Measure & Integration
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Lecture - 03 A
Sigma Algebra Generated by a Class

We had started looking at the concept of algebra of subsets of a set x. We will look at some more properties of that today and after that we will start looking at what are called sigma algebras of subsets of a set.

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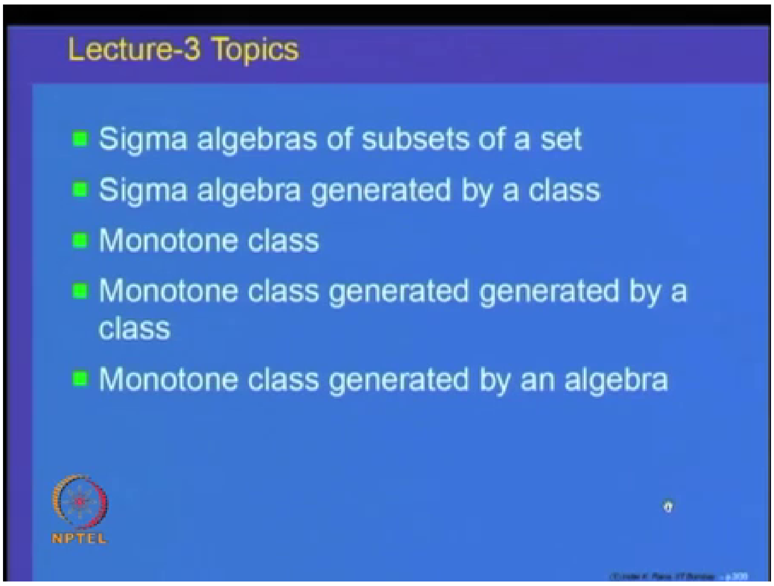

And then come to sigma algebra generated by a class of subsets of a set x. And then go on to look at what is called a monotone class; the monotone class generated by a class and then look at a monotone class generated by an algebra. So, let us just recall what we had started looking at, namely the algebra we said an algebra of subsets.

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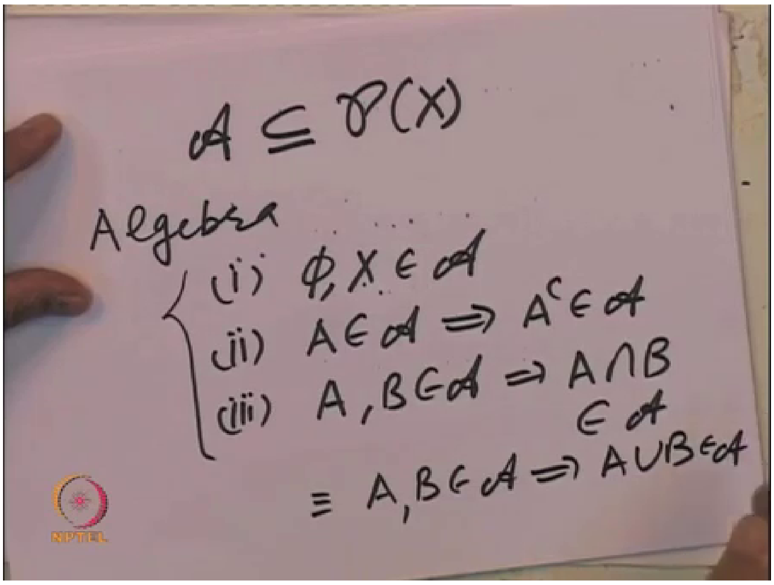

So, a class of subsets a contained in P of X that is the power set of X is called an algebra. If it had the properties 1, the empty set, the whole space is a member of A. Secondly, whenever A belongs to A implies, it is complement, is also inside the class A, that is the class A is closed under the operation of compliments and the third property was whenever A and B belong to A, that implies a intersection B belongs to algebra. So, these are the three properties that define a class A to B, an algebra and keep in mind this property, because of compliments, this can be equivalently stated as A and B belonging to the class A implies A union B also belongs to the class A.

So, this is what we are defined as a collection of subsets of set X to be an algebra and then we looked at various properties of algebras for example.

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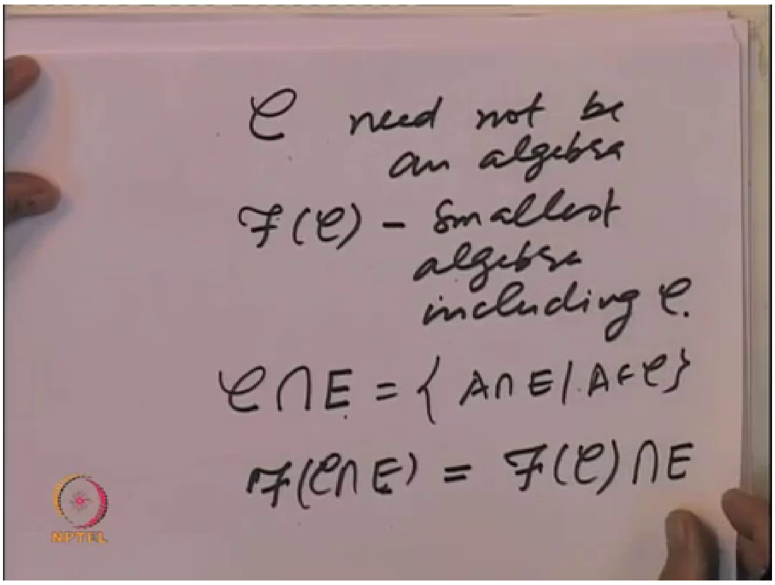

