Measure & Integration
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Lecture – 27 A
Integration on Product Spaces

If you recall we had started looking at the computation of a product measure of a set E in the product sigma algebra, and we had shown this can be computed via the sections of the set E integrating the sections and taking the measures. So, let us recall this result and then we will continue to generalize this result for functions which are non negative and integrable functions.

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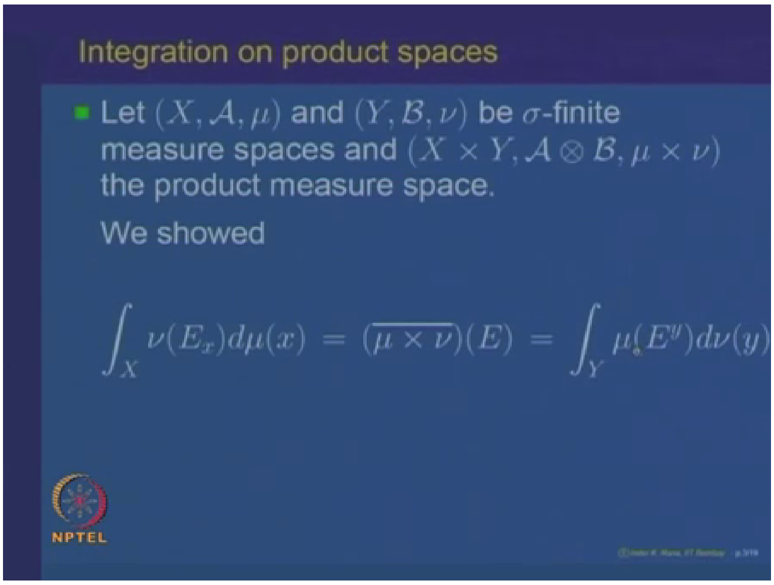

So, let us recall the result that we had proved in a last time namely if XA mu and YB mu are 2 sigma finite measure spaces and the product sigma algebra a product spaces X cross Y, A times B and mu cross nu is a product measure space.

Then we showed that for any set E in the product sigma algebra the measure mu cross nu of E can be computed, by either taking the section of the set E at a point x and that gives as a subset of the set y. And we showed that this is in the sigma algebra you can take the measure of this section. So, this we can say if the function of the variable x and then we can integrate out this function with respect to mu to get the product measure. Or equivalently you can take the y section of the set E that gives you a subset of x which is a measurable set and then you can take the measure of mu measure of that that gives the function of y, and then integrate out that with respect to y to get the measure of the set E.

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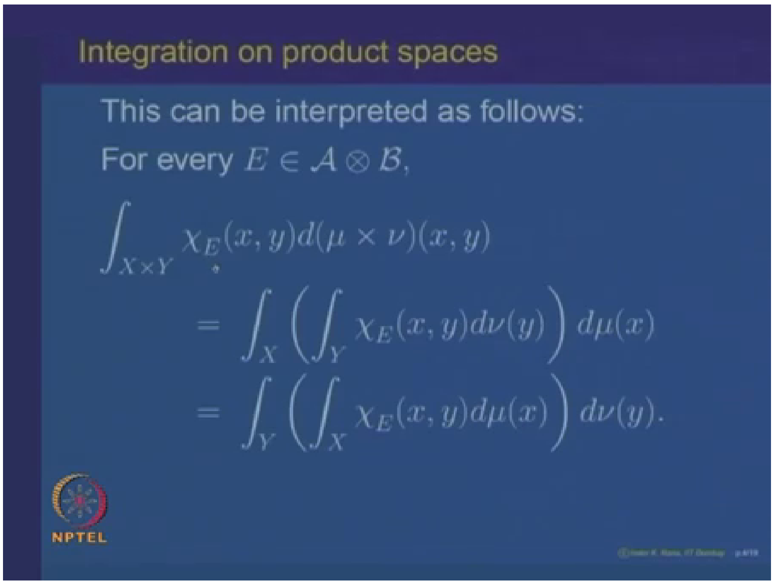

So, this result we want to reinterpret as follows see the measure of the set E x can be written as the integral of the indicator function of the set E with respect to the product measure. So, mu cross nu of the set E is nothing, but the integral of the indicator function of the set E, on one hand on the other hand if you look at the x section or the y sections they are nothing, but the indicator functions of the set E again. So, the when you take the integration with respect to y; that means, you are fixing x. So, you have to looking at the section of E at x. So, nu of ex is nothing, but the integral over y of the indicator function of E with respect to the variable y and similarly the other variable.

