**Measure & Integration**
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**Lecture – 26 A**
**Computation of Product Measure – II**

In the previous lecture we had started looking at how to compute product measure of a set in the product sigma algebra. We had shown part of theorem and we will continue looking at the proof of that theorem in this lecture. So, let us just recall what we have been doing.

So, we were looking at computing the product measure, so we will continue that study today.

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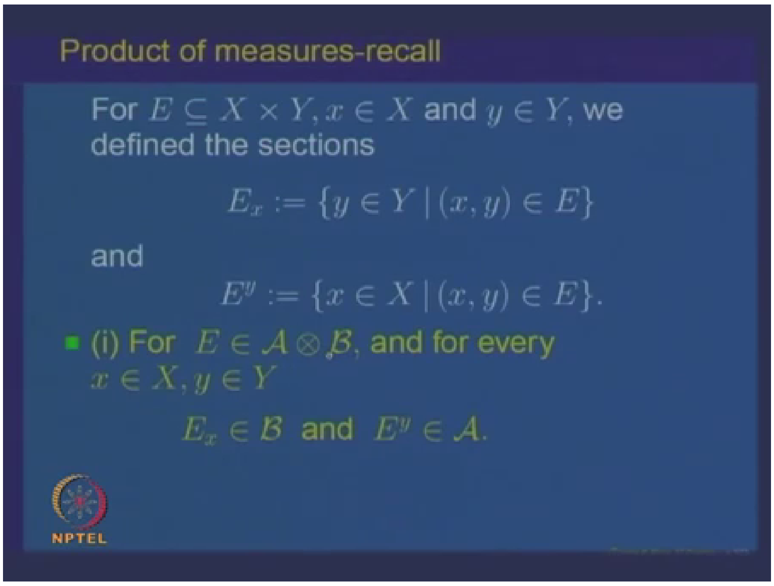


So, let us just recall the settings we have a sub set E contained in the product set Z cross Y, and for any element x in X and y in Y. We defined what is called the x section E x and E y in the previous lectures. And then we claimed that for every set E in the product sigma algebra set sections E x is a element of the sigma algebra B, and the section at y is an element in the sigma algebra A. So, this we had proved. So, I am just recalling them.

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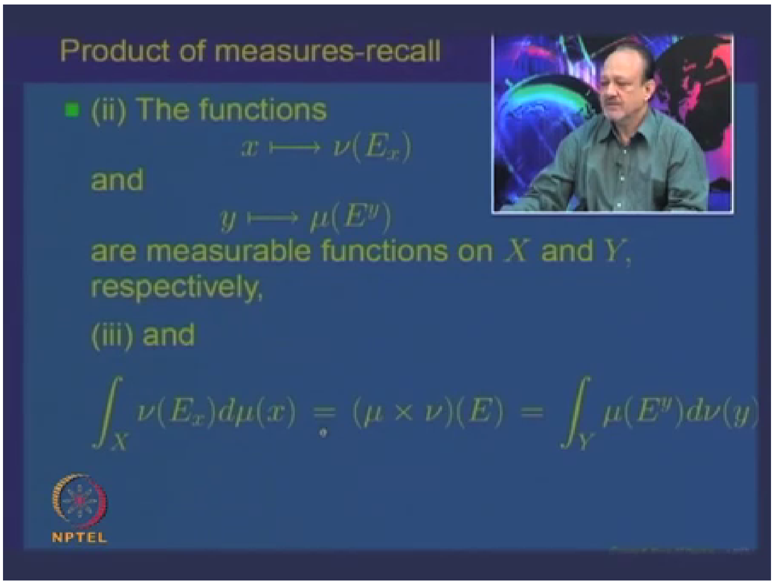


And then we proved that the functions x going to nu measure of E x ex is a subset of is an element in the sigma algebra B, and nu is measure define there. So, we can compute what is a nu of E x, and the claim is that the function for every x the image being nu of E x this is a function defined on x and the claim it is a measurable. And similarly function y going to the measure of the y section is a measurable function on the set y with respect to the sigma algebra B.