We proved one thing that a class C need not be need not be an algebra, but you can generate an algebra out of it. So, this is the smallest algebra including the given collection C and then we went on to prove that if you take a collection C and restricted, it is elements to a set E, which is defined as all elements of the type A intersection E where a belongs to C, then, if you generate algebra out of this collection C intersection E, which is the algebra of subsets of E generated by this collection C intersection, we showed this is also equal to the algebra generated by C restricted to E.

So, these are the various ways of generating more algebras out of the given algebra, what is the advantage of having an algebra is the following.

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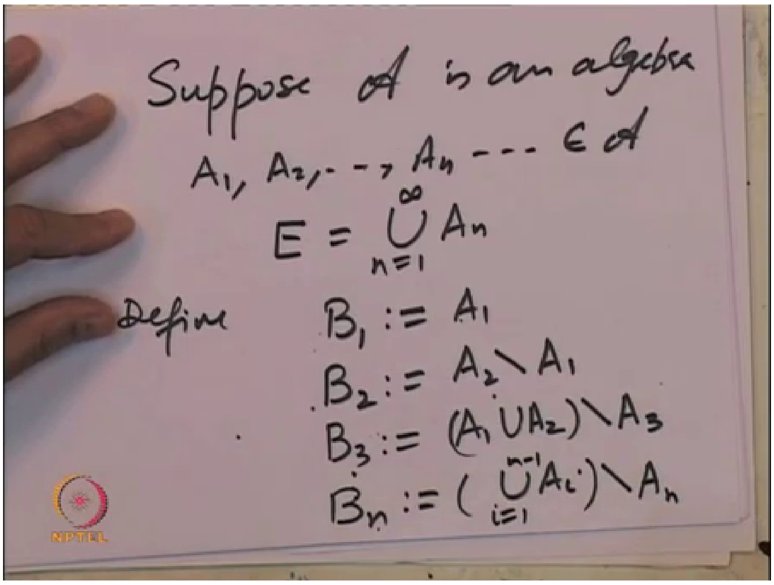

So, let us suppose that you have got A is an algebra and then let us take a sequence A 1 A 2 A n inside A, a collection of elements of A and let us take their union E equal to union of A n's n equal to 1 to infinity of course, this E need not belong to the algebra.

Because, algebra is only closed under finite unions; however, there is something nice one can be do. Let us define B 1 to be equal to A 1 itself. So, let us define B 2 to be equal to A 2 and remove from it. The set elements which are in A 1 and similarly, let us define B 3 to be A 1 union A 2 and remove from it, the elements which are in A 3 and so on.

So, you will define B n in general to be equal to union AI, I equal to 1 to n remove from it the elements, which are n minus SOB, B n is defined as the union of elements up to n minus 1 and remove off from it, the elements which are in A n right. So, what are the: so we have generated a new sequence out of the given sequence; so B n.

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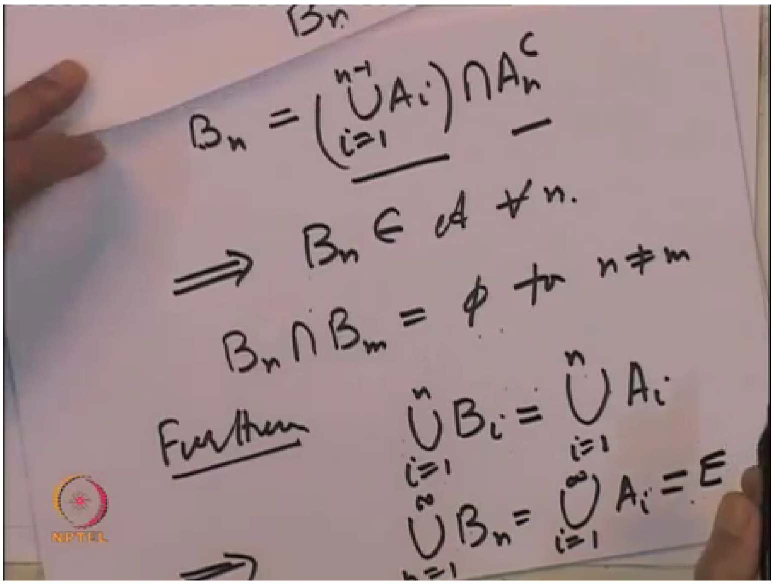