So, as we had mentioned the importance of the result this result lies in the fact that, the indicator function of a set E is a function of 2 variables. And to find it is integral with respect to the product measure what we can do is we can fix one of the variable say x, this becomes a function of one variable y. And we one shows that this is integrable with respect to nu. So, when you integrate out with respect to nu the variable y, this is a function of x, which again can be integrated with respect to mu and this integration gives you the integral of the indicator function of e.

So, the importance thing is that here when you integrating with respect to y the variable x is fixed. So, this is only a function of the variable y. So, for every x fix you treated as a function of the variable y, integrate that out. And then integrate out that integral with respect to the other variable. And similarly we can interchange x and y and get the result. So, we what we want to show is today is that this result is true for all non negative measurable functions on the a product space X cross Y.

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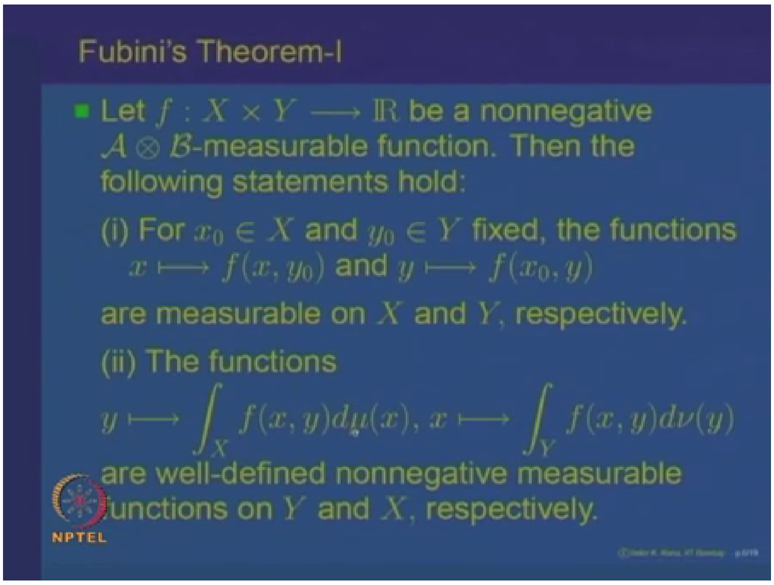

So, the theorem we want to prove is the following namely, if f is a function on the product space X cross Y, and it is a non negative function which is so f is a non negative measurable function on X cross Y is measurable with respect to the product sigma algebra.

Then the following statements hold namely if you fix one of the variables say x. So, if x naught is fixed then consider the function f of x comma y naught. And similarly y goes to f of x naught y. So, in the for the function f of xy either you fix the variable y at y 0, or you fix the variable x at x 0 and treated as a function of one variables only then this functions are measurable on x and y respectably. So, what we are saying is that for a function of 2 variables, if it is a measurable with respect to the product sigma algebra then fixing either of the variables gives you a function of other one variable which is measurable on the corresponding basis with respect to the corresponding sigma algebras and these are non negative functions.

So, you can integrate them out. So, if you integrate this function x going to f of xy naught with respect to mu then that gives you a function which depends on y. So, the function y going to integral over x of fxy d mu x. So, integrate out the variable x this gives you a function of y. And similarly you integrate out fxy with respect to the variable y you get a function with respect to x, so the claim is these 2 are well defined non negative measurable functions on the respective spaces. And finally, these are non negative measurable. So, you can integrate them out with respect to the corresponding variables.

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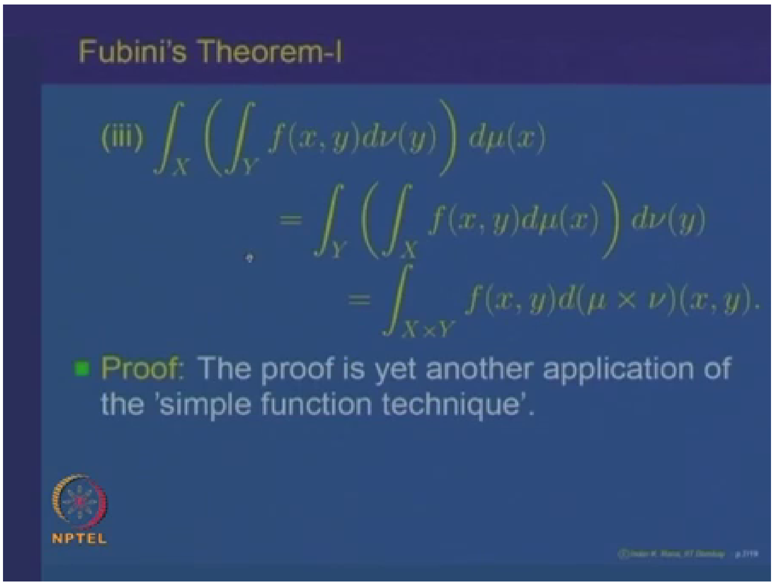

