Measure & Integration
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Lecture – 25 B
Computation of product Measure – I

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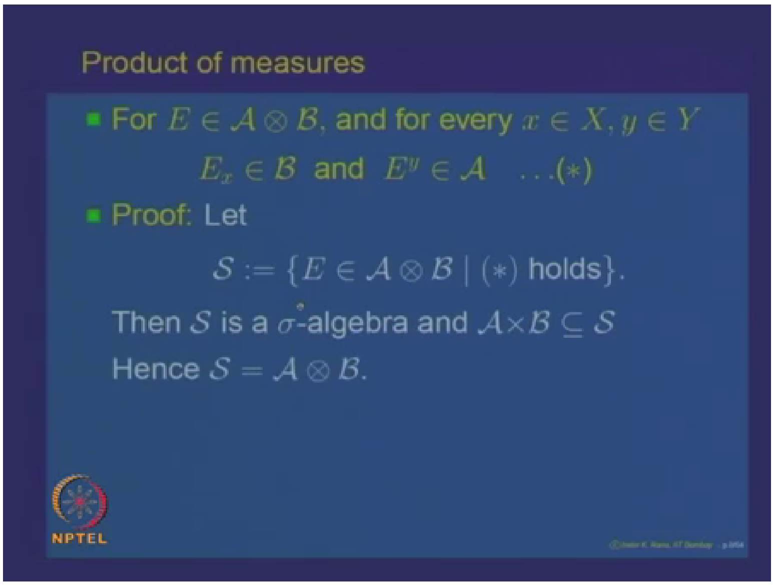

So now using these properties, we will prove namely if E is a set in the product sigma algebra A times B and X and Y are elements x is in X and y is in Y then the claim is that the section Ex belongs to B and the section Ey belongs to A. So; that means, for every x in X look at the subset of Y, which is the section of E at a point X that belongs to the sigma algebra on B, whenever E is element to the product sigma algebra. And similarly the section at Y is a subset of X, and our claim is that this belongs to the sigma algebra a

So, these are the 2 properties we want to check, for every set E belonging to A cross B. Now, here is the technique of proving all these results in the product sigma algebra. Basically we will apply the monotone class sigma algebra techniques; namely we will, whenever we want to show a property holds for A cross B elements in A cross B, we will collect together all subsets for which this property is true, and try to show that we will collect sets for which this property is true in A times B, and show that collection includes rectangles, and this collection is A sigma algebra. So, once this collection is A sigma algebra and includes rectangles, it will include the product sigma algebra A times B.

So, that is what i had called as the sigma algebra technique. So, we will apply that technique here. So, let us define the collection S to be all subsets in A times B; such that this property, which we are calling A star. So, Ex the section at X belongs to the sigma algebra B, and the section at Y belongs to the sigma algebra A. So, what we want to prove. We want to prove that this S is equal to A times B. So, to prove that S is equal to A times B, we will prove 2 things; namely S is a sigma algebra, and it includes rectangles, and that will prove, that it actually is equal to A times B.

So, let us prove these properties. So, what we are given is, we are given that the set E belongs to the sigma algebra. So, set E belongs to the sigma algebra A times B.

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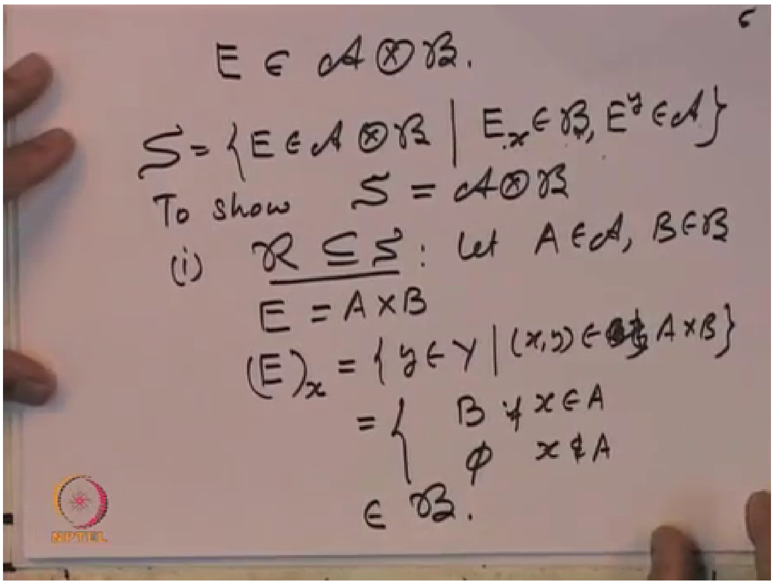

So, let us. So, S is the collection of all the subsets E belonging to A cross B; such that the section Ex belongs to B, and the section Ey belongs to the sigma algebra a. So, to show S is equal to a product sigma algebra B naught, it is already a subset of A cross B.