Let us observe that B n, which is equal to union of this AIS minus An, what is this looks like. So, it is union AI, I equal to 1 to n minus 1 intersection A n compliment, because removing A n is same as taking it is intersection with an compliment. So, this. So, now, observe each AI is an element in the algebra. This is a finite union of elements in the algebra A n compliment is in the algebra, because A n is in the algebra, algebra is closed under compliments. So, this implies that each set B n is an element of the algebra A for every n. So, that is one observation and. Secondly, let us observe that B n intersection Bm, is empty for n not equal to m.

Because what we are doing, B 1 is A 1 B 2 is from A 2 remove A 1. So, B 1 and B 2 are going to be disjoint and B 3, it is A 1 union A 2 minus A 3, we are removed, what is an A 3. So, this B 3 is going to be disjoint from B 2 and B 1 both. So, in general it is quite obvious that B n's are pair wise disjoint their elements in a further. Here is an important consequence, the way we have constructed if I take union of BIS, I equal to 1 to n; what is that equal to it is precisely B 1 is A 1 B 2 is A 2 minus A 1. So, what is B 2 union B 1 union B 2, that is same as A 1 union A 2?

And similarly, B 1 union B 2 union B 3 is same as A 1 union A 2 and union A 3. So, that is same as B 1 union B 2 union B 3. So, for every n union of BIS, I equal to 1 to n is same as union of BIS, I equal to 1 to n. So, that implies. So, as a consequence, this implies that union of Bn, n equal to 1 to infinity is same as union I equal to 1 to infinity of AI which was our set E. So, what we have shown, we have shown that if we start with a count any countable union of elements in the algebra E need not be an algebra, but E can be represented as a disjoint union of sets B n and each B n is A algebra.

So, what we are saying is we have proved a theorem. It is going to be quite useful and that is advantage of being inside in algebra.

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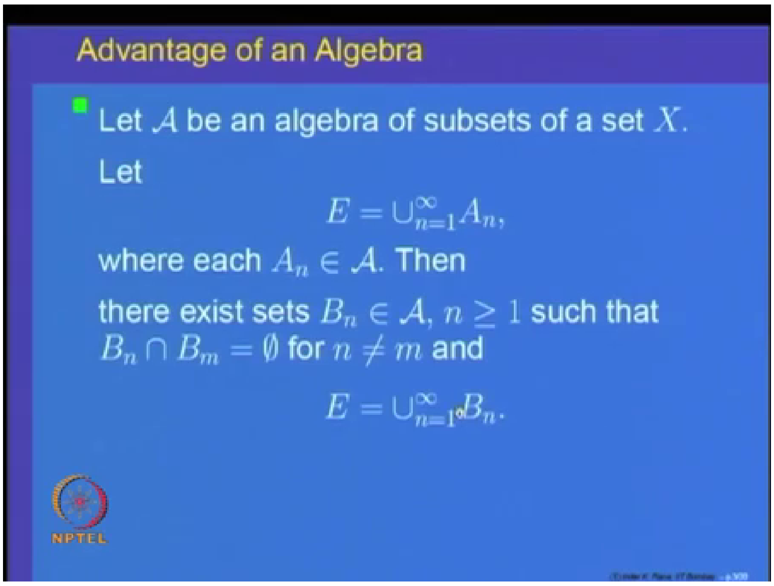

Let us recall once again, if A is an algebra of subsets of a set X and a set E is union of A n's n equal to 1 to infinity, where each A n belongs to A then there exists disjoints sets. So, there exist sets B n belonging to the algebra, which are pair wise disjoint and their union is equal to B n's. So, any countable union in algebra can be represented as a countable disjoint union. So, that is a advantage of being in a class, which is algebra. So, that is nice we will see an applications of this next time.

So, let us now start with a class, which is slightly more stronger than algebra. So, that is called a sigma algebra.

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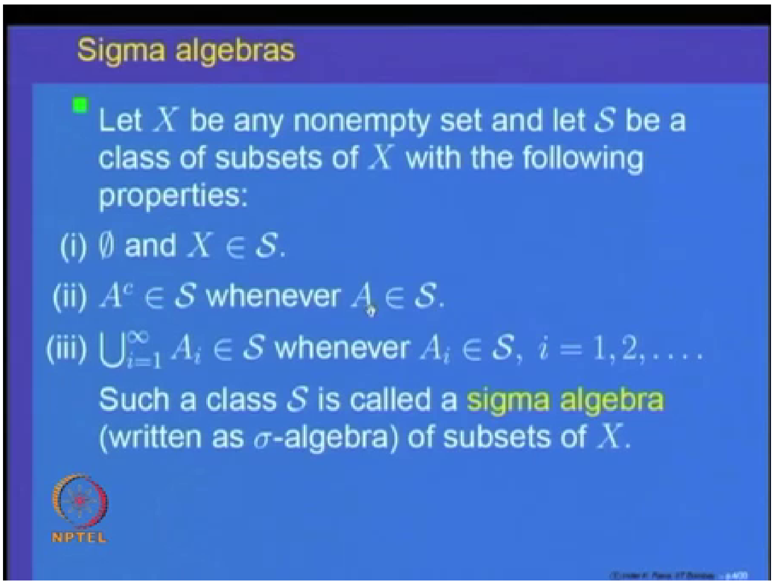