So, if you integrate out then. So, first integrate with respect to y and then with respect to x that is same as integrating first with respect to x, and then with respect to y and both are equal to the integral of the a function f of xy with respect to the product sigma algebra. So, this gives an extinction of the earlier result and so, it says that for non negative measurable functions. If you want to integrate with respect to the product sigma algebra a product measure then you can do it one variable at a time. So, this integrals these 2 integrals are called the iterated integrals. So, the claim is that for a non negative a measurable function the integral with respect to the product measure is equal to the 2 iterative integrals.Once again the importance being you are integrating one variable at a time. So, let us prove this result. So, this proof is going to be built up step by step and this is what I call as the simple function technique. So, the idea is that when f is the indicator function of a set this result is true by the earlier resultant product measures. And then by looking at because all the integral everything involve involves a integrals. So, and integration being a linear operation, we will get that from that this result is true for non negative a simple measurable functions and once it is true for non negative simple measurable functions a application of monotone convergence theorem, we will give us that the result is true for all non negative measurable functions on the product space.

So, that is a approach basically we are going to follow and this is what I call as the simple function technique, when we want to prove something for non negative measurable functions on a measure space verify it for the indicator functions verify it for the a non negative simple measurable functions. And then verify it for the limits of non negative simple measurable functions. So, let us prove this.

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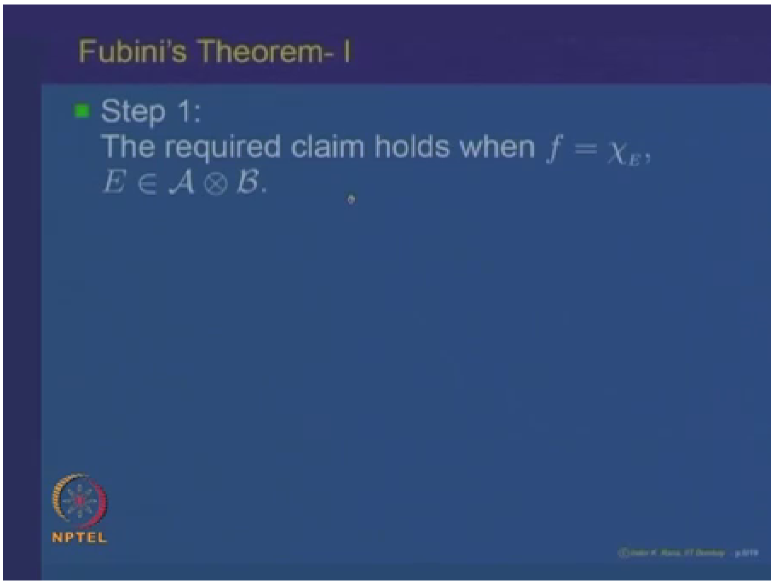

So, the step one is at the required claim holds when f is the indicator function of a product a indicator function of a set E in the product sigma algebra.

So, let us look at that that was what we have already shown then E is so step one.

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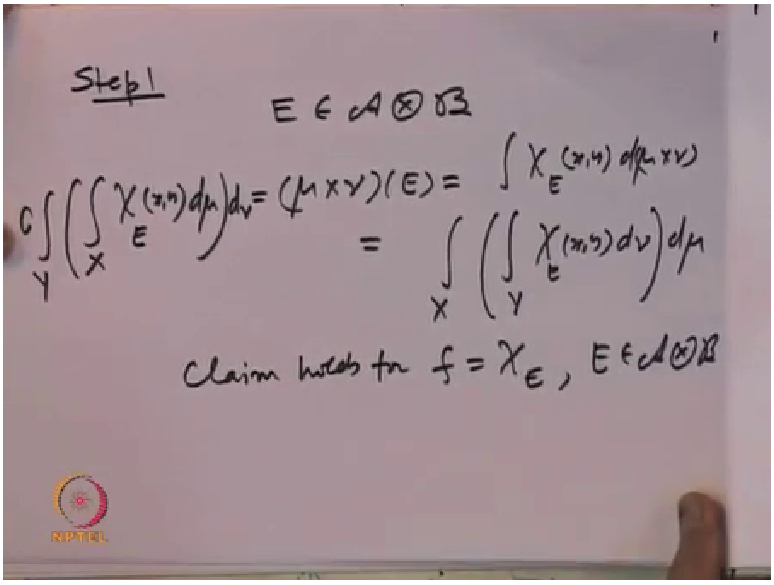

When E is an element in the product sigma algebra, we had already shown that the product measure mu cross nu of E on one hand is equal to you take the indicator function of E and integrate out with respect to x d mu. And then integrate out that with respect to the variable y. So, d nu and that is same as you first integrate out the indicator function xy with respect to d nu; that means, keeping the variable x fix you are integrating with respect to y, and then compute the integral of that with respect to the variable x. So, these 2 are equal and the middle thing if you recall the set it is equal to the indicator function integral of the indicator function of E with respect to the product sigma algebra. So, this is precisely saying that the claim holds for f equal to the indicator function of a set E, E belonging to the product sigma algebra. So, this is step one, and now from here we want to go to step 2. So, let us. So, step 2 is let us take a function. So, step 2.