So, this 2 we have proved and we wanted to prove finally, the third one that if we integrate these functions with respect to mu, and with respect to nu these are non negative measurable functions and we can integrate them. So, the claim is that the integral nu E x d mu x is same as the product measure mu cross nu of E and it is same as the integral of the y section with respect to y. So, this is the step we were trying to proved in the previous lecture.

So, to prove this what we said let us look at the class of those sub sets E in the product sigma algebra for which this is true.

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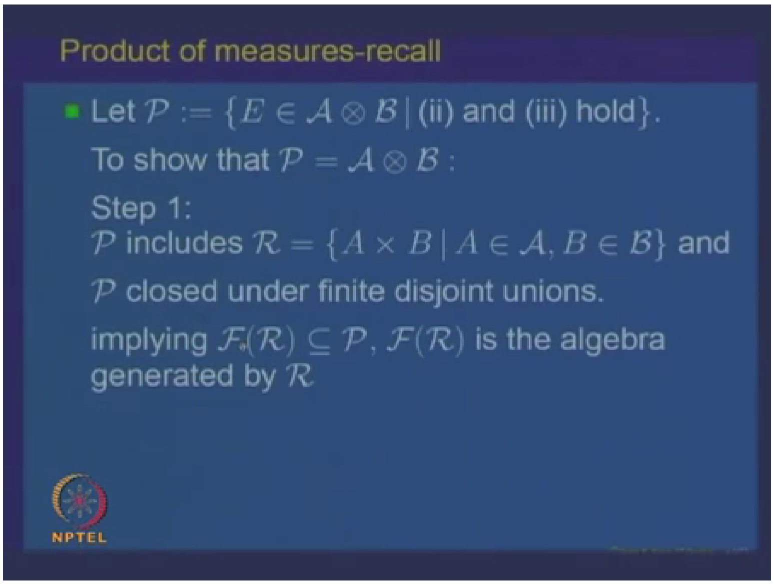


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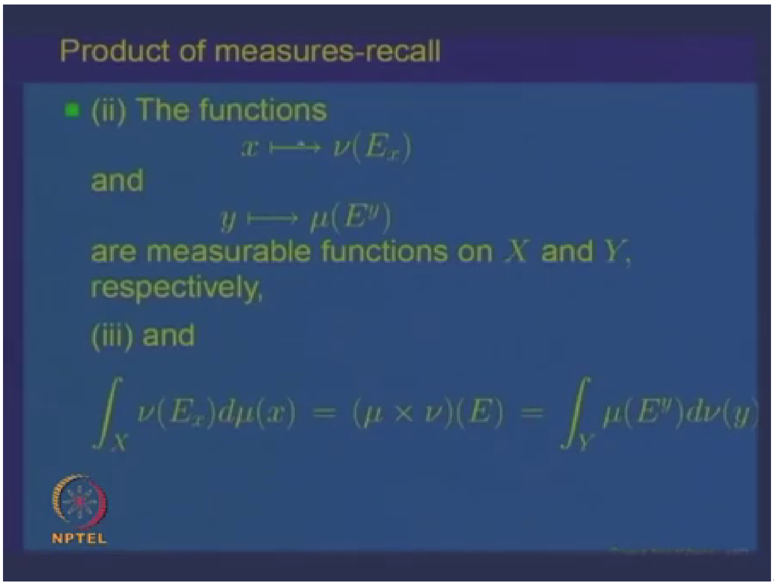


So, we constructed the class P all those subsets in the product sigma algebra such that the previous 2 claims namely this a claim 2, and claim 3 both hold namely x going to nu E x and y going to mu E y are measurable functions.

So, P is the family of all subsets A cross B such that the property 2 and 3 hold. So, what just recall what are the properties 2 and 3? Property 2 is that x going to nu E x and y going to mu E y these are non negative measurable functions. And the property 3 says that the integrals of nu E x with respect to mu is same as the integral of mu E x with respect to nu, and both are equal to product measure of E.

So, the both these properties holds for a set E, then that set is in the collection P. So, our aim is to prove that P is equal to the product sigma algebra A cross B. Already observed in the previous lecture to show this the first step is to prove that this class P is includes the rectangles so that is one. So, that we had proved and also we had proved that this class P is closed under finite disjoint unions. So, once this class is P is closed under finite disjoint unions and includes the rectangles rectangles form a semi algebra. So, the algebra generated by it looks like the class of sets which are finite disjoint union of rectangles.