So, we will follow 2 things; one, let us check the properties of this first, is the rectangles R inside S. So, to check this property let us take a rectangle. So, let A belong to A and B belong to B, and let us take the rectangle E which is equal to A cross B. So, if we recall, we had calculated; what is the section of E at x. So, that is all Y belonging to Y such that X, Y belongs to B X, Y belongs to sorry A cross A cross B.

So, now X comma Y can belong to A cross B, only when X belongs to a, and in that case Y should belong to B. So, this set is equal to. So, if X belongs to A . So, for all X belonging to A, this set is equal to B. So, the section is equal to B. If X belongs to A and if X does not belongs to a, then in no way the X comma Y is going to, belong to. this is empty set, if X does not belong to a.

So, for a rectangle we have already seen. I am repeating the steps which we have done earlier; namely for a rectangle A cross B, the X action is either B or empty set. So, in either case this belongs to the sigma algebra B. So, the property that Ex belongs to B, is true. A similar argument we will show that Ey also belongs to a. So, this proves that the rectangles are inside the sigma algebra S.

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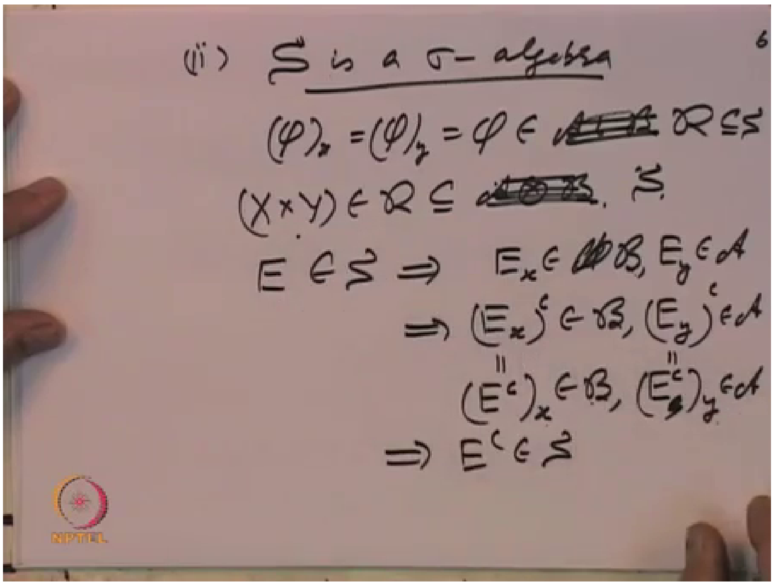

So, the next step we want to check is the following. So, second step we want to check is that, this collection S, S is a sigma algebra. So, this is what we want to check.

So, for that the first property, look at the empty set. The sections of the empty set; either X, section is same as the section Y, and that is empty set, and that belongs to both A and B. and similarly if i look at the whole space; that is X cross Y; that is actually a rectangle which is inside A cross A times B. Sorry which is A times B, and already rectangles are inside s. So, both the whole space, and the empty set are inside S and A and B. And hence it is also a rectangle. So, actually we should say that this belongs to A rectangle and which is part of S.

So, empty set and the whole space, both belong to S. The next property, let us take a set E belonging to S, and show that its complement also belongs to it, but E belongs to S, implies the sections E X belong to a. sorry Ex belongs to B, and Ey belongs to A right; that is a definition of s. So, let us just recall. So, what was the definition of the set S. The definition of the set S is all subsets A cross B. So, that Ex belongs to B, and Ey belongs to a.

So, by the definition this is true, but Ex belongs to B, and B is A sigma algebra Ey belongs to a, and A is a sigma algebra. So, that implies that Ex complement belongs to B, and Ey complement belongs to a, because of the properties of sigma algebras, that A and B are both sigma algebra. So, there must be close under complements, but on the other hand, this set taking section, and the complement as now we just. Now we observed it is same as i can take the complement first, and then take the section. So, that should belong to B.