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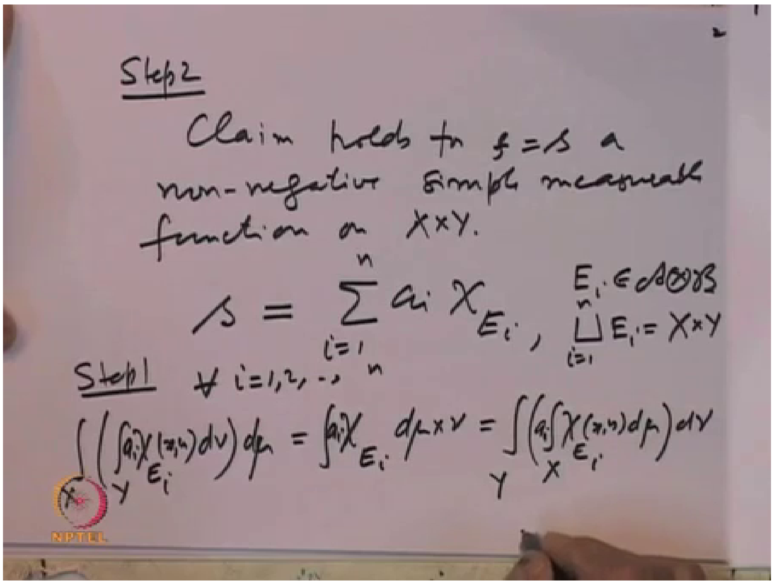

The required claim holds for non negative simple measurable functions for, s f equal to s a non negative simple measurable function on X cross Y.

So, let us take. So, what does a function look like? So, a non negative simple function s on the product space looks like sigma ai, indicator function of some sets ei, where i is one to n and this E i's are sets in the product sigma algebra a times b and the union of eiz pair wise disjoint and their union is equal to X cross Y right and now by step one what does step one say step one says for each E i the claim holds. So, step one says for every i equal to 1 2 and so on, n the integral of the indicator function of E i with respect to d mu cross nu on one hand, it is equal to the integral over x integral over y of indicator function exy d nu and d mu and also equal to the integral over y integral over x of the indicator function of E d mu and then d nu. So, this is what we know.

Now, let us this observe because this is true for every i and integration is linear. So; that means, if I multiply I can multiply throughout by ai. So, if I multiply by ai. So, I can multiply here by ai. So, this is ei. So, if I multiply. So, this is ai and I can multiply by ai and I can multiply here by ai. And then take the summation. So, sum over i then. So, summation over i, summation over i, and here will be summation over I right now this summation integration linear I can take the summation inside. So, when I take the summation inside the integral and then again take it inside, I will get summation of ai times indicator function of E i integration with respect to nu and then integral with respect to mu is equal to this I take it inside that will be summation of ai.

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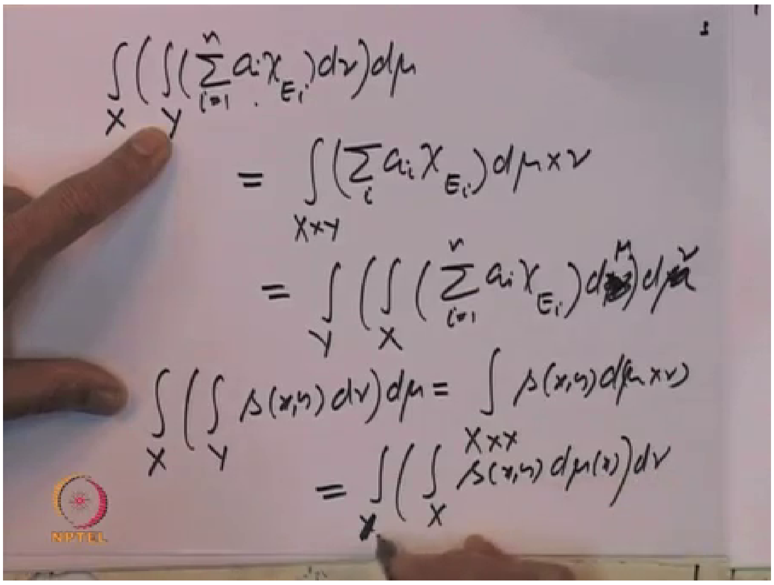

So, let us just write that this is more of writing than understanding. So, when I do this summation and take the summations inside, I will get integral over x integral over y of summation i equal to one to n ai indicator function of E i d nu d mu is equal to. So, that is equal to and this summation inside will give me integral over X cross Y of summation ai indicator function of E i d mu cross nu. And the last one we will give me integral over y integral over x of summation I one to n ai indicator function of E i d mu d nu.

