Measure & Integration
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Lecture – 26 B
Computation of Product Measure – II

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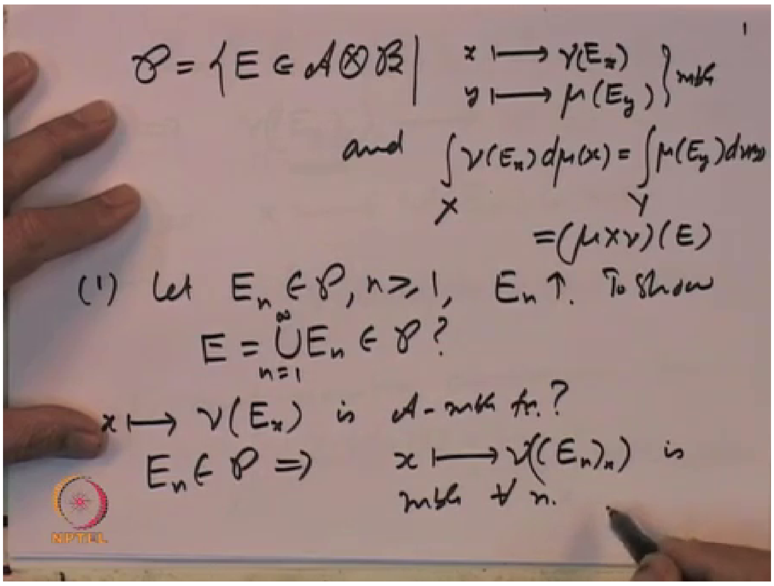

So now we have got E n decreasing. So, because E n belongs; so, what we said first thing was that because E n belongs to p. So, this is a measurable function. So, that is a property of the set E n being in the class p. So, increasing or decreasing is not coming into picture. So, this step will carry over and then if E n is increasing to E.

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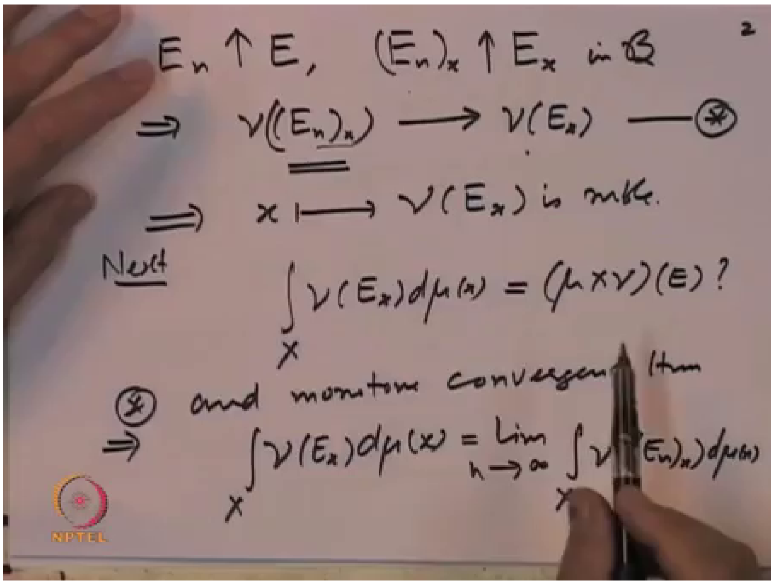

So, now we have got E n is decreasing to E. So, this thing will change. So, if E n are decreasing then of course, it is true that the sections E n x will be decreasing to the set, the section of the set E at x. So, if the sections E n x will decrease through the set Ex. So, that step also will be ok.

Now, we want to say that when E n decrease to E. So, we want to say that here we use a property that, for the increasing case we said whenever a sequence increasing nu of Ens converge. The corresponding result we know it is not true for decreasing sequences. So, here the proof trying to copy the proof, for the increasing thing will fell down because these steps will not this equation star will not hold, to make this star hold, we have to put extra condition that the measures are finite because if measures are finite then E n decreasing to E will imply measures converge.

So, if E n mu and nu are finite say for example, if nu is finite. Then E n sections of each E n that is a decreasing sequence. So, nu of E n will converge. So, to carry over the proof the similar case, we have to put an extra condition. So, for this claim to hold we have to assume that mu and nu are finite.

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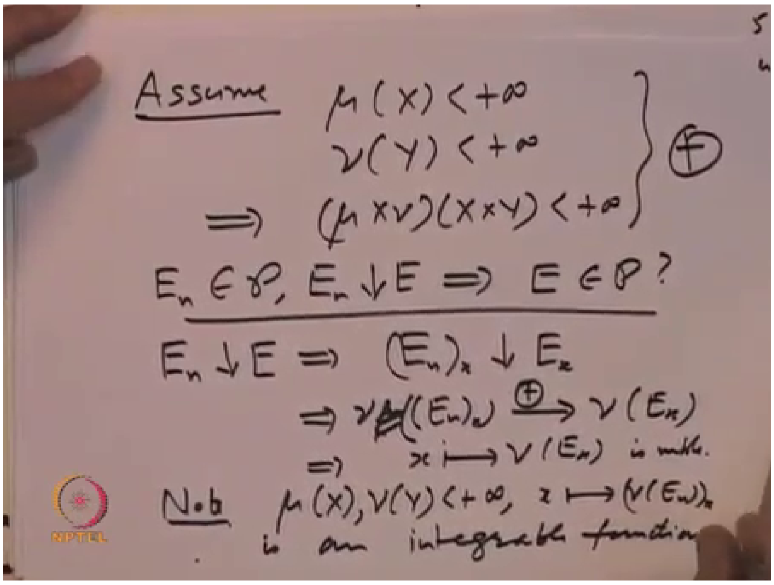

So, let us assume that mu of x is finite and nu of y is finite, of course that implies mu cross nu of x cross y is finite. So, under this conditions we want to show that if E n belongs to p, E n it decrease to E that implies E belongs to p. So, to show that we just now we can repeat the steps.

So, E n let me just go through the proof again for the decreasing case also, to emphasize where exactly we will be using the finiteness condition. So, E n decrease to E. So, that implies that the sections E n x decrease to E of x. So, that implies that mu of E n x oh sorry a nu of E n x, because E n x is a subset of b the subset in b. So, this converges to nu of E x. So, this is the stage we will be using this condition plus. So, under this condition plus that mu and nu are finite, this holds now each E n belongs to p. So, each one of them is a measurable function. So, that will imply that x going to nu of E x is measurable.

So, this is a measurable function and we have got nu of E n x decreases to nu of E of x. So, earlier we use monotone convergence theorem to conclude that nu of ex integral of nu of ex must be limit, but here it is a decreasing sequence. So, we cannot use monotone convergence theorem. Also here, but let us note so here is a observation, note that because mu of a x is finite, nu of y is finite. So, this function mu x going to each of the functions nu of E n x is an integral function. Why is that because it is a decreasing sequence.