And similarly, here the section Y and then complement is same as taking the complement first, and then taking the section that should belong to a. So, this set is same as this. So, for every set E in S, if i look at the E comply, set E complement it section at X belongs to B, and it sections at Y. Sorry this is E complement. So, at Y belongs to a. So, that implies that E complement also belongs to s. So, S is closed under taking complements. And finally, to show it is a sigma algebra, I have to show, it is also closed under say countable unions.

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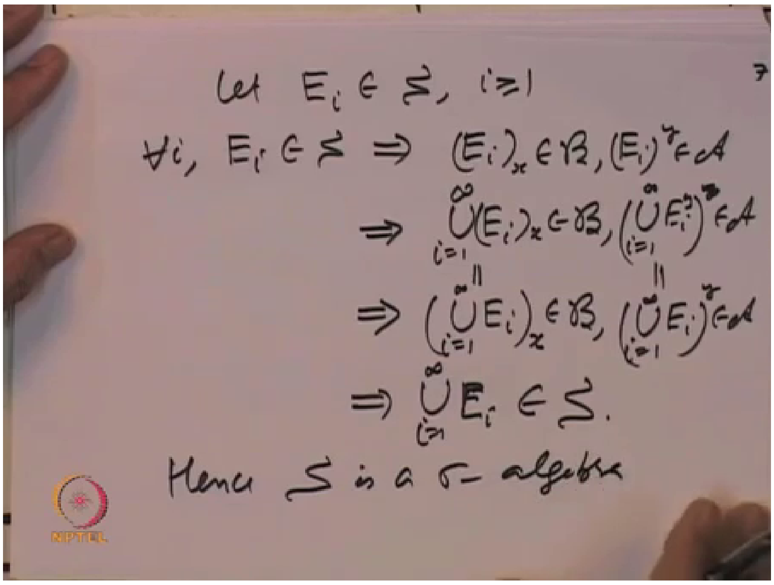

So, to show that; so, let Eis belong to S; say S bigger than or equal to 1, but each Ei belonging to S implies. So, for every i Ei belongs to S, implies that Ei section at X is in the sigma algebra B, and Ei section at Y belongs to the sigma algebra a. So, this property is true, but once again A and B both are sigma algebras. So, that implies that the union of Eis sections at X i equal to 1 to infinity, belongs to B. and similarly the corresponding one, the union i equal to 1 to infinity of Eis section at Y belongs to a. So, this is true.

But that implies by the fact that this set taking the sections, and taking the union is same, as first taking the unions, and then taking the sections, just now we observed that . So, that belongs to the sigma algebra B. In similarly this set, first taking the, first taking. Sorry this was Eis at Y, because union belongs to a. So, that is same as now i can write as the same as union 1 2 infinity of Eis at section, at Y belongs to a.

So, that implies that union of the set union of Eis, it section at X belongs to B and it section at Y belongs to a; that means, this belongs to the calculation x. So, that proves. So, hence S is a sigma algebra. So, S is a sigma algebra, and we know that rectangles are inside s. So, that implies that A cross B is inside S. S is the subset of already A times B. So, all these are equal; that means, the property that for every set in the product sigma algebra. So, this property that for every set in the product sigma algebra, the X section belongs to B, and the Y section belongs to A is true.

So, let me once again emphasize the fact that we are looking at this, proves which are nothing, but application of the technique, call the sigma algebra technique.

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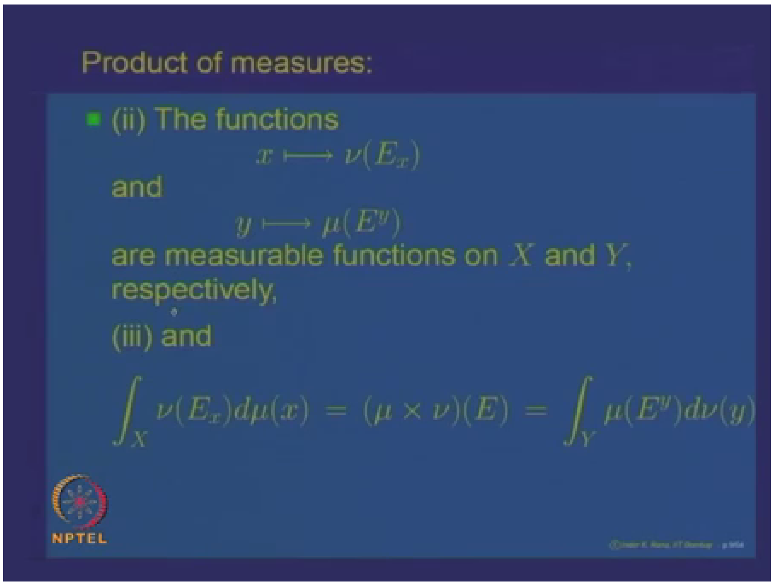

So, now let us go to the next property, namely we want to check the property that we already know, that for every X Ex is a section of E at x. So, if E in the product sigma algebra, this set E lower X E section of E at X is in the sigma algebra B. So, nu of that set makes sense, because nu is defined on the sigma algebra B. And similarly now the section of E at Y is in the sigma algebra a.