And P being closed under such operations will get that as a consequence of this that the algebra F R generated by this rectangles is also inside P. So, as a consequence of step one we get the algebra generated by the rectangles in inside the class P.

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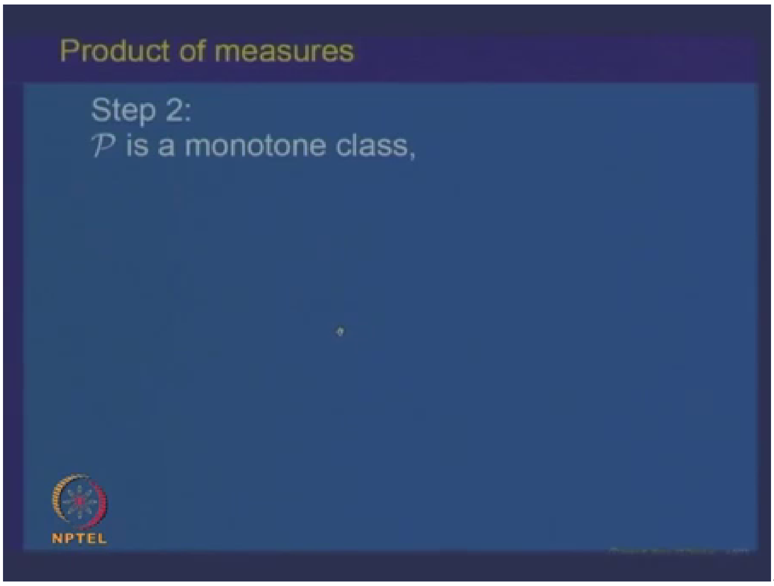


So, the second step we wanted to prove that this class P is a monotone class. And the reason for that to prove that is a monotone class is a following this class P it is directly it is difficult to show that it is a sigma algebra. Because if you could show directly that P is a sigma algebra it includes algebra generated by rectangles. So, then it will include the sigma algebra generated by it that directly root is not possible.

So, we follow the monotone class result namely if we are able to show that P is a monotone class and F R being is inside it the monotone class genertaed by F R will be inside P. And F R being algebra the monotone class generated by an algebra is same as the sigma algebra generated by that class. So, we will get that the sigma algebra generated by rectangles will be inside P, and that is precisely what we want to show and that is a times B because the sigma algebra generated by rectangles is the product sigma algebra A cross B.

So, to complete that proof we have to only show that the class P is a monotone class. So, let us start proving that P is a monotone class.

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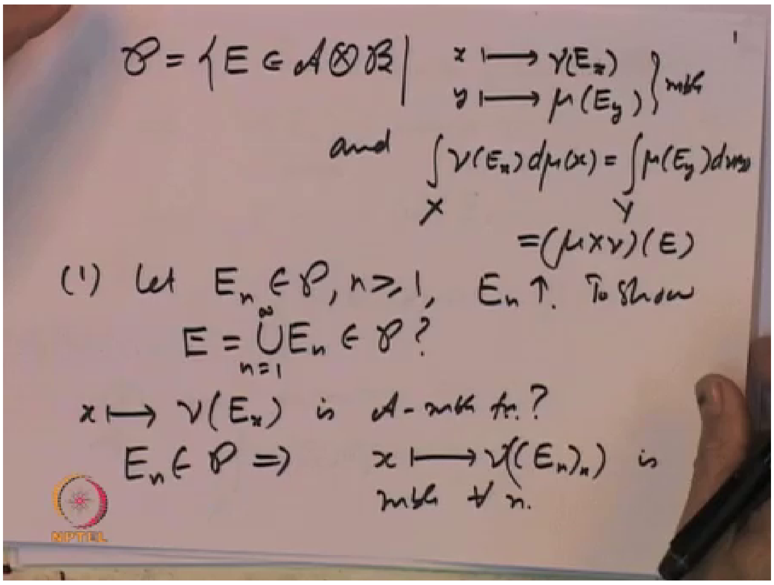


So, P is the class of all the those subsets E belonging to the product sigma algebra A times B, such that if we look at the set x going to take the E take it section x that is subset of the set y in the sigma algebra B. So, nu of that make sense. So, we get this function. So, this is measurable and the function y going to mu of E y that is E is measurable.