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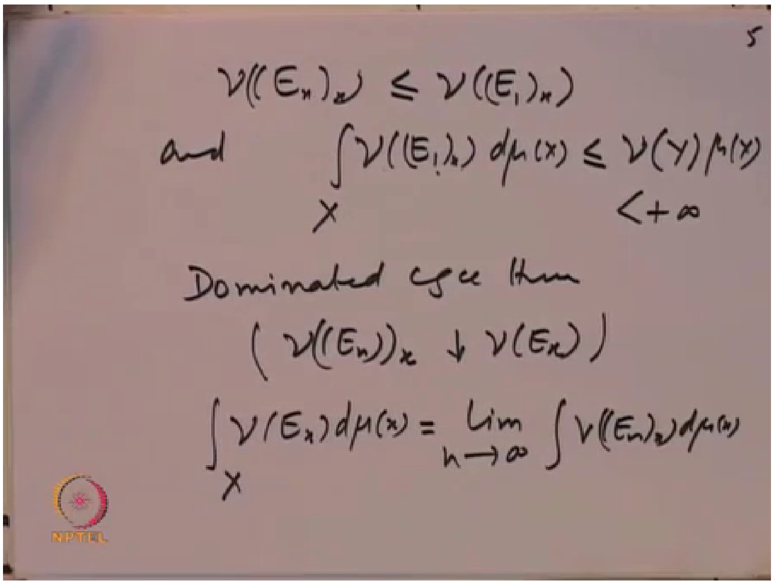

So, let us observe nu of E n x for every n, if I look at this non negative function it is less than or equal to nu of E 1 of x and nu of the section E 1 x integral over x d mu x right, is less than or equal to nu of E 1 E 1 x is less than or equal to nu of y and So this is integrant is less than nu of y. So, integral of 1 d mu x is less than mu of x right. So, which is finite; so, nu of E 1 x is a integral function on the measure space y b nu and each nu of E n x is less than or equal to. So, each nu of nu E n of x is integral.

So, we can apply a dominated convergence theorem. So, dominated convergence theorem applied to the fact that, nu of E n x is a sequence of non negative integral functions and they are decreasing to the integral to the function nu of E x. So, this is also a integral. So, implies by dominated convergence theorem and this observation that the function nu of ex d mu x over x, is this function is integral and its integral is nothing but the limit n going to infinity of integrals nu of E n x d mu x. So, for the decreasing sequence the proof differs in both the steps, first of all when we want to say that E n r are decreasing the sections decrease.

So, the finiteness conditions allow us to say that nu of ex is a limit of these functions and that implies that this is a measurable function. So, finiteness says and this function is measurable because of this fact and in fact, the finiteness conditions says this is a sequence of integral functions decreasing to the function this. So, dominated convergence theorem can be applied and that gives us this limit is equal to. So, nu of E x integration with respect to mu is limit of and now the proof is as before this E n be being in the collection p. So, this integral is nothing, but measure of mu cross nu of the set E n. So, that is limit n going to infinity of measures of the sets E n and once again E n s are decreasing to E and mu cross nu is a finite measure. So, that will imply so this is equal to Mu cross nu of E, again using the fact that mu and nu are finite.

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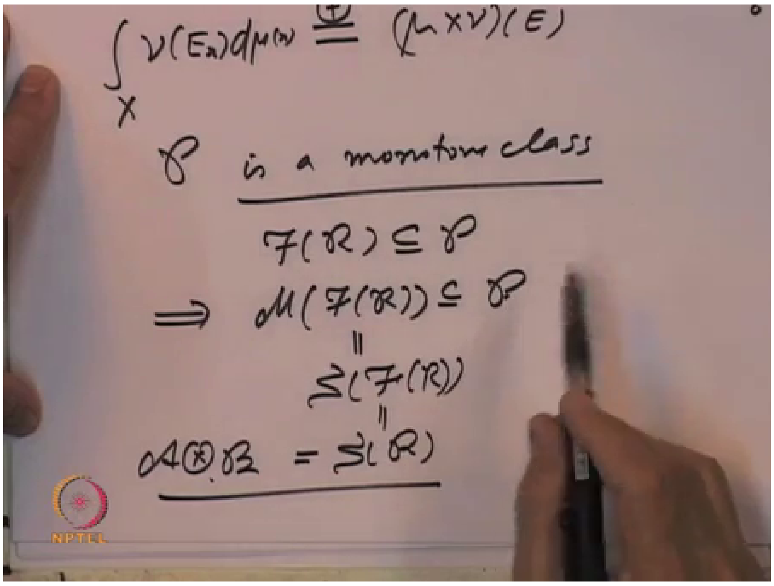

So, we get the conclusion, that again using finiteness condition namely that the integral nu of ex d mu x over x is equal to. So, we have already shown it is the class p is close under increasing sequences, now we have shown it is closed under decreasing sequences. So, p is a monotone class. So, that proves that p is a monotone class. So, as a consequence of the fact that p is a monotone class, the consequence of this would be namely that we already have f of r is inside the class p and p is a monotone class. So, that will imply that the monotone class generated by f of r will also be inside p.

But this is nothing, but the sigma algebra, generated by the class r of rectangles or same as the sigma algebra generated by rectangles and that is same as the product sigma algebra. So, that will prove that the product sigma algebra is equal to p, namely that the required conditions hold for the corresponding.