So, measure of this section mu of this section make sense, but both nu of Ex depends on X, and mu of Ey depends on y. So, this gives us 2 functions; X going to nu of Ex, and Y going to nu of Ey. The first 1is a function on the set X, and the second 1is a function on the set y. So, we want to prove that both of these are measurable functions, and clearly these are non negative functions. So, they are non negative measurable functions on X and y. So, their integrals make sense with respect to, this is a function on x. So, its integral with respect to mu make sense, and this is a non negative measurable function with respect to y.

So, its integral with respect to the measure nu make sense. So, we want to claim that the integral of the first function with respect to mu is same as the product measure of the set E, and which is same as the integral of the second function with respect to nu. So, that will give us a nice way of computing the product measure, namely the product measure of a set E can be computed either by taking its sections with respect to X, finding the size of those sections; that is the nu measure of the sections with respect to x.

And then summing it up; that is taking integrals with respect to mu or we can interchange the roles of X and Y. We can take the sections of E with respect to Y, first take its measures with respect to mu and then add up. So, take integrals with respect to nu. So, we want to prove that this property 2 and property 3 hold for every subset E of product sigma algebra A cross B. So, once again this proof is going to be an application of the sigma algebra monotone class technique, and you will see how effective these techniques are.

So, what we will do. we will collect together all the subsets of A cross B for which this 2 properties are true, and will try to show rectangles are inside it, and hence everything is inside it

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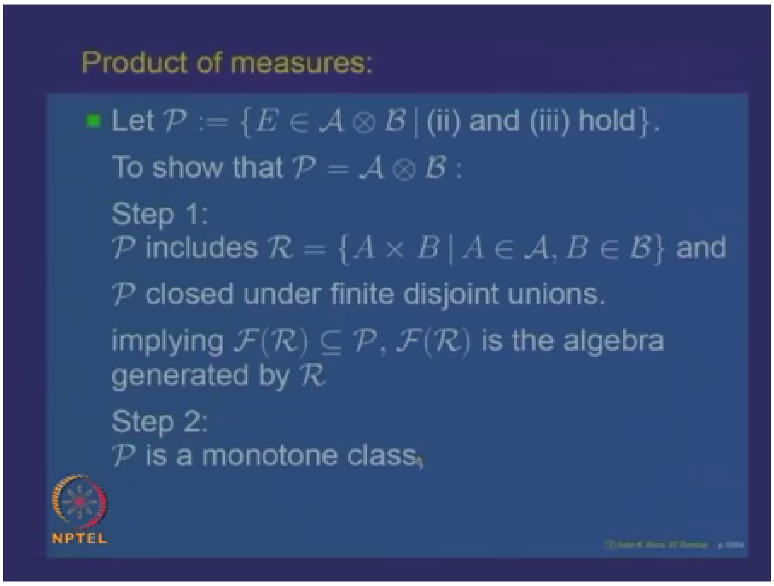

So, let us look at let us look at the collection P of subsets of E cross elements in A cross B. So, that property 2 and 3 both hold. So, what is going to be our technique. So, what is a problem to be proved. So, the problem is to show, that this P is equal to A times B. So, to show that we will do the following. first we will show that rectangles are inside A cross B. So, that is 1that the set of all rectangles are inside the class, this collection P and we will show the second step; namely this collection P is closed under finite disjoint unions.

So, what will that prove? You recall we had shown that R is a sigma algebra, and if the collection P which includes R is closed under finite disjoint unions; that means, finite disjoint unions of elements of the rectangles also will be inside p, but finite disjoint union of rectangles is nothing, but the algebra generated by this semi algebra r. So, that will prove. So, this step will imply that the algebra generated by the rectangles is inside the class p. So, this first step is to conclude that the algebra generated by rectangles is inside p, and the method is to show that R is inside it and F of r. So, and it is closed under finite disjoint unions.