So, this is just using the property that for every indicator function of a set the result is true and integration is linear. So, the result is true for finite linear combinations of this things also. So, this gives us and this is precisely my function s. So, this says integral over x integral over y of sxy d nu d mu is equal to the integral over the product space. So, this is the function s. So, s of xy d mu cross nu and that is also equal to the other related integral over y integral over x of s xy, d mu of x d nu over we have already done it this is over y. So, over y that should be nu actually and this should be mu sorry because this was over this was over x. So, that should be you know that was that was d mu and this is d nu.

So, this is over x. So, d mu and d nu. So, that is so they says that the result holds. So, this proves us second step namely the claim holds for f a non negative simple measurable function.

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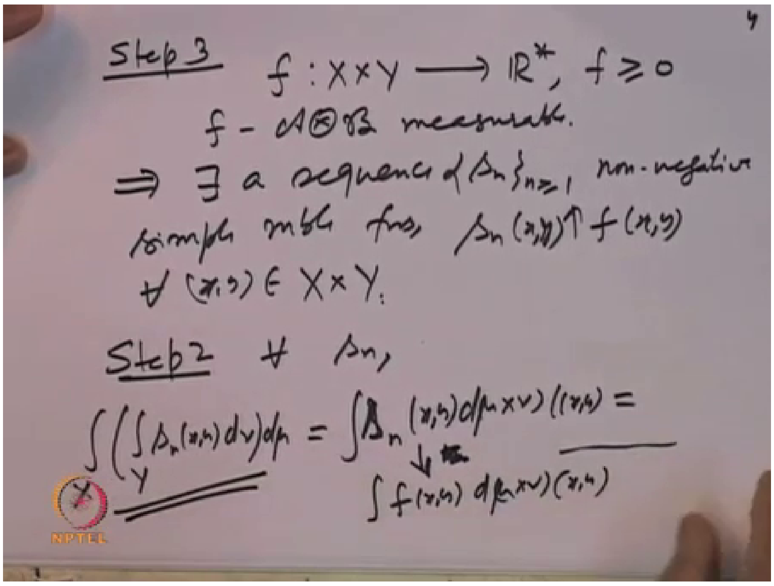

So, now, from here to go to general non negative functions step 3, says we should be able to prove the result when f is so let us take f on X cross Y, to r star f non negative f a times measurable with respect to the product sigma algebra. So, now, we look at the characterization of a non negatives a measurable functions f being non negative measurable implies that there exists a sequence, Sn of non negative of non negative simple measurable functions Sn of x increasing to f of Sn of xy increasing to f of xy for every xy belonging to the product set X cross Y.

So, that is why because of the fact that, this is by the fact that for every non negative measurable function there is a sequence of a non negative simple measurable functions converging to it, now by step 2. So, by step 2 we know that for every Sn the corresponding result holds. So, what does it mean; that means, for every Sn the integral of the non negative function Sn xy, d mu cross nu xy is equal to the iterated integral. So, let us write say for example, integral over x integral over y Sn of xy, d nu d mu and similarly the other iterative integral.

Now, what we are going to do is observe the fact that for every Sn this result is true, and Sn is a non negative sequence of a non negative simple measurable functions increasing to f. So, by the definition of the integral this one converges to the integral of f x y, d mu cross nu of xy. So, this is by the fact of monotone convergence this is not really monotone convergence theorem, this is actually the definition of the integral. So, if f is a non negative measurable function, then it is integral is defined as the limit of any sequence of non negative simple measurable functions increasing to it.

So, that is by the definition on the other hand we will compute this integral and so it is the corresponding iterated integral of the function f. So, let us look at this integral. So, integral over x.