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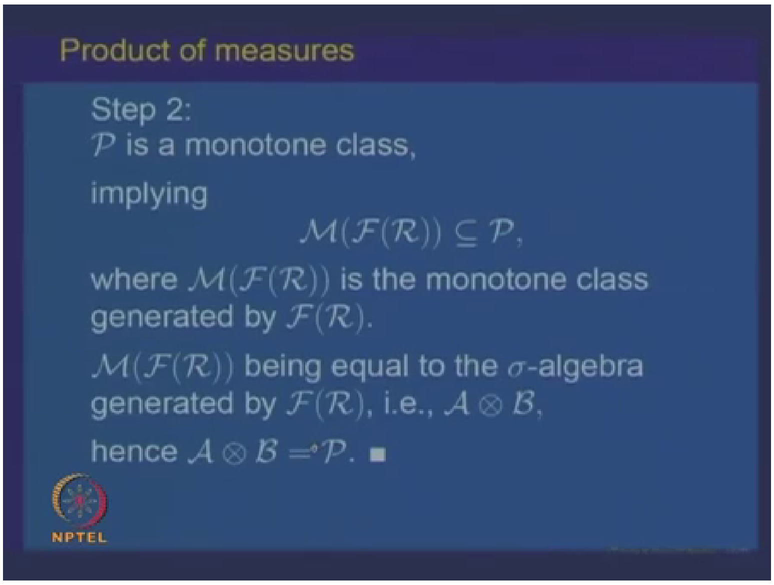
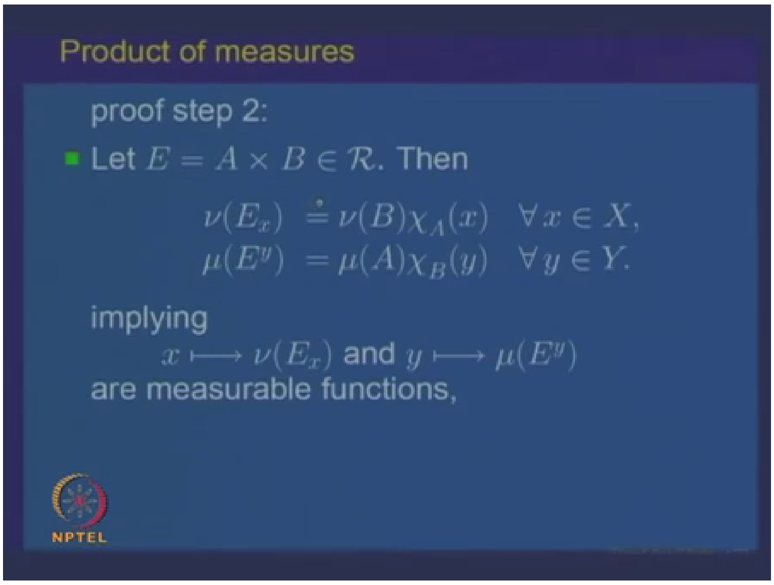


So, that proves a step 2 namely, that p is a monotone class and that implies that the monotone class generated by f of r is inside p and hence that will prove that monotone class generated by f of r is a algebra. So, the monotone class generated by the algebra, is precisely the sigma algebra generated by E. So, A cross B will be inside the class p and hence everything is inside. So, a b cross b is equal to p.

So, this is a theorem where we have used very sensibly, the fact that when mu and nu are finite in that case, we can extend that argument of the increasing to the case of decreasing also. This also illustrates the technique the monotone class sigma algebra technique. So, we have proved the theorem required claim, that p is a monotone class under the conditions mu and nu are a finite.

So, now with the usual arguments one can extend it to the case when it is sigma finite. (Refer Slide Time: 10:38)



So, let us see that, but before doing that let me just go through the proof of the step 2 again, to illustrate the basic facts. So, the first thing we looked at was if E is a product set A cross B. So, I am just revising the proof of step 2 to highlight the important points in the proof. So, A cross B belongs to r is a rectangle then nu of ex. So, that is the first step, the showing that r is the class in r will includes a rectangles. So, there we use the fact that, if you take a set which is a rectangle. Then intersection is nothing, but either the set A or the set B or the empty set according to the point x or y.

So, nu of ex is thing, but nu of b times the indicator function of x because if x does not belong to a then this is 0 and the section is just b and similarly mu of E y is mu of a times the indicator function of b. So, these 2 facts prove that, x going to nu of E x and y going to nu E y for rectangles are measurable functions and if we integrate because this.

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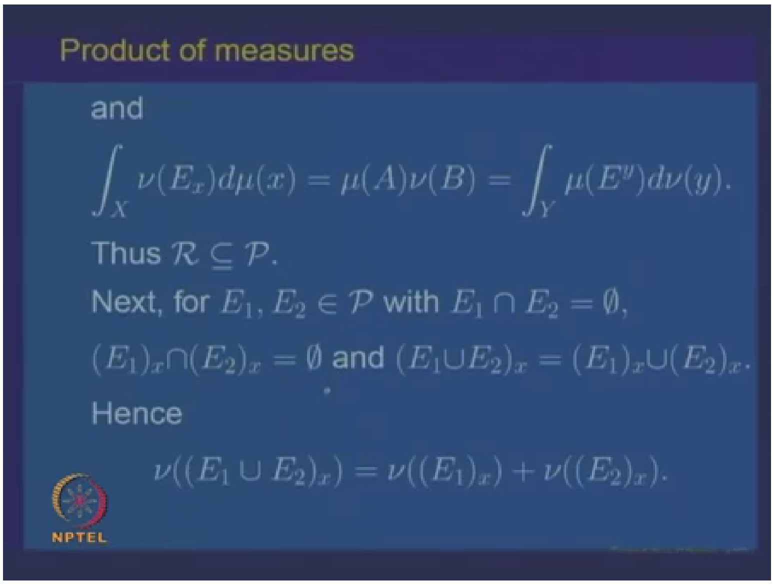

So, integral of nu will be equal to nu of B into nu of A. So, that is the product measure of the product set A cross B. So, that says the rectangles are inside. So, that is the straight forward argument, which says rectangles comes inside a p. Showing that p is a close under finite disjoined unions is also a straight forward because that follows from the fact that, if E 1 and E 2 are 2 sets in the class p which are disjoined, then the sections are disjoined of these 2 sets.

So, and the sections of the union is equal to union of the sections. So, as a consequence of this the nu of the section of the union. So, E 1 union E 2 section at x nu of that is addition, nu of E 1 x plus nu of E 2 x because the sections are disjoint and E 1 and E 2 both belong to p, imply these 2 are measurable functions and hence the sum of measurable functions is measurable. So, this becomes measurable. So, that is a straight forward proof of the fact that if E 1 and E 2 belong to p, then even intersection and their disjoint, then the union also belongs to p and finally look at the integral.

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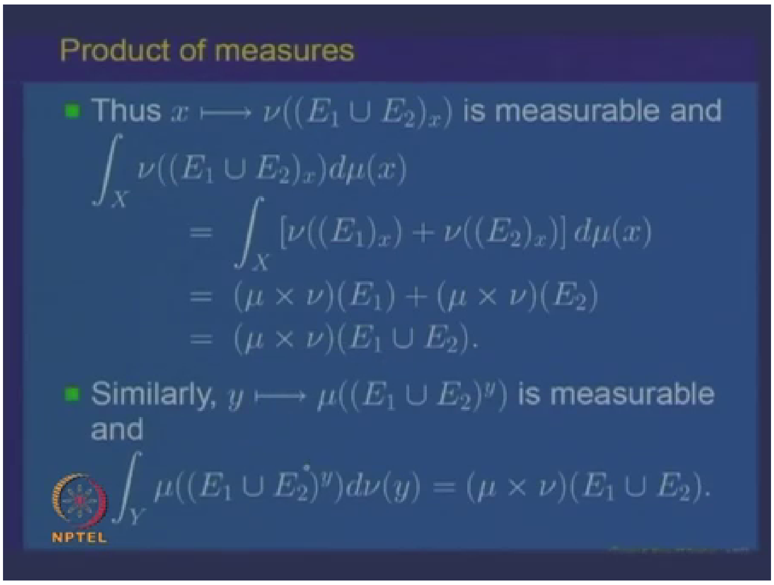

So, integral of nu of the section of the union because that splits into 2 parts. So, nu of E 1 union E 2 is nu of E 1 x plus nu of E 2 x with respect to mu.