So, let us prove this step 1first. So, we have got the collection p.

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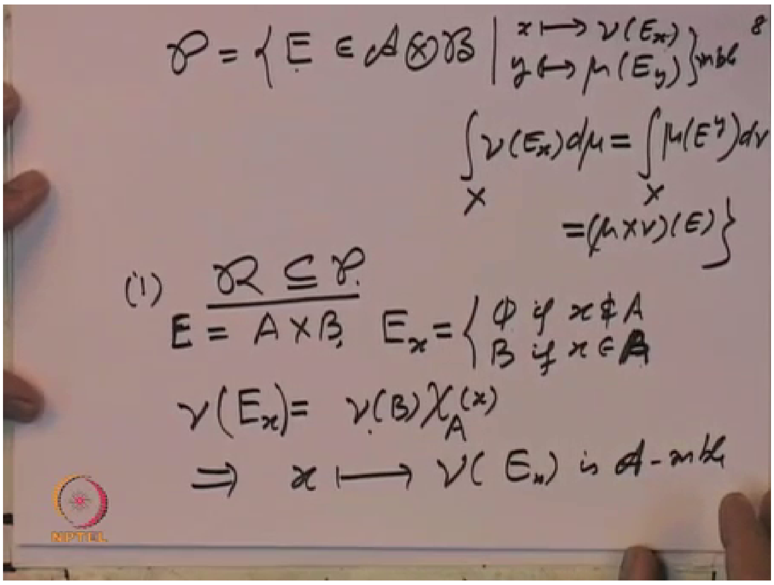

So, P is the collection of all subsets E belonging to A times B; such that those 2 properties hold. So, what over the 2 properties, the properties 1that X going to going to nu of Ex and Y going to mu of Ey. These 2 are measurable functions and that the integral of nu Ex with respect to mu is same as integral of mu Ey with respect to d nu. So, this is over X and this is over Y, and both of them are equal to the product sigma algebra, namely mu cross nu of e.

So, E is essentially what we are saying is. we are looking at the sets E in the product sigma algebra for which the required properties hold. So, we want to show. So, the first thing is, we want to show the rectangles are inside p. So, let us take. So, to prove this let us take A rectangle e. So, E is equal to A cross B right, where A belongs to the sigma algebra a, and B belongs to the sigma algebra B. So, let us look at you recall what was the sections. The section Ex was equal to empty set, if X does not belong to A and it is equal to B, if X belongs to B.

So; that means, this Ex is nothing, but when X does not belong to A it is empty set. So, what is going to be nu of that; that is going to be 0, Ex is going to be the set B. So, it is going to be. So, it is nu of B into the indicator function of A at x. So, this is what is important that for a rectangle A cross B we have already computed the sections X, section was empty set, if X does not belong to a, and it is B, if X belongs to B. So, nu of Ex. sorry nu of Ex is going to be nu of empty set which is 0, if X does not belong to a, and if X belongs to if. Sorry; if X belongs to a. So, this should be X belongs to a.

So, if X belongs to a, then it is nu of B, and here nu of A is one. So, this equality whole, because if X belongs to a, this value is 1, and indicator function of A at X, x does not belongs to A is 0. So, we have got this equation namely nu of Ex, which we want to show is measurable is nothing, but the indicator function constant time. So, indicator function of a set in the sigma algebra A is in the sigma algebra. So, that implies that X going to nu of Ex is a measurable. So, this is a measurable function.

And similarly, if we take the corresponding section with respect to y. So, let us write that also. So, if we look at E of y. So, that is, we are writing it up actually.

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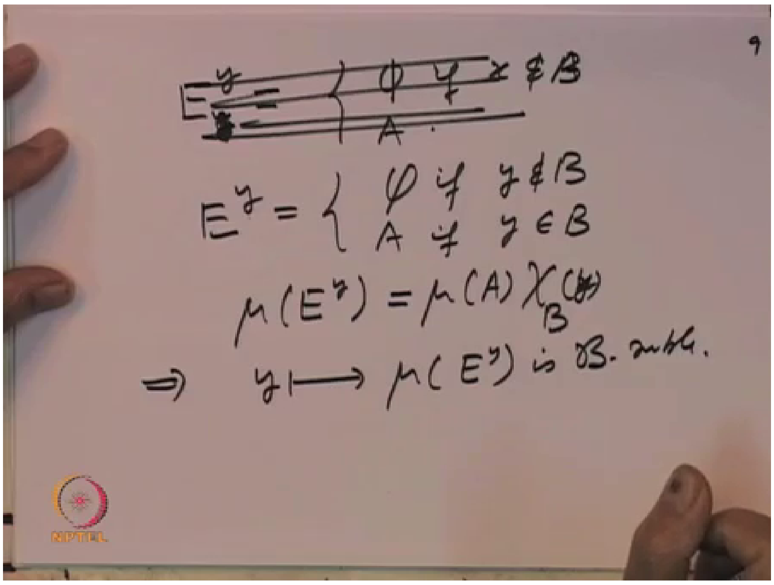