So, let us start with a collection X is a nonempty set and let S be a class of subsets of the set X, with the following properties 1. Empty set and the whole space are elements of it like in a semi algebra, like in an algebra A compliments belong to S, whenever the set A is in the S; that means, the collection S is closed undertaking compliments as in the case of an algebra. So, these 2 properties are same as were the case for an algebra. The third property is the one which distinguishes it from an algebra. We want that whenever sets AIS are in S I equal to 1 2 3 and so, on; that means, whenever you take a countable collection of sets in S their union I. 1 to infinity AI is also belongs to S; that means, the collection S is closed undertaking countable unions also. So, such a collection we are going to call it as a sigma algebra. Sigma indicating that is closed under sequence of unions.

So, let us just emphasize once again, a sigma algebra of subsets of a set X is a collection, which includes the empty set in the whole space. It is closed undertaking compliments. So, if A belongs to S, A compliment belongs to S and whenever you take a sequence AI of elements of S, their union is in S; that means, S is also closed undertaking countable unions.

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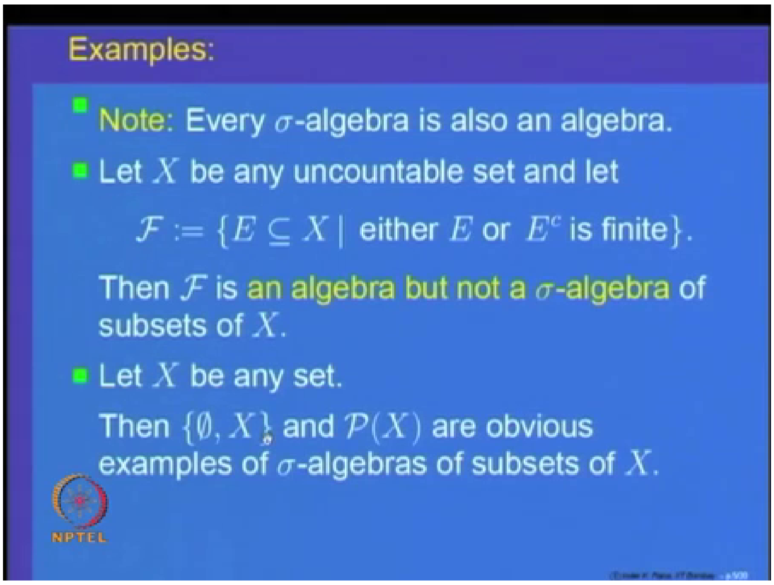

So, such a class is called a sigma algebra 1, obvious example every sigma algebra is also an algebra, because sigma algebra means it is a collection which is closed under countable unions and algebra only requires finite unions of course, both algebra and sigma algebra are closed undertaking compliments and empty set and the whole space are always members of both of them. So, every sigma algebra is also an algebra.

Let us look at an example of X, an uncountable set and let us look at the collection of all those subsets of X be such that either the set is finite or it is compliment is finite. So, an element E is in this collection F, if either the set is finite or it is compliment is finite. We are already shown that this collection F is an algebra.

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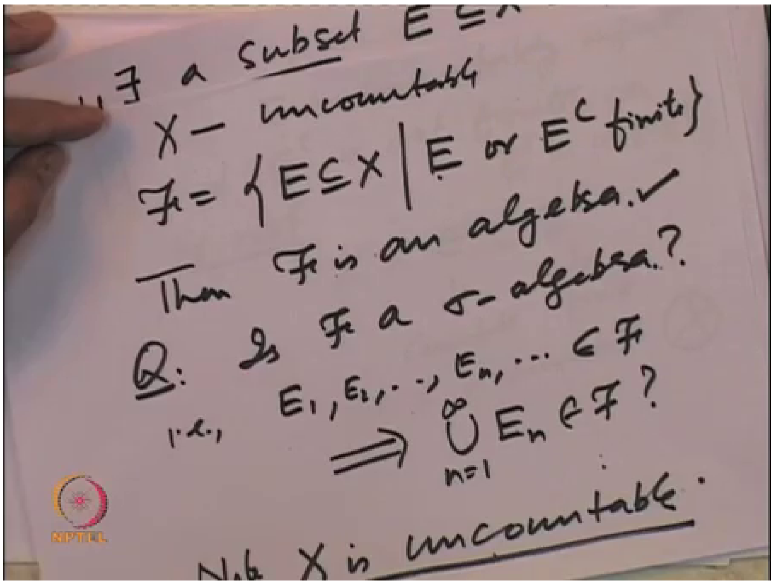

So, let us recall what we have already shown that, if I take this collection F of subsets E of X, such that E or E compliment finite, then we already observed that F is an algebra. So, the question is F A sigma algebra.