So, the internal splits into 2 parts. So, that is mu cross nu of E 1 because E 1 belongs to p and this is mu cross nu of E 2 because E 2 belongs to p and now using the fact that mu cross nu is a measure that gives us this equal to mu cross nu of E 1 union E 2 and the similar thing will work for the y sections. So, proving that rectangles are inside the class p and p is closed under finite disjoint unions, is rather straightforward computation.

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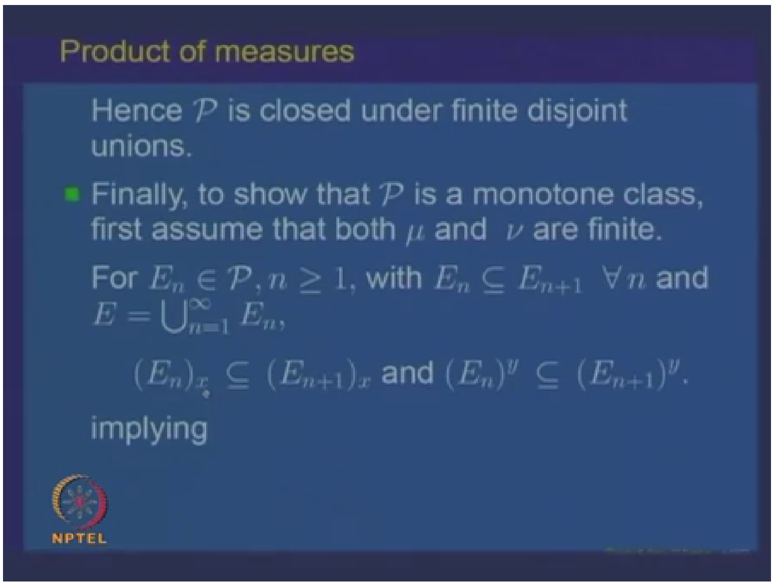

The problem arises, when we want to show that p is a monotone class. So, there we first assume that mu and nu are finite. So, once mu and nu are finite, we want to show it is closed under increasing union and decreasing intersections.

So, take a sequence of sets E n which is increasing. So, a simple fact that if E n are increasing the sections are increasing, mu and nu being measures imply mu of the sections E n we will converge to mu of E.

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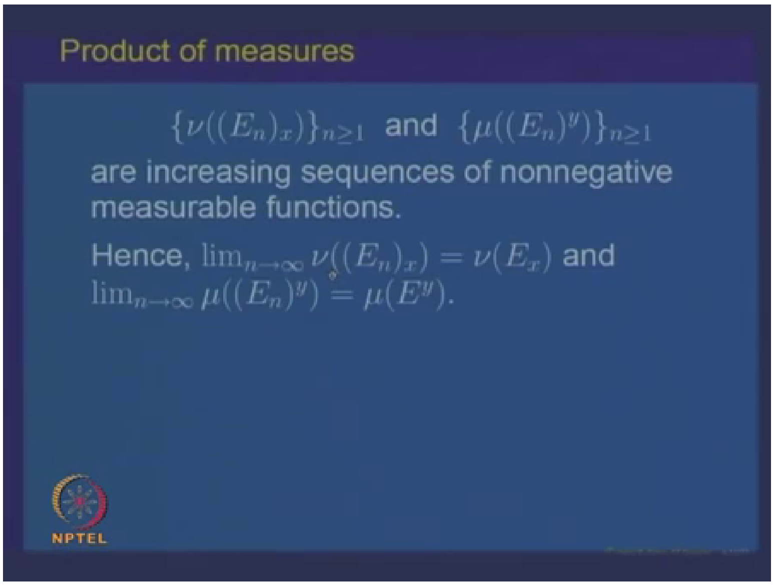

So, mu of E x and nu of E y are limits of measurable functions. So, they become a measurable. So, straight forward till now no finiteness condition has been used. So, this is true whenever mu and nu are any 2 measures but for the decreasing part, where we will need the finiteness condition. So, for the increasing part everything goes straight by a monotone convergence theorem, application gives you nu of E x is limit of that and that is equal to the product measure and everything is right.

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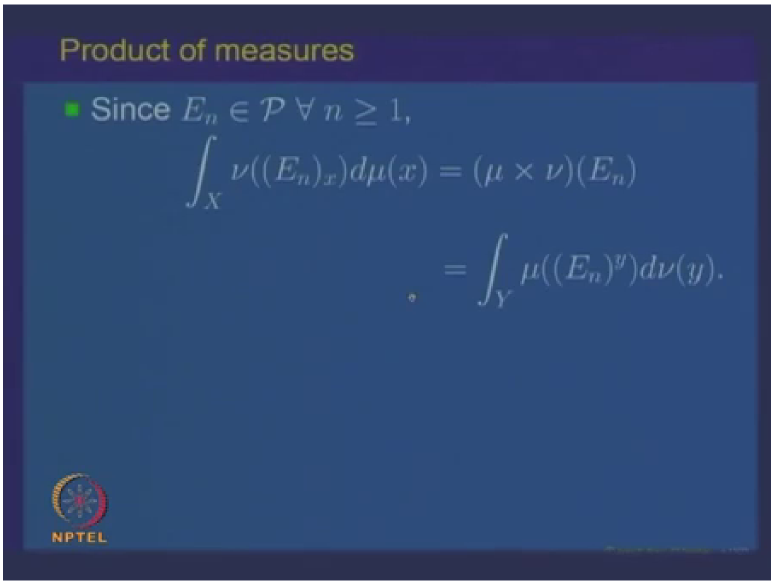

So, let us look at the part where we find the difficulty arises.