So, let we write follow the same rotation the section of E at y. So, E superscript Y is equal to, it is empty set, if X does not belongs to A and it is equal to a. I am sorry let me write it properly. So, the section Ey is equal to empty set if Y does not belongs to B, and it is A if Y belongs to the set B, because Y is A point in x.

So; that means, that mu of Ey is going to be equal to mu of the set A times the indicator function of the set B at A point at the point y. So, as a function of Y it is just the indicator function of the set B at the point Y multiplied by a constant. So, that will imply that Y going to mu of Ey is B measurable. So, that proves the first thing; namely the 2 wanted to show that rectangles are inside. So, what we have shown here for a rectangle, the first property namely X going to nu of Ex, and Y going to nu. Y are measurable with respect to the corresponding sigma algebras, or let us compute the integrals of these things ok.

So, nu of Ex is this function. So, what is its integral with respect to mu. this is a constant, and this is a indicator function. So, it is nu of B into mu of a. So, from this equation star. So, let us write that from the equation star it follows.

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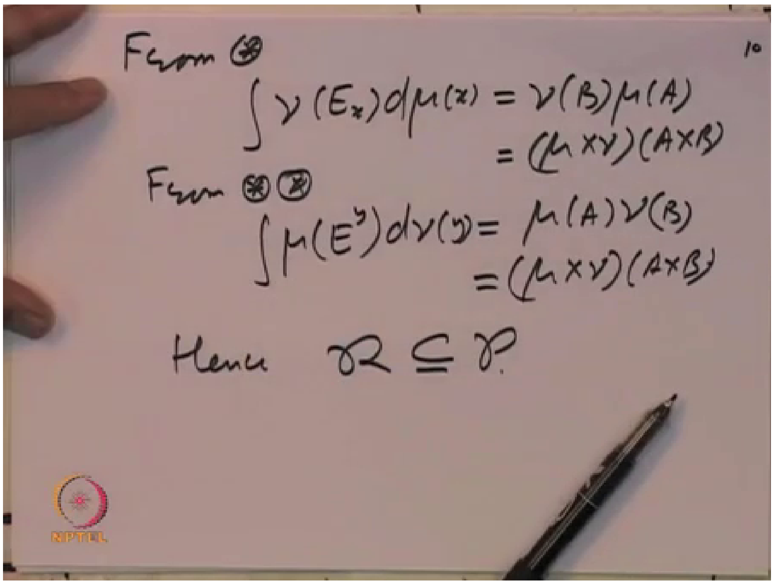

So, from star integral of nu of Ex d of. So, this is the property star d of mu X is equal to, this is integral of this quantity. So, that is nu of B times mu of a. So, that was the property star, and similarly let us look at the integral of the other function

So, we want to compute integral of the function mu of Ey so, but mu of Ey is equal to this quantity. So, let us call it as double star. So, once we integrate this, what we will get is, integral of mu of Ey with respect to Y d nu Y is equal to mu of A into nu of B. So, from the equation double star will have integral mu of Ey d nu of Y is equal to mu of A into nu of B. So, in either case, these integrals are, which is nothing, but the product. So, that says. So, this is equal to the product sigma algebra product measure mu cross nu of the rectangle A cross B, and similarly here, this is the product mu cross nu of A times B.

So, this proves. So, hence what we are shown is at rectangles are inside the collection P of the sets, for which we wanted to prove the required claim holds. So, what was the second step we wanted to prove? We want to prove that this collection are p. So, claim. So, the second collection. So, here is the second thing in the step one. So, we have proved part of the step one; namely we have proved that P includes R, P includes rectangles. So, the next part of the proof requires us to show that P is closed under finite disjoint unions.