So; that means; so, does it have the property? So, that is E 1 E 2 En belonging to F does this imply always that union of En S n equal to 1 to infinity also belongs to F that is not true for the following reason.

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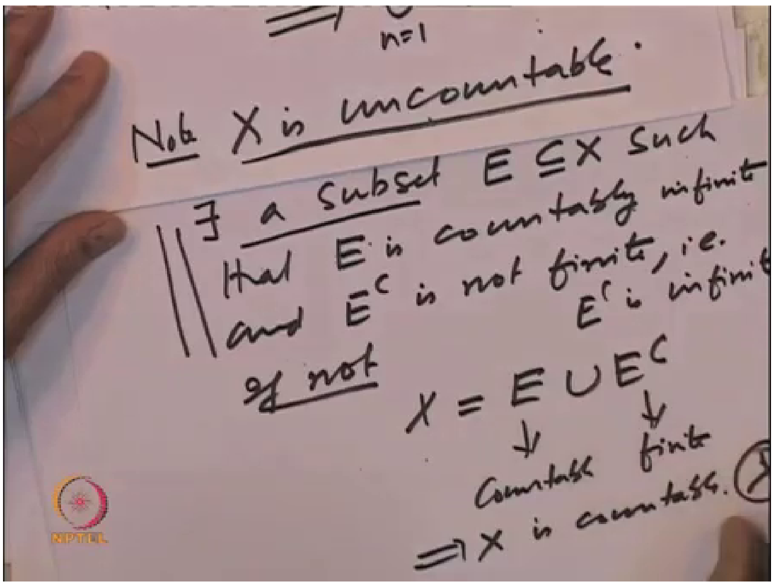

Because; so note X is uncountable. So, as a consequence of this, there exists a subset E contained in X such that E is countably infinite and E compliment is not finite, because if this is not true then what will happen, we will have X which is equal to E union E compliment this is countable and this is finite. So, that will imply X is countable which is not true, which is a contradiction. So, whenever you got a set X which is uncountable, there always exists a subset of it such that E is infinite and it is compliment is not finite, that is E compliment is infinite.

So, we have got a set E, which is countably infinite and it is compliment is not finite, so; that means, what. So, since E is countably since E is countably finite.

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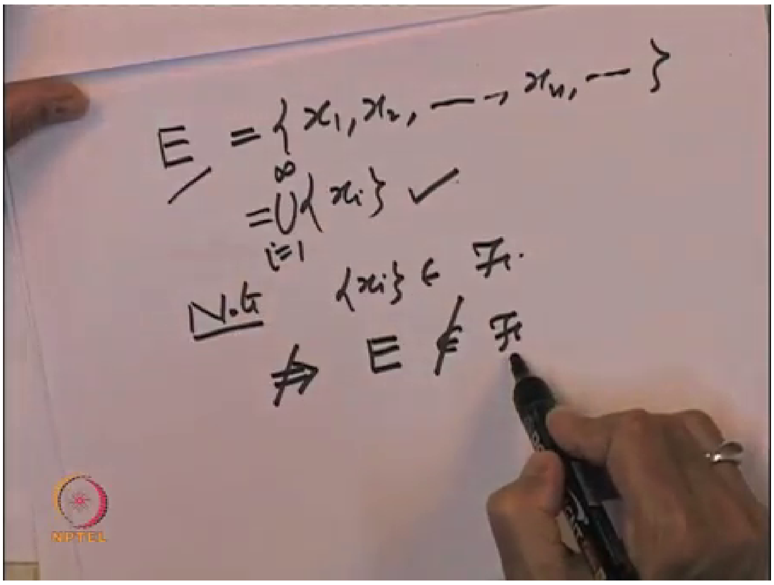

So, I can write E equal to x 1 x 2 x n and so on. So, it is a countably infinite set. So, I can write it as a sequence I can enumerate the elements of it. So, that is equal to single turn x I union I equal to 1 to infinity and note let us observe that single turn x I is an element in the algebra F, because single turn is a finite set. So, that implies that and that does not imply E, which is a union of this elements belong to F E does not belong to F why E does not belong, because if E has to be belong to this collection F E should be either finite or E compliment is finite both of them are not true.