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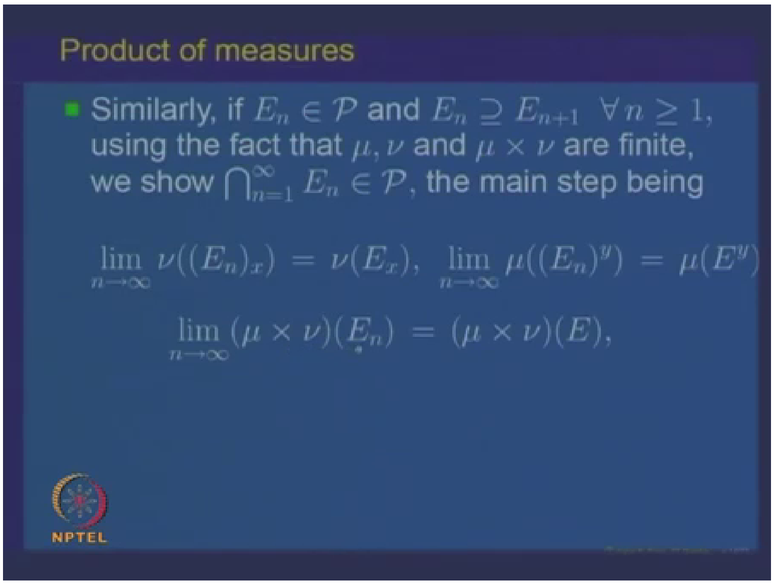

So, difficulty arises, when we want to show that if E n belongs to p and E n are decreasing, then the set E which is the intersection of E n also belongs to p. So, here the main step is to conclude that, nu of E n x is equal to nu of ex. So, for that we need finiteness condition because whenever sets a sequence of sets is decreasing to a set, then measure of the sets need not converge to measure of the limiting set unless the measures are finite. So, finiteness condition will give us that and then instead of monotone convergence theorem, we can apply the dominated convergence theorem to conclude that mu cross nu of E n is equal to corresponding integral.

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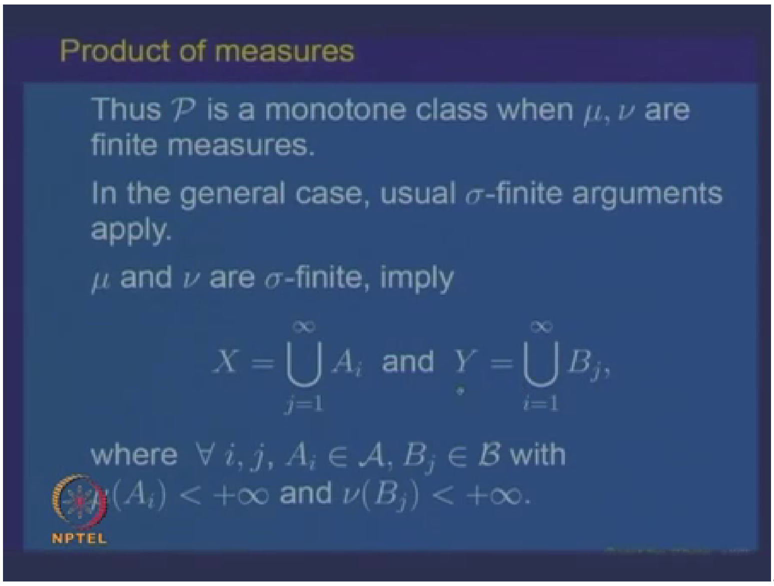

So, that will prove that mu and nu being finite p is a monotone class, but still we are not concluded the proof for the general case. So, for the case one can apply the usual sigma finiteness criteria, namely whenever 2 measure the sigma finite the whole space can be cut up into finite number, countable a disjoined pieces each of finite measure and on each the result holds. So, put them together to get the result holds for the whole space. So, let us see the argument how it works because mu and nu are sigma finite. So, x can be decomposed into a disjoined union of sets A i and y can be decomposed into a union of sets B j. Such that a disjoint union such that, mu of each A i is finite and nu of each B j is finite.

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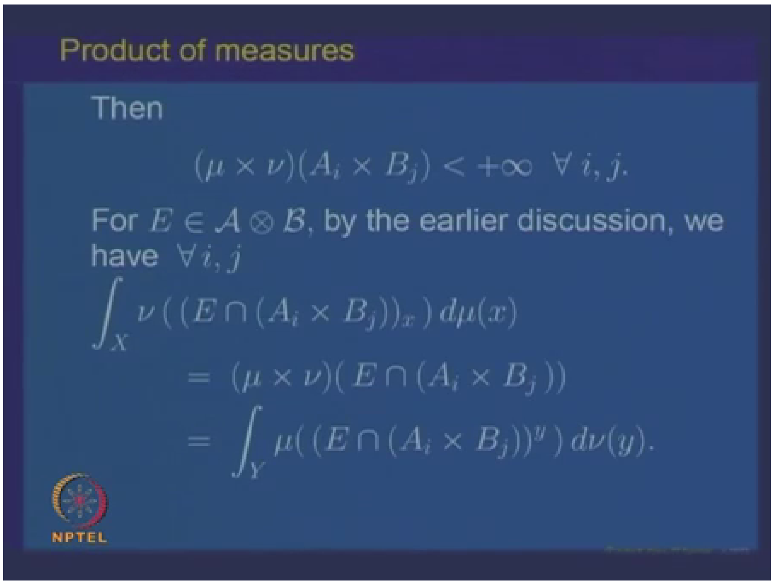

So, using that we can write down that mu cross nu of A i cross B j is finite because this is nothing, but mu of A i times nu of B j. So, as a consequence on each of these pieces our earlier results was hold the p was a monotone class. So, let us see how that is used to prove for a general set E in A cross B, for a set in the sigma algebra A cross B. Note that the integral of the measure nu of E intersection A i cross B j x d mu x because each nu of each of the sets has got a finite measure right.

So, we are applying the earlier result on the piece A I times B j. So, for every I and j using the earlier case, we have that the integral over x of the x sections of E intersected A i cross B j is nothing, but mu cross nu of E intersection A i cross B j and that is equal to the mu integral of the y sections of the corresponding sets. So, this step follows basically from the fact that mu cross nu of A i intersection B j is finite and for any set E. E intersection this rectangle A i cross B j, on that rectangle a mu and nu are finite. So, this earlier case gives us the result and now we have to only sum both sides with respect to I and j.