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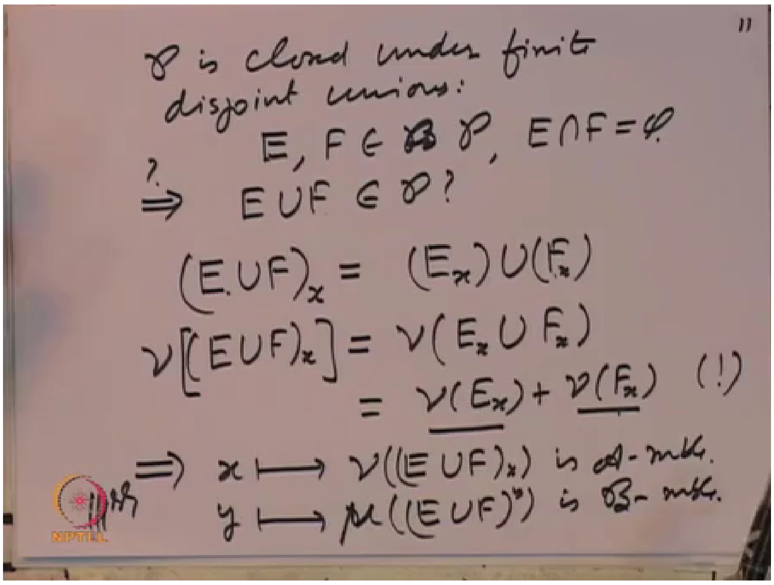

So, let us proof that. So, P is closed under finite disjoint unions. So, for that. So, let us take 2 sets E and F which belong to P, which belong to the collection P. So; that means, what. So, that implies that for E and F the corresponding results are true, and E and F are disjoint; that is also given to us intersection is equal to empty. So, to show. So, we want to show that E union F belongs to p. So, that is what we want to show.

So, let us start with looking at the sections of E union F, it section at A point X by the definition properties of the sections. The section of the union is union of the section. So, it is union of Ex union of Fx. So, what is going to be these are the sections. So, what is going to be nu of the union E union F section. So, what is the nu of that. So, that is now as E and F are disjoint, these sections are going to be disjoint. So, it is nu of the disjoint union of the sections Ex union F of X these being this joint. so; that means, this is equal to nu of Ex plus nu of Fx; that is nu of Ex plus F of x.

So, here is something for you to think, and conform that if E and F are disjoint sets, then their corresponding sections are also disjoint, and hence this property is true, now E and F both belong to p; that means, this is a measurable function of X, and this is also a measurable functions of X, and we have proved that some of measurable functions is measurable. So, this will imply that X going to nu of E union F section at X is a measurable.

So, this is a measurable function. similarly Y going to nu of mu of E union F section Y is B measurable . So, to check that P is closed under finite disjoint unions. We have check the first property; namely if E and F are 2 disjoint sets in P, then X going to nu of E union F section and Y going to mu of the section E union F at Y are both respectively measurable functions.

Now, let us check the next property; namely that the integral property is true for the union. So, for that what we want to do, is the following. So, we want to integrate.

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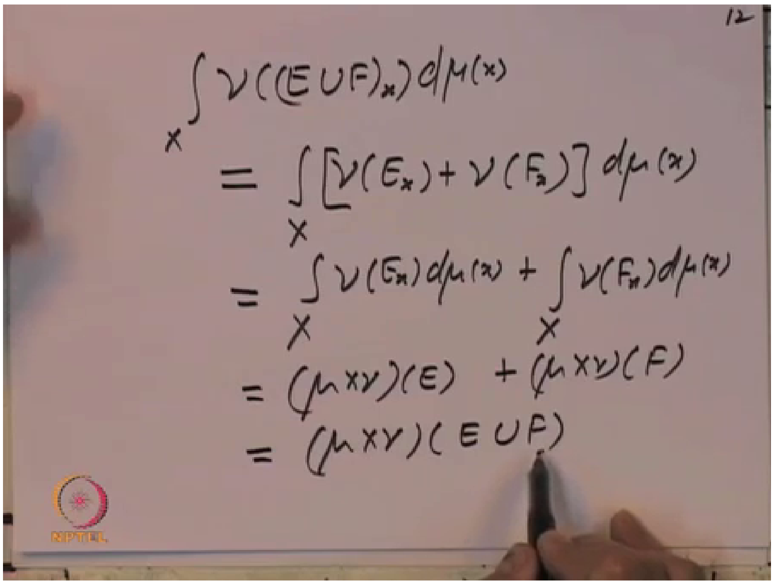

