integral over x integral over y f plus x y of d of nu y d mu x is finite. Now, so here is an important observation that we have earlier proved that if the integral of a function is finite, then the function must be finite. So, here we are using integral finite implies function finite almost everywhere. So, this we had already proved. So, this fact we are going to use now. So, look at this inner this integral with respect to mu of this function is finite, so that implies that the function. So, this as a function of X, X going to integral over Y of f plus x, y d nu y is finite almost everywhere almost everywhere with respect to x. So, we have used the fact that integrable function implies that the function is finite almost everywhere.

And now once again for almost all x this is finite, that means so this also implies that the function, so y going to f plus x, y is finite almost everywhere, and of course, integrable because this integral is finite almost everywhere. So, is a function which is integrable and finite almost everywhere. So, implies I can integrate out and this is a nonnegative function, it is integrable and nonnegative integrable function. So, we have for this already seen this is equal to so that we have already seen that for nonnegative measurable function this is equal to this integral.

So, similarly the function for x going to f plus x, y is finite almost everywhere and integrable is also integrable; and the corresponding results also hold for similar results holds for f minus so that means what so all those four functions are finite and integrable. So, we can integrate them out and we have the results corresponding results for so these are all integrable.

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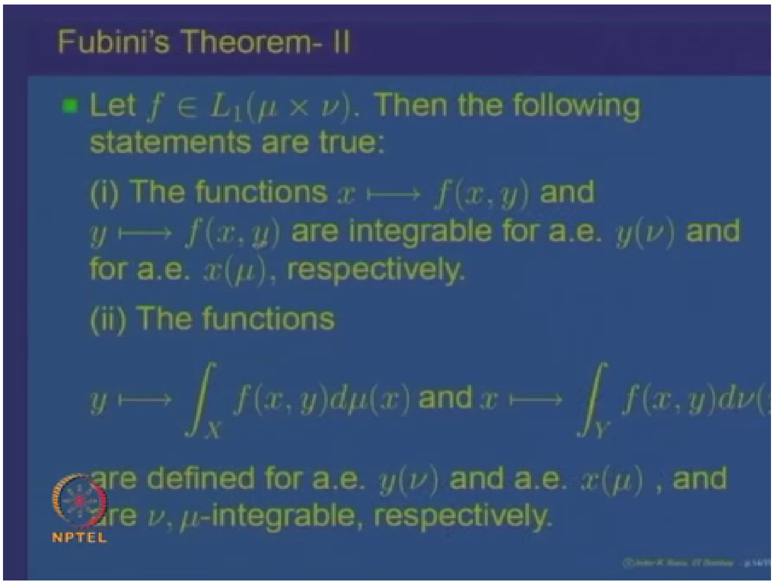

So, that is the first part that these functions are integrable almost everywhere and correspondingly almost everywhere with respect to x and y, and these functions are also defined and are integrable.

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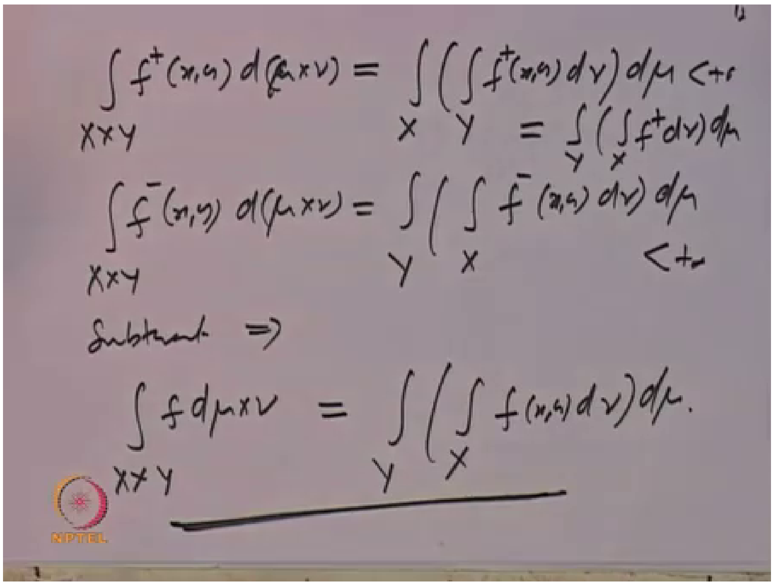

So, that means, we have the following, so they are all integrable and finite and for f plus x y of respect to X cross Y, the mu cross nu, we have got this is equal to is iterated integral with respect to x with respect to y of f plus x, y d nu d mu. And similarly for the negative part we have x, y d mu cross nu X cross Y is equal to integral over y integral over x of f minus x y d nu d mu. So, all these are finite quantities because f plus and everything is integrable. So, these are all finite quantities this is finite and this is finite. So, I can take the difference of the two. So, the difference of the left hand side, so subtract second from the first. So, and use the fact that integrals are linear. So, the difference subtract implies subtraction, and also the corresponding identity is for the other one interchange thing. So, this is also equal to integral over Y integral over X of f plus d nu d mu right for nonnegative that is true.

So, we will subtract this from this. So, we will get integral of f d mu cross nu X cross Y, because integral of f plus minus integral of f minus is integral of f is equal to the iterated integral of f plus with respect to x y minus the iterated integral of f minus with respect to the same iterated integral. So, that will give you y integral of respect to x of f of f plus minus f minus, so that is f of x, y the nu d mu. So, that will prove that the for the integrable function the double integral is equal to iterate integral one of them the other proof is similar.

So, basically this result that for integrable functions the corresponding interchange of integrals hold is basically coming from the previous result namely that the corresponding result holds for nonnegative in simple nonnegative measurable functions. So, what we have proved is to Fubini's theorem-1 and Fubini's theorem-2, Fubini's theorem-1 says that for nonnegative measurable functions the double integral the integral over the product space can be computed by integrating one variable at a time; and similarly this can also be done for functions which are integrable. So, we will continue this Fubini's theorem a bit more and then specialize it for integrals for R 2, R 3 and so on.

Thank you.