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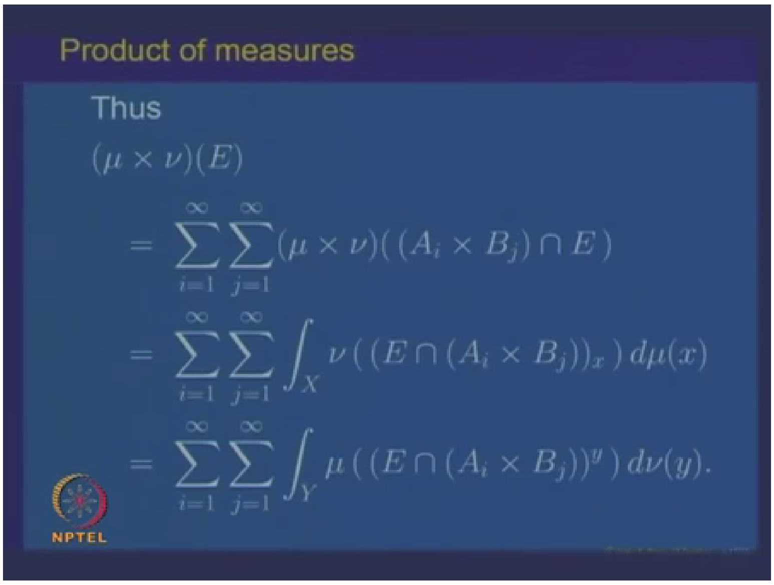

So, let us look at mu of mu cross nu of E, is equal to because the whole space is equal to a mu in our I and j, of the rectangles A i cross B j that is a partition. So, mu cross nu of E can be written as, using countable additively of the measure mu cross nu. As summation over I summation over j mu cross nu of the pieces A i times B j and now for each one of this piece we know the result holds.

So, I can write this as a integral of the x sections or as integrals of the y section. So, this term mu cross nu of A i cross B j intersection E, is equal to this integral or this integral because of the fact that for the finite case the result holds and now using the fact that if you look at the section E intersection A i cross B j of x this section is nothing, but mu of ex times Ai cross.

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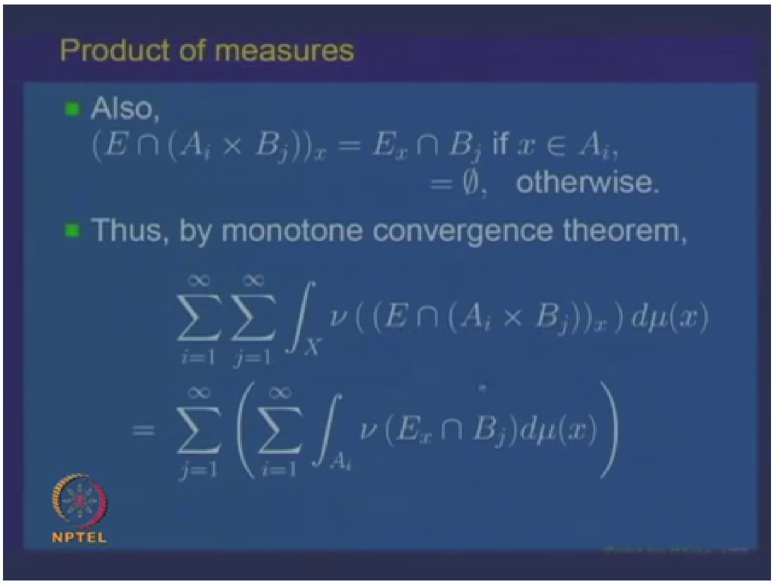

So, this is a the small observation, that you look at set E and take its piece inside the rectangle A i cross B j and take its section ok.

So, this section is going to be equal to the section of E intersection with B j. Of course, if x belongs to E j and the facts does not belong to E j then there is not going to be any intersection. So, this is going to be empty set. So, this is a observation and that observation can be used in this part that, if x does not belong to A i then this thing is going to be 0. So, using that we can write that sum. So, this sum which was integral over x of E intersection this can be written as. So, this set is nothing, but nu of E x intersection B j because that is the only place where the section appears when x belongs to A i. So, this is integral over A i of mu ex intersection B j.

So, this integral is equal to this because of this fact and now the summation over I means that this integral is over x.

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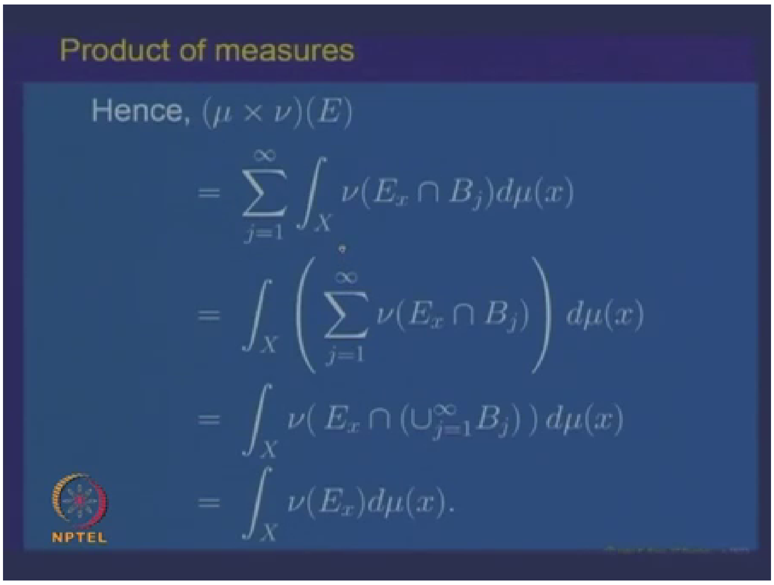

So, this summation you can transform into integral over x and now you can interchange the 2 integral and the summation again. You will be using fact here that this is a integral which depends on j. So, you can push it out and take it inside, basically you will be applying implicitly a monotone convergence theorem to say that this is equal to I. Can take the integral sign x right and because this is a sequence of a functions which are non negative measurable and So on.

So, here is an application of monotone convergence theorem, which helps you to interchange summation and the integral sign. So, summation goes inside and now summations over B j are disjoined. So, that gives you over the whole space y. So, that is just E x. So, you get that mu cross nu of E is equal to the integral of the section nu of ex d mu x. So, you see that almost every step, we are using sum theorem at the other to justify the facts. So, this is the case for the x sections and the similar result will hold for a y sections.

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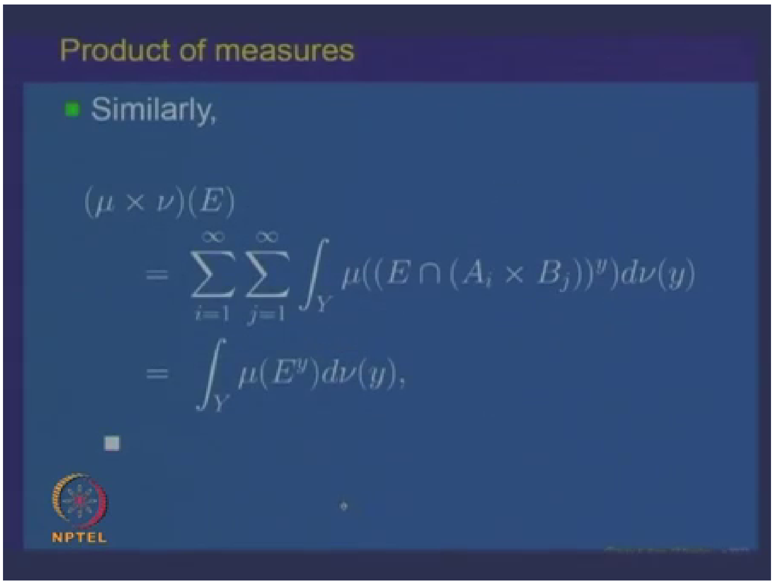

So, that will prove that mu cross nu is also equal to integral over the y sections and that will complete the proof of the fact that, one can reduce the result in the case of a sigma finite. So, from finite to sigma finite is a almost a straight forward, in the sense that we split the whole space into countable number of pieces a finite measure.