So, let us integrate nu of E union F section, this is a measurable function. So, with respect to mu we can integrate this. So, this over the set X; of course, this we know by the, just now we proved that the nu of. So, here is nu of E union X union F at X is nu of Ex plus nu of F of x. So, let us use that property and. So, this we can write as x. So, the integrant nu of E union F of X is equal to nu of Ex plus nu of F of X d mu x

So, now by the using the properties of the integral this we can split it as integral over X of nu Ex d mu X plus integral over X of nu of F of X with respect to d mu X. Now, because E and F both are inside the class, means inside the collection P for which this property integral of the section nu Ex d mu X is nothing, but mu cross nu of E, and the second integral is nothing, but mu cross nu of F, and now by the fact that E and F are disjoint and mu cross nu is a measure, this is nothing, but mu cross nu of E union f.

So, what we have shown is that, if I integrate nu of E union F section with respect to mu; that is the product measure of the set E union F A corresponding result will also hold, when I take Y sections; namely we can show that integral of mu E union F at y. So, similarly. So, let me just write the argument; that is the corresponding result will be similar.

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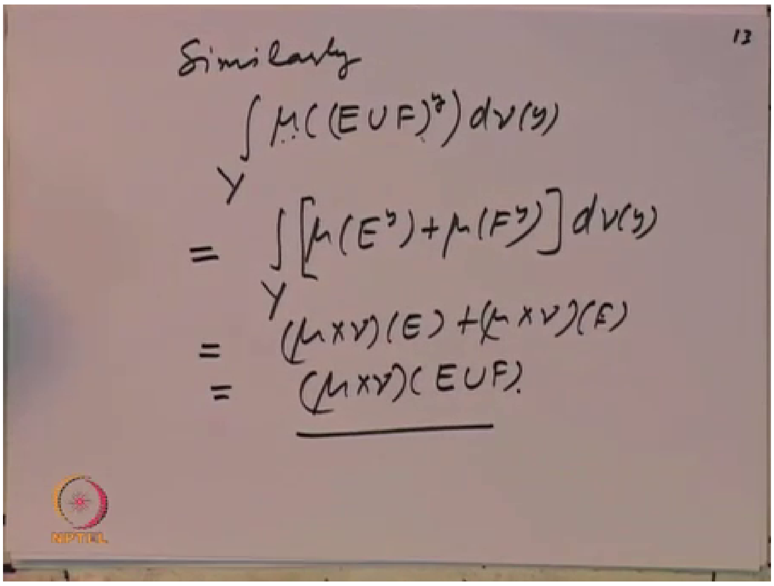

So, similarly if I integrate over Y and mu of E union F section at Y d nu of y.

So, if I take the section of E union F with respect to Y, take its mu measure. So, that is a measurable function, and its integral with respect to nu that we just. Now we observed that this section is nothing, but mu of E Y plus mu of Fy and that was, because the section E union F section is same as E section union F section, and there disjoint some measures adopt, and that is equal to d of nu Y. And now once again as before we can write this as mu cross nu of E plus mu cross nu of F, and by again using the property of that mu cross nu is a measure E and F are disjoint. This is mu cross nu of E union f.

So, that proves the second part of the property namely that not only P includes rectangles. In fact, P is closed under finite disjoint unions. So, as a consequence of this, because finite disjoint union of elements of A semi algebra give us the algebra generated by that is semi algebra. So, as a consequence of step one, we have gotten that the algebra generated by rectangles is inside the class P, where Fr is the algebra generated by rectangles.

So, now. So, our next step should be that, trying to show that this P is actually A sigma algebra, but once tried to do that, 1tries to show that P is a sigma algebra, 1lands into problem, 1is not able to show that it is closed under arbitrary unions. So, that will be a problem. So, 1 modifies the arguments and in. So, instead of showing that P is a sigma algebra one, tries to show that P actually is, at least a monotone class. So, once 1tries to show that P is a monotone class it includes an algebra.

So, it will include the monotone class generated by the algebra, which is the sigma algebra. So, that is A root we will follow. So, from here onwards our technique will be the monotone class technique. So, we will try to show that P is a monotone class. So, it will include the monotone class generated by the algebra F of R which is same as the sigma algebra generated by R, and that will complete the proof. So, the second step we will do it in the next lecture. So, today's lecture we have just concluded that the class P, for which we want to prove the required claim holds, includes the algebra generated by rectangles. So, we will continue the proof in the next lecture.

Thank you.