So, basically if I take a set E, which is countably infinite then it is a countable union of elements of F and it does not belong to F. So, F is not going to be an algebra. So, what we are saying is if we look at whenever X is X is uncountable and look at this collection of sets E contained in X such that E or E compliment is finite, then it is an algebra and it is not a sigma algebra of subsets of X. So, this collection that we taken we have proved that it is an algebra of subsets of X, but it is not a sigma algebra of subsets of X. So, every sigma algebra is an algebra, but every algebra need not be a sigma algebra.

So, that is the observation that we get from here. Let us look at some more examples of sigma algebras, let X be any set then; obviously, the empty set and the whole space put together. The two elements that collection is a sigma algebra because they are only two elements. Their union belongs closed under compliments and so, on. And of course, the collection of all subsets of X the power, set of X also is a sigma algebra of subsets of X, because it is closed under all kind of operations right. So, empty set and the whole space put together is an example of a sigma algebra power. Set it is a example of sigma algebra of subsets of any set X, these are called obvious examples of sigma algebras of subsets of X.

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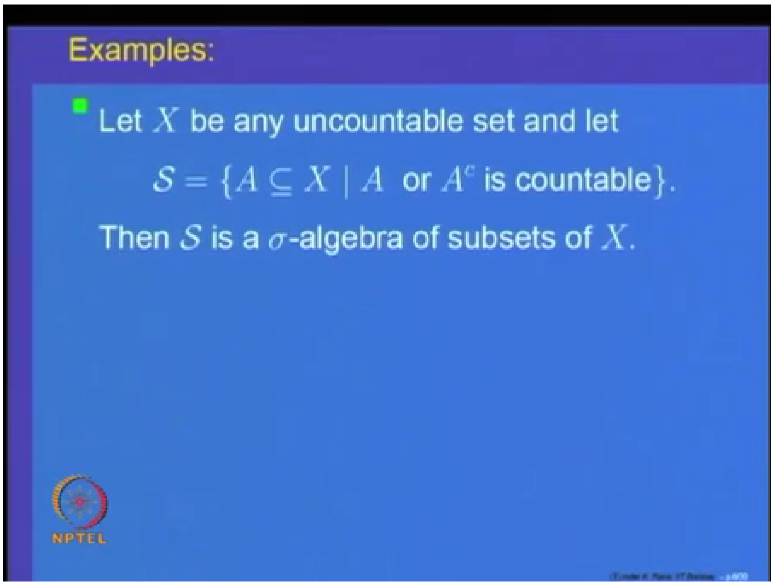

Let us look at some nontrivial examples of sigma algebras of subsets of X. So, let us take X, an uncountable set and let us take S to be a subset. Although subsets of X such that A or A compliment is countable and the claim is S, is an algebra of subsets of the set X. So, let us try to prove that, this collection S is an algebra, is a sigma algebra of subsets of X.

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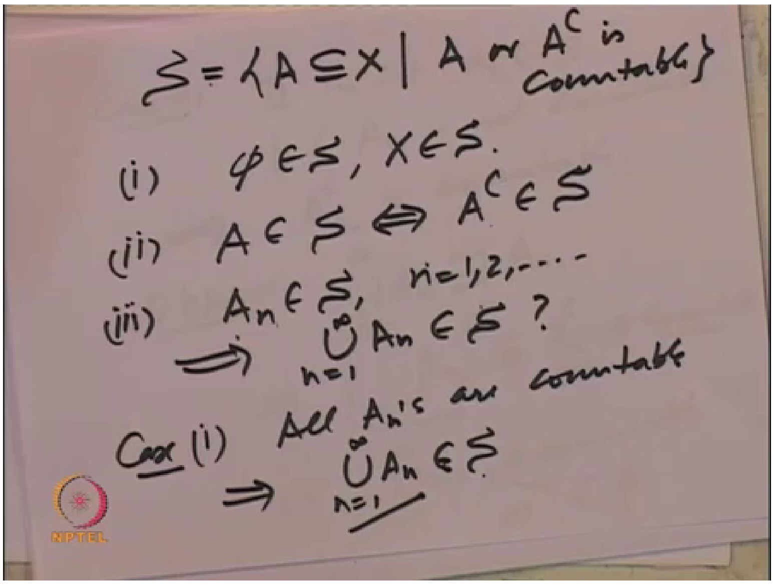

So, what is S? S is the collection of all those subsets a contained in X saying that A or a compliment is countable.

So, first observation empty set belongs to S, because empty set is taken to be a finite set. So, it belongs to S does X belong to S. Yes, X belongs to S, because it is compliment, is empty set and hence, that belongs to S. So, empty set and the whole space both belong to S, clearly if A belongs to S, then this implies actually it is, if and only if a compliment belongs to S right, because our defining condition is symmetric with respect to A and a compliment. So, let us check the third property that if A n belongs to S n equal to 1 2 3 and so on then this implies union of A n's n equal to 1 to infinity also belongs to S. So, let us check that property.