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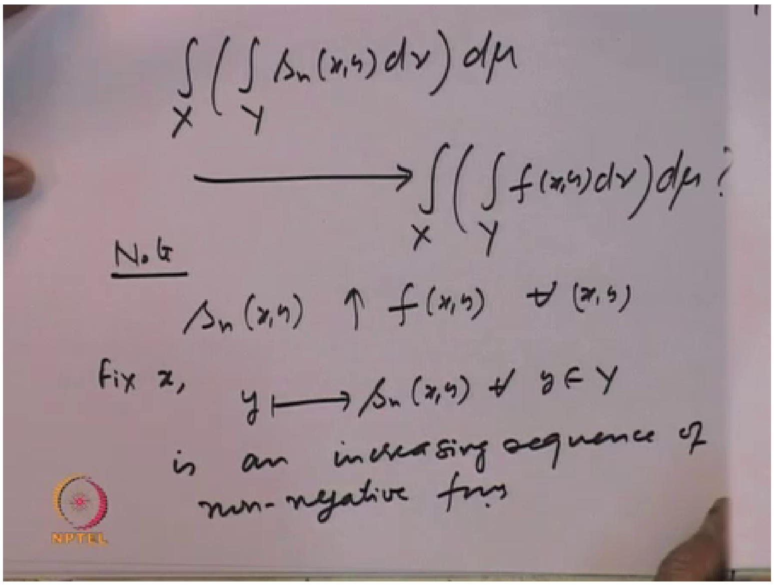

Integral over y Sn of xy d nu d mu. So, this they want to show claim that this converges to integral over x integral over y of f xy d nu d mu. So, this is what we want to show and once that is shown. So, on one hand this converges right. So, this converges to this iterated integral on the other hand this converges to this integral of f. So, these 2 will be equal and we will be through.

So, let us try to prove that this iterated integral converges to this iterated integral. So, for that let us observe first that sn. So, note. So, here is the are the steps for proving this so first of all, let us try to prove that integral over y with respect to nu convergence to corresponding integral. So, for that let us note that Sn of xy is increasing to f of xy for every x and y. So, if you fix if we fix x, then we get a function y going to Sn of xy for every y belonging to y. And because Sn itself is increasing. So, this sequence of functions we already know these are measurable functions, these are measurable functions for every simple function we had seen that if you fix one of the variables the other one is a measurable function for simple functions that is true.

First of all, this is clear that for a fix x, this is an increasing sequence of non negative functions. So, this is a sequence of increasing non negative functions. And each one of them is a measurable function on y and so second observation is that each one of them.

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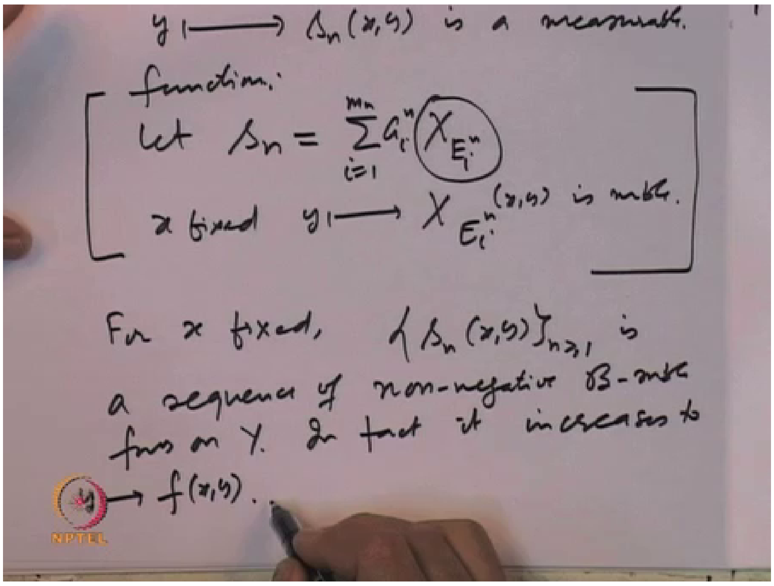

So, the function y going to Sn of xy is a measurable function measurable function. Well that is in a sense obvious because so to see this we note that. So, for this so observation is so let us say sn. So, let Sn be equal to something, so sigma. So, let us say it is a i n indicator function of E i n for some i equal to 1 to some indexing set 1 to mn.

Then for every fix for every fixed x. So, for x fix y going to the indicator function of E i n xy is measurable. So, that we have already observed the that is a measurable function while computing the product measure we saw that this is so each one of them is each one of them is measurable with respect to the product sigma algebra because the indicator function of a set and for every fixed x this will be a measurable function on y. So, that will be the sections. So, that will the measurable scalar multiple of a measurable function is measurable and the sum of measurable functions is measurable.

So, this is a obvious fact that the for a simple measurable function a Sn, if you fix one of the variables in the other variable it will become a measurable function. So, this is a measurable function. So, this so; that means, for x fix. So, what we have gotten is for x fix the sequence Sn, xy over n is a sequence is a sequence of non negative b measurable functions on y. And also it is increasing. In fact, it increases to. So, what does it function it increases because Sn increases to f. So, when we fix one of the variables x this is going to increase to the function f of xy x fixed as a function of y. So, increases to the function y going to f of xy.