So, both this functions are measurable. And the property that if you integrate nu of E x with respect to mu. So, we are integrating over x, this is same as the integral over y of the second function mu of E y with respect to d nu E y. And both of them are equal to the product sigma algebra mu cross nu of E. So, this the collection of all those sets E in the product sigma algebra this holds, and we want to show that P is a monotone class.

So, let us look at the first property. So, let E n belong to P E n B collection of sets in the class P such that E n is increasing. So, to show that the set E which is equal to union of E n's also belongs to P. So, this is what we have to the first to show that P is monotone class, we have to show it is closed under increasing unions and decreasing inter sections. So, that is the 2 properties we have to check.

The let us check a sequence E n E P which is increasing a let us E is the union of this E n's. So, the claim is that E belongs to E n. So, what we have to do? We have to look at the corresponding, so what is the first property we have to check. So, to check that E belongs., we have to look at nu of E x. So, the first thing we have to show is that this is measurable is a measurable function ok.

So, to do that let us observe in the following. So, this is what we have to show. Now E n each E n belongs to P. So, implies that x going to nu of E n it section at x is measurable for every n. So, this is what is given to us. And we want to come to nu of E n.

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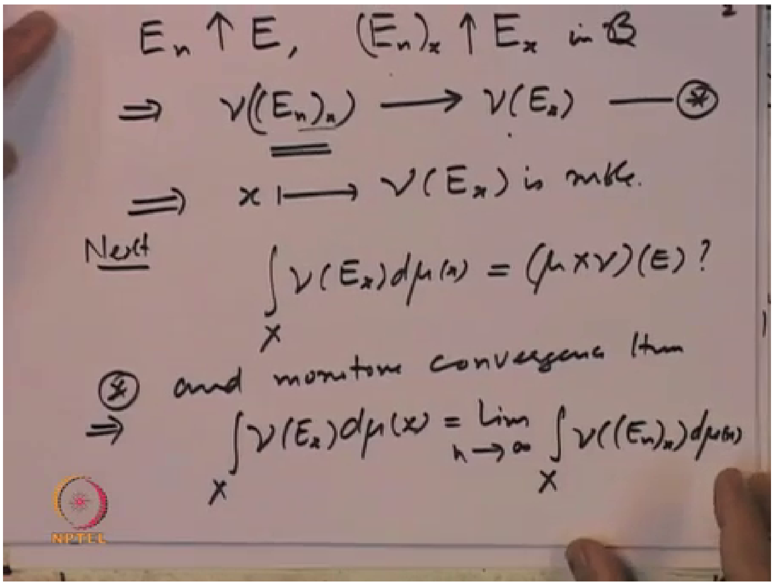


But for that So, let us observe that as E n is increasing to E the sections E n x is a increasing sequence of sets increasing to E of x. So, this is the sequence of sets in the sigma algebra B.

So, that we have already seen that if a is a subset of B then the section of a is the subset of section of B. So, that will prove that is a E is the sections are increasing and the increase to the union. So, union of the sections. So, union of E n at x this is same as union of each E n and hence this is increasing to x.

So, this is a simple observation using the properties of the sections. So, E n x is increasing and now you recall that nu being a measure if a sequence of sets increasing to another set. So, that implies that nu of E n X the sections that will increase that will converged to nu of E x. So, that proves that nu of E n x increases now each one of them is a measurable function. So, nu of E x is a limit of measurable functions.

So, that implies that x going to nu of E x is measurable. So, basically what we have saying is the because the nu of E x. So, the function x going to nu of E x is a limit of the functions nu of E n of x. And that comes from the fact that because E n is increasing to E. So, the sections E n x increase to the section E x and; that means, in the sigma algebra B and nu being a measure nu of E n x misconverged to nu of E x.