So; obviously, like in the case of finite and compliment finite, we have to divide it into cases; the first case is. So, case one all AIS or all A n's are countable, but that will imply that union of A n's is also countable and hence, belongs to S, why union of an S countable, because countable union of countable sets is countable, that is a set theory property. So, A n. So, this set is countable. So, it belongs to S. Let us look at the second possibility, the second case.

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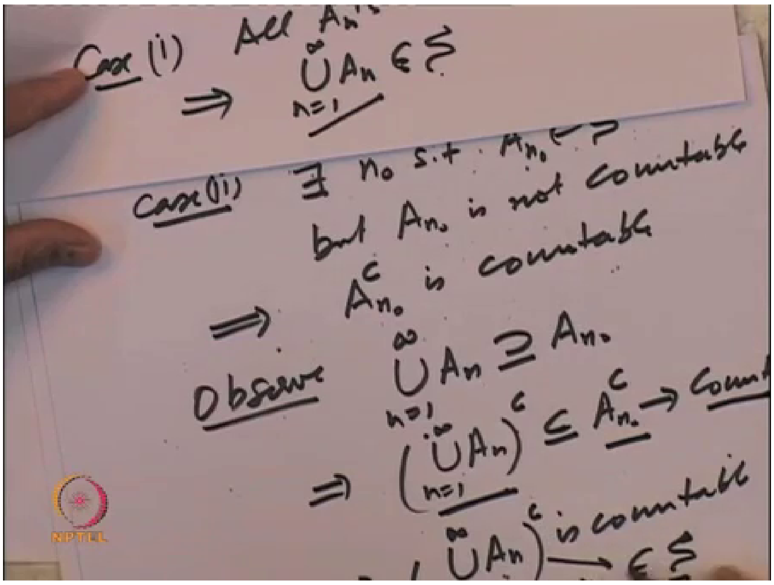

So, what is the possibility not all A n's are countable; that means, there exists some n naught such that A n naught belongs to S, but A n naught is not countable so; that means, what A n belongs to S not countable; that means, an naught complement is countable right by the definition of S and now, observe that union of A n's n equal to 1 to infinity includes the set A n naught, because that is 1 of the members. So, that implies that union of A n's n equal to 1 to infinity their compliments is contained in A n naught compliment and this is countable. So, this set it is compliment, is subset of A countable set. So, that implies union A n n equal to 1 to infinity is countable and hence. So, implying that this belongs to the classes.

So, this is a set whose compliment belongs. So, this is a set, which is whose compliment is countable so; that means, the set must belong to S right. So, this is contained in A and this is countable; that means, this set is countable and hence, because the compliment of this set is countable. So, this belongs to S. So, what we are shown is let us look at the set collection S of all subsets of A that A or A, compliment is countable then, this collection is a sigma algebra of subsets of X and let us observe way that we have not used anywhere the fact that X, the underline set is a countable set, this is true for any actually.

So, what we have shown is that if X is any set and let us take the collection of all those subsets of X, which are either countable or their compliments are countable then that form the sigma algebra of subsets of X and let us observe one thing that we have not used anywhere the fact that the set x is uncountable. So, this property even remains true when X is any set of course, the collection s still remains as sigma algebra, but it will its nature will change in the sense that for example, if X is a countable set you can try to prove yourself, then this collection of those subsets of X we say that a or a compliment is countable in. In fact, that will be in all subsets of the set X. So, it is a non trivial example only, when X is a uncountable set. So, we have given example of a set X of a collection S of subsets of a set X and which is a sigma algebra.

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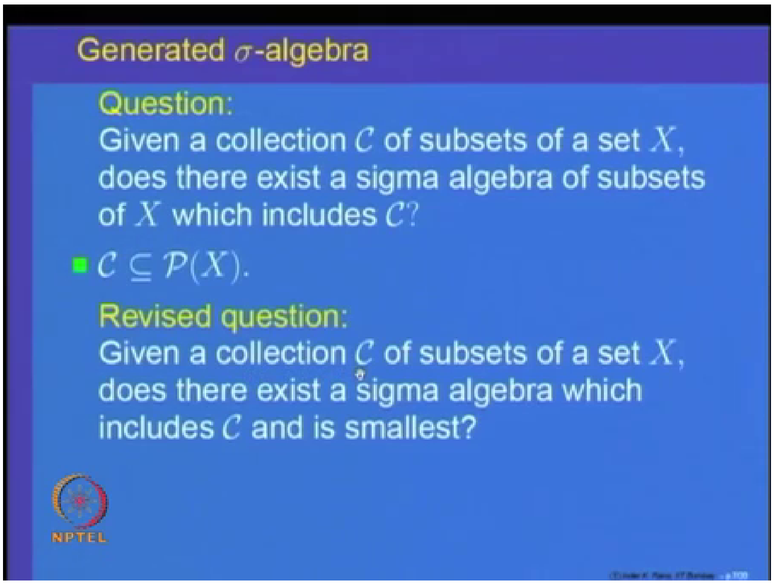

So, next let us ask the question given a collection C of subset of a set X. It may not be an algebra or it may not be a sigma algebra. So, the question arises can we say that we can find a sigma algebra of sub sets of the set X, which includes this collections C. So, in some sense, this collection C may not be closed under compliments or may not be closed under taking countable unions. So, we would like to enlarge it. So, that it becomes a sigma algebra. So, obvious examples are if you take all subsets of the set X, then that itself is a sigma algebra and that include C, but that is a very trivial example of a sigma algebra which includes C. So, we will like to modify our question that given a collection C of subsets of a set X does there exists a sigma algebra of subsets of X, which include C and is the smallest. So, let us and the A n'swer is yes and it is something similar to what we have done in for the case of algebras.

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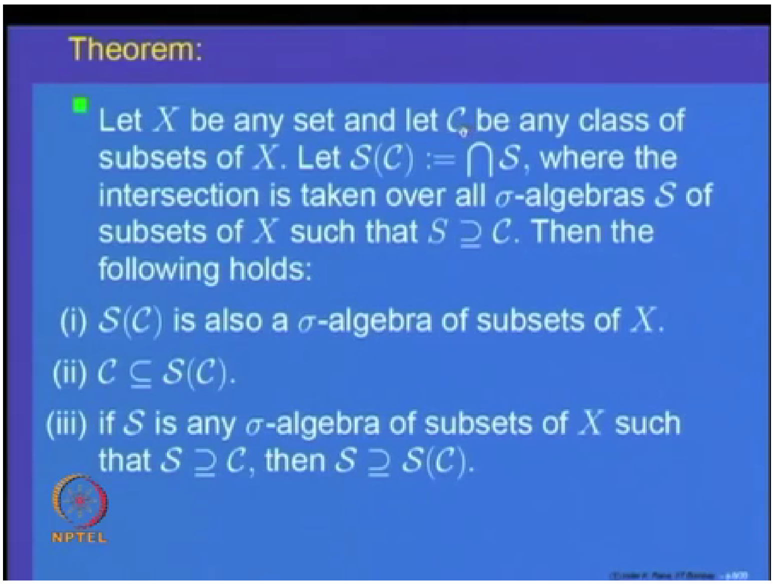

So, let X be any set and C be any class of subsets of set X. Let S of C denote the intersection of all the algebras S of subsets of X, which include C, then the claim is that this collection S of C is a sigma algebra of subsets of X. It includes the class C and it is the smallest. So, that will show that given a collection C of subsets of a set X. You can always find a sigma algebra of subsets of X which includes C and which is the smallest. So, let us check these three properties A 1 by 1. So, S of C is defined as the intersection of all the algebra S, say that S algebra and S includes C.

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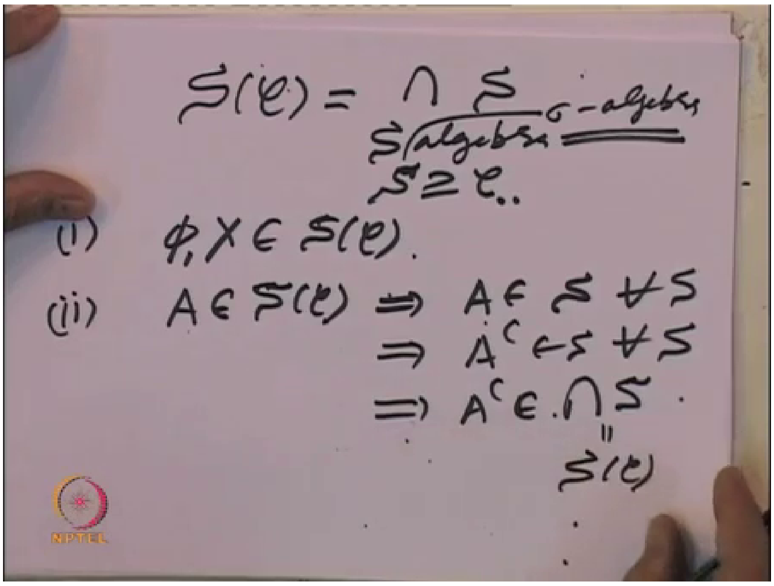

So, the first property we want to check that the empty set and the whole space belong to S of C. So, that is obvious from the fact that empty set and the whole space will belong to every algebra S which includes C right, because it is A S, is the algebra. So, empty set and the whole space belong to the every element S in this collection was intersection we are taking. So, the intersection also will have that property. So, that is a obvious property like observed in the case of algebra generated the second thing. Let us