So, is a perfect setting for application of monotone convergence theorem. So, by monotone convergence theorem we get the following. So, by monotone convergence theorem we get. So, this applied monotone convergence theorem.

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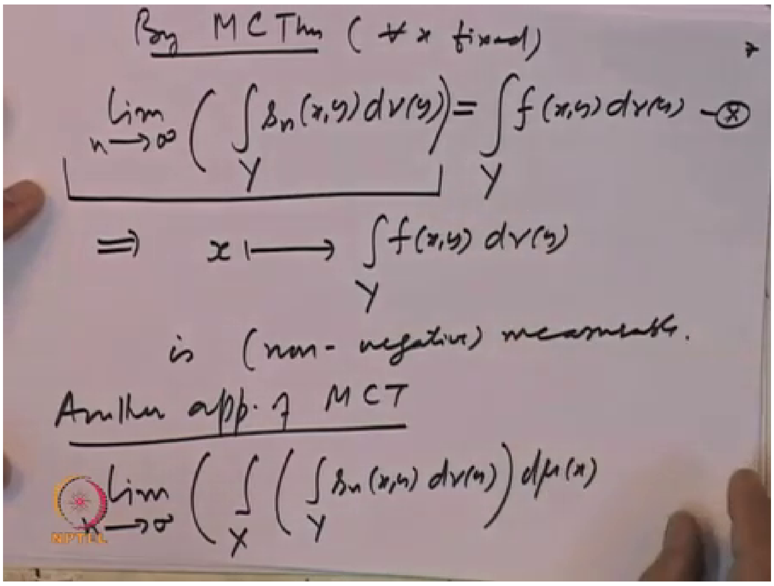

So, by monotone convergence theorem the integrals Sn xy d nu y over y limit of that must be equal to integral of f xy with respect to d nu y. So, this is what we get moves on. Of course, for every x fixed. So, for every x fixed we get that this limit must be equal to this right. So; that means, what; that means, this function. So, this is a function of x. So, that implies that if I look at x going to y f of xy d nu y, if you treat this as a function of x then it is a limit of this functions limit of non negative simple integrals of a non negative simple functions.

So; that means, that this function is measurable. So, that implies. So, this implies that this function is non negative measurable, it is a non negative measurable function. And this is a non negative measurable function and it is a limit of this sequence of non negative measurable functions. So, by star I can apply again another application of again another application of monotone convergence theorem. So, this limit of this on the left hand side the limit of integrals of Sns with respect to nu that also is a limit of measurable functions. So, this itself is a measurable function with respect to x and because Sns are increasing this integral are also increasing. So, this is so if you call. So, this limit is equal to this.

So, now what we are saying is another application of monotone convergence theorem to the fact that the if you look at the sequence of measurable functions this is a measurable function. So, integral of that. So, let us write. So, the function x going to this is a non negative measurable function. So, what is left to you proved you want to integrate this with respect to mu now. So, that is what we are saying that through this we apply monotone convergence theorem. So, we will get that limit of this functions is this function. So, integral limit of the integrals. So, this is limit n going to infinity integrals of this functions. So, integral over x of this functions this functions are Sn xy d nu y d mu x.

So, this limit of this must be equal to a integral of this with respect to x again by monotone convergence theorem. So, let us write that.

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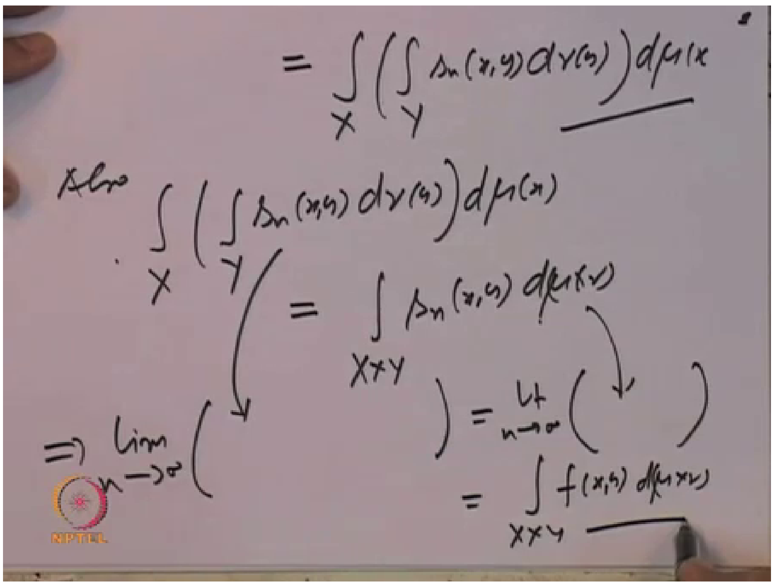