So, on each piece we apply and then sum it up to go back to the original piece. So, we have proved the theorem namely, a how to compute the measure of a product Set.

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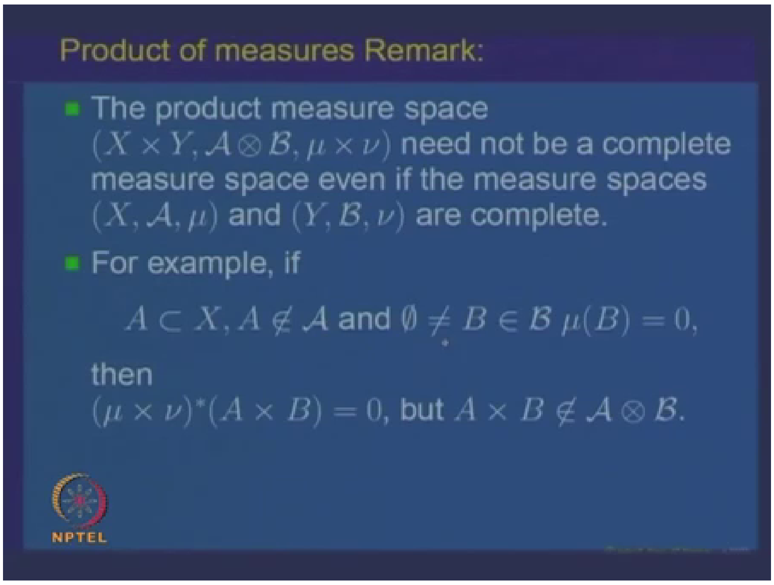

So, let us a observe one thing here namely, even if we start with measure spaces x A mu and y B nu to be complete. The product measure space which we have denoting by x cross y, A times B mu cross nu need not be complete. Because how do you do we get this measure mu cross nu on a cross B. we looked at the product mu cross nu on rectangles and extended it and defined the outer measure via that and then looked at the measurable sets mu cross nu and that included this sigma algebra.

So, this A times B the product sigma algebra is not the sigma algebra with respect to which of all mu cross nu measurable sets. So, it may not be complete. So, for example, you can take any set A in x such that a does not belong to the algebra A and take any non empty set B of measure 0, then the outer measure of mu cross nu will be equal to 0 because mu of b is equal to zero. But the rectangle a cross b does not belong to product sigma algebra because A does not belong to A. So, in case one wants to look at the completion of this so, that is possible.

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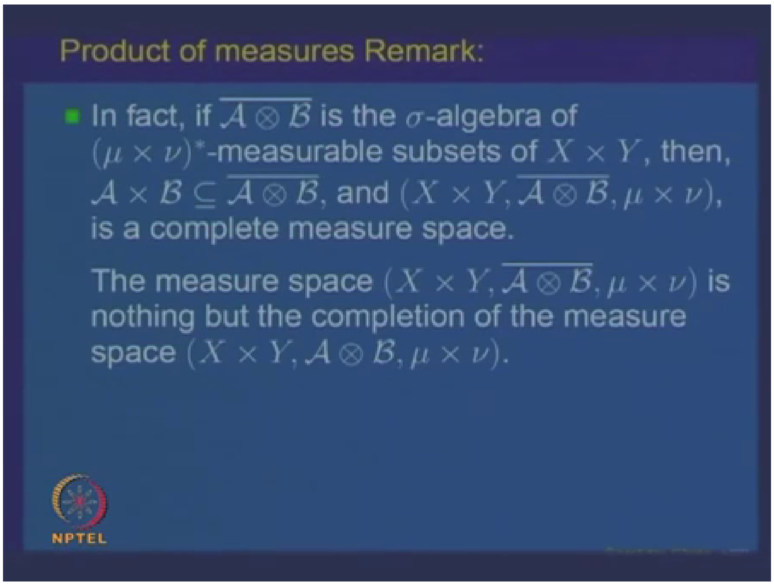

So, if you look at the sigma algebra A times B bar and denote that to be the sigma algebra mu cross nu, measurable subsets the product space. Then of course, the product sigma algebra is inside it and that will be a complete measure space. So, you can say that x cross y and mu cross nu measurable sets, as before is the completion of product measure space x cross y A times B, mu cross nu. So, this is just a small observation which we should keep in mind that, the product sigma algebra which is a sigma algebra generated by the rectangles need not be giving you a complete a measure space.

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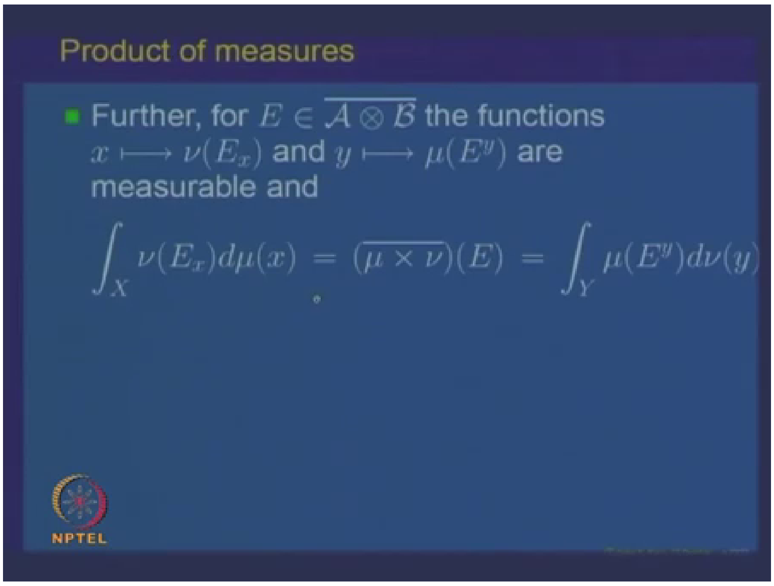

However, one can always complete it and the corresponding result holds for sets in A times B that is a small technical result. Which one can prove that, we had proved this result for sets in the product sigma algebra namely you can integrate the sections and get back the product measure. So, this also applies to any set E in the product sigma algebra; that means, in the completion space also the corresponding a result holds. So, this is the way we can compute the product measure of a set in the sigma algebra. I want to go about to an interpretation of this result, which leads to a very important a result in integration of product spaces. So, what we had was the result namely.