And each one of them being measurable because I it is in the collection P. So, each is a measurable function. So, limit of measurable function is measurable. So, that proves one part that x going to nu of E x is measurable. So, next what we have to check is a following we have to check that integral of nu of E x d mu x over is equal to nu cross nu of E.

So, this is want we want to check. So now, once again let us go back to the earlier fact that we saw that nu of E n x the sections these measurable functions. These are actually non negative measurable functions and their converging to the function nu of E x. And that is a increasing sequence of measurable functions. So, this is nu of E n x is a increasing sequence of non negative measurable functions converging to a measurable function nu of E x.

So, we can apply our monotone convergence theorem. So, that says so by, once again this property star and monotone convergence theorem, theorem apply, and apply and they give us as a consequence that integral of nu of E x d mu x over x because nu of E x is a limit of increasing sequence of non negative measurable functions. So, integral of nu E x must be equal to limit n going to infinity of the integrals of the corresponding sequence of non negative measurable functions and they are nu of E n section at x d mu x.

So, this is a application of monotone convergence theorem. Let us observe that E n belongs to the class P. So, property 2 of that says that if I integrate nu of E n sections with respect to mu.

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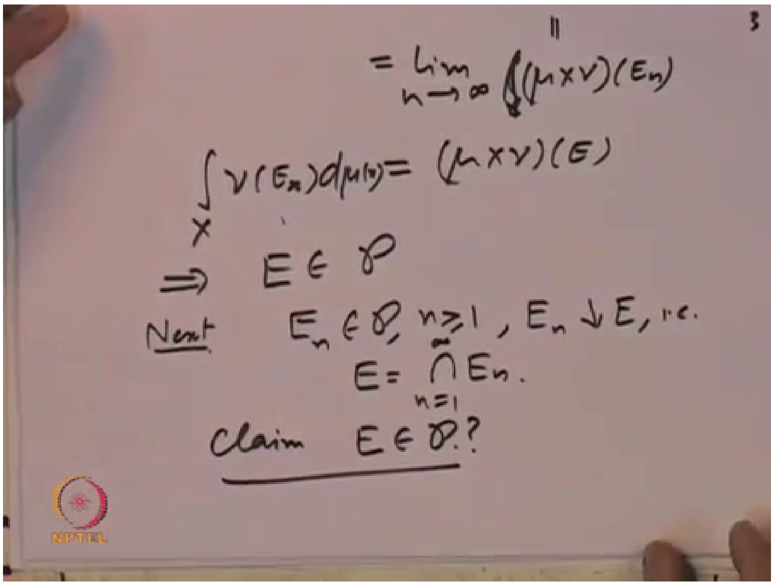


This integral is equal to the product measure mu cross nu of E n. So, that is because E n belongs to the class P. So, by the third property of the collection of sets in p; that means, nu mu cross nu of the product measure of E n is a integral of the sections with respect to x. So, we can say that this integral is equal to limit n going to infinity of mu cross nu of E n.

So, once that is true we want to look at this limit once again let us observe that E n is a increasing sequence of sets in the sigma algebra A cross B, and mu cross nu is a measure. So, once again using the property of measure that if a sequence of sets is increasing then the measure of limit of to the measure of the sequence is equal to measure of the limit. So, that is equal to mu cross nu of E.

So, once again we have use set that E n is increasing to E and mu cross nu is a measure, sothis limit must be equal to mu cross nu of E. What we get is that this limit is equal to this. So that means, we get that mu of integral over x nu of E x d mu x is equal to mu cross nu of E. So, we have proved that if E n is increasing to. So, this implies that E belongs to the class P. Because we showed that if E n is increasing to E then both the properties hold for this ok.

Now we want to do this a similar thing for decreasing. So, next let us considered E n belonging to P and bigger than or equal to 1 and E n's decrease to E. That is E is equal to intersection E n's n equal to 1 to infinity. So, claim we want to claim that E belongs to P. So, this is what we want to check. So, we can try to copy the proof for the increasing case. So, let us back to the proof of the increasing case and let us see can be carry over the proof by sayings similarly. So now, we have got E n's decreasing. So, because E n's belong so what we said first thing was that because E n's belong to P. So, this is a measurable function.