So, this is equal to integral over x integral over y of Sn xy d nu y d mu x. So, this limit is equal to this, but on the other hand we had seen that on the other hand we had seen that, this iterated integral for Sns that is equal to the double integral. So, this is one thing of one observation also. So, let us look at the other fact. So, what is the other fact. So, also we have that integral over x integral over y Sn xy d nu y d mu x. For simple a non negative simple measurable functions the claim holds; that means, this is equal to the double integral the integral over X cross Y of Sn xy with respect to the product measure mu cross nu.

So, this is because we have already proved in step 2 that the result holds for non negative simple measurable functions. And now if I look so this, this result is equal to this so limit of this must be equal to limit of that. So, implies the limit n going to infinity of this left hand side must be equal to limit n going to infinity of the right hand side. So, this one and that is coming here, but limit of the left hand side we have already seen is equal to the limit of the left hand side we have already seen it is equal to this. And what is the limit of the right hand side Sn being a sequence. So, we have already seen that Sn is a sequence of non negative simple functions increasing to f. So, this must be equal to integral X cross Y of f xy d mu cross nu.

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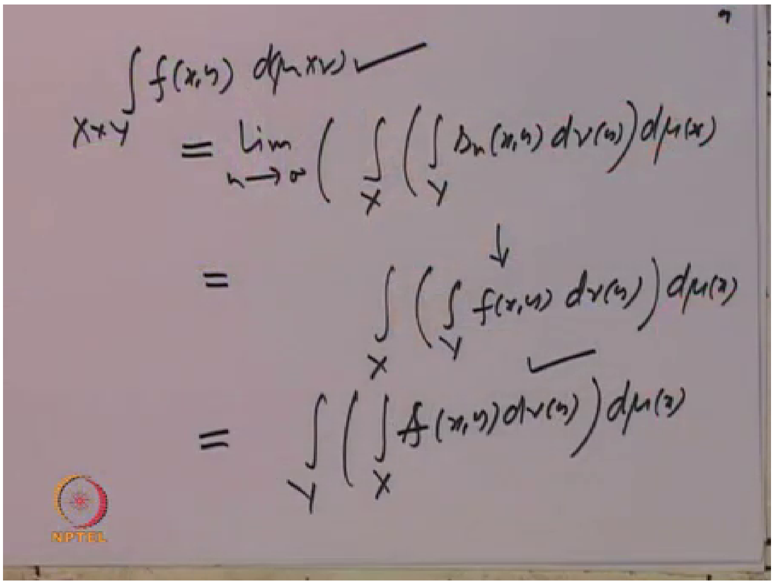

So, that proves that this must be equal to this. So that proves that a step 3 proves that there is the integral of f xy with respect to integral of let us just see here we proved that.

So, what we have shown is this limit must be equal to this, and what was that limit of that quantity. So, what we have shown is that integral of d mu cross nu is equal to over X cross Y is equal to a limit n going to infinity of integral over x integral over y of Sn xy d nu y d mu x. So, this is what we have proved, just now that this this limit or non side was this other side was this so limit of these 2 quantities must be equal.

So, this is what we have proved, but this quantity let us see what is this so note that see Sn for every y fix was increasing. So, let us this look at the sequence for every x fix that is the increasing sequence of non negative measurable functions increasing to the function f of xy. So, monotone convergence theorem says this inner integral convergence to integral of y f of xy d nu y right. That is what we had already observed and then again this is a sequence of non negative simple a non negative measurable functions a application of monotone convergence theorem gives us that integral of this limit of that must be equal to d mu of x.

So, that says that the double integral of the non negative simple function is equal to the iterated integral of the non negative measurable function iterated first respect to nu. And then with respect to mu and we can interchange x and y. So, same arguments you will imply. So, that will say that this is also equal to integral over y integral over x of f of xy d nu y d mu x. So, basically, let us just go through the ideas in the proof that basically this proof is an application of the fact that integral for a non negative simple function is built from the limits of integrals of non negative simple measurable functions. And that fact is used very effectively because we know that the corresponding result is a true for indicator functions and integration is linear.

So, that allows us to say that from the indicator functions you can go to non negative simple measurable functions, by just taking scalar multiplications and a additions of characteristic functions. So, that will give us that the result is true for non negative simple measurable functions. And then just an application sum suitable applications of monotone convergence theorem, we will give us that the integral of a non negative measurable function on the product space can be computed via the iterated integrals.