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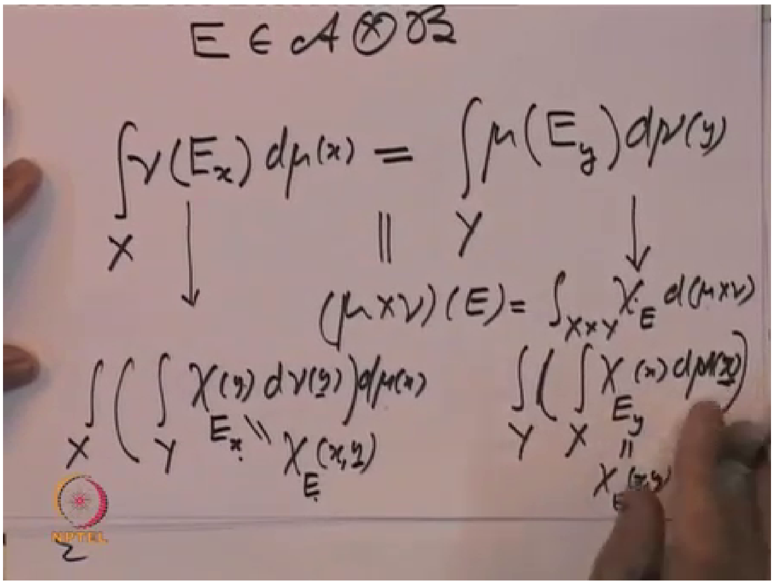

So, what we had shown is for every set E, in the product sigma algebra A times B we can take its section with respect to every point x that gives us a set in the sigma algebra B. So, we can define nu of that and that becomes we show it is a non negative measurable function. So, I can integrate this over x with respect to mu, on the other hand I can also take the section of E with respect to every point y and then take gets measure. We showed that this sections belong to A, take it is measure mu of E y and we showed that that is a non negative measurable function and I can integrate it over y d mu of y right and we showed that, these 2 are equal. In fact, both of them are equal to the product mu cross nu of E.

But a simple observation that the measure of a set is the integral of the indicator functions. So, what is this, I can write it as integral over x this mu of E x, I can write it as integral over y of the indicator function of E x y, d nu y and similarly this thing. I can write it as integral over y mu of E y. So, that way I can write as integral of over x of the indicator function of E y x d nu of y and then we should have d mu of x. So, integral sorry this is E y. So, this is d mu of y. So, this is E y. So, there should be d mu of x and then d nu of y ok.

And this product thing, I can write as integral over x cross y of the indicator function of E d, the product measure mu cross nu. So, we get a integral representation of this result namely that, I can take the indicator function of the set E. So, but note that this function the indicator function of ex y is nothing but, see this is nonzero when y belongs to E x; that means, x comma y belongs to E. So, this is just the indicator function of E x comma

y. So, and similarly this is also the indicator function of a E x y. So, everywhere it is indicator function of E. So, what we are saying is look at the indicator function of the set E and integrated with respect to y. So, keep x fix and integrate with respect to y, that depends on x integrate with respect to x or take the indicator function of E, then integrate with respect to x. So, keep y fix. So, that integral depends on y and integrate it over y. So, that is another number that you will get.

And it says both of them are equal to integral of the indicator function of the set E with respect to the product measure mu cross nu. So, let me just rewrite and show it to you in the form in the slide.

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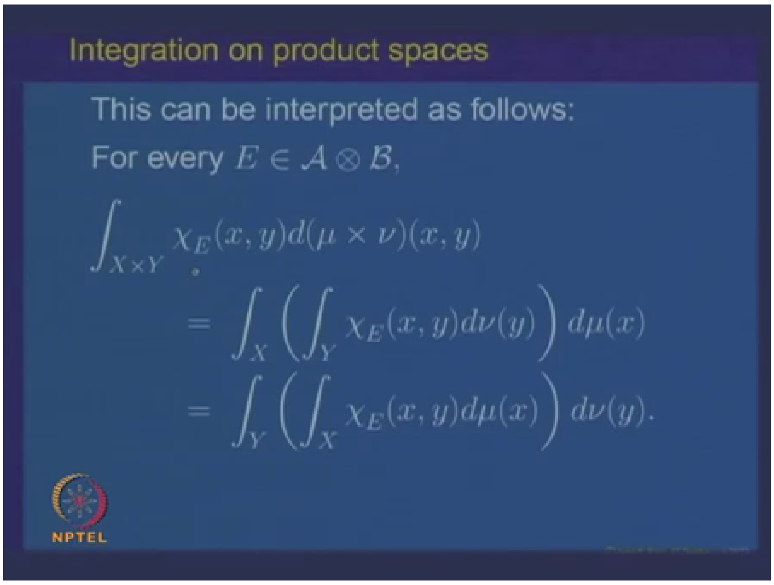

So, what we are saying is the result that we proved just now, for every set E in the product sigma algebra A cross B. I can rewrite the result in the form of integrals that namely, it is same as saying that the integral of the indicator function of E with respect to the product measure mu cross nu. Is same as look at the indicator function it is a function of 2 variables, for a this function of 2 variables I can fix an x if I fix an x and vary only y then this indicator function becomes a function of one variable y. So, it says let me integrate this function, indicator function of E for a fixed x with respect to y. So, this integral is can be computed it and this integral depends on x and says that is a measurable function and its integral can be taken with respect to x with respect to the measure mu and that is same as that integral.

And similarly instead of fixing the first variable x, I can fix the second variable as y I can fix this as y. So, then this becomes a function of x, I can integrate it with respect to x. I get a number which depends upon y and that function is integral with respect to y and that integral is also equal to the original one. So, the result of computation of a product measure of a set E in the set A cross B can be written in terms of the integrals, of indicator function over the product set. So, basically this illustrates that to integrate the indicator function which is a function of 2 variables, I can integrate as 1 variable at a time.

So, this is an important result which is leads to an important result in integration that given a function of 2 variables, if you want to integrate it with respect to the product measure then this gives a hint then possibly what one can do is fix 1 variable of the 2 variable function. So, it becomes a function of 1 variable integrate it out the 1 variable and then it becomes a function of the other variable, integrate out that variable also you get the integral with respect to the product measure

So, we will prove this in the next lecture. Namely that this result can be extended to non negative measurable functions on product spaces and eventually, it can be extended to integral functions. So, that leads to the important theorems in the theory of integration on product spaces called fubinis theorems. So, we will continue looking at that in the next lecture.

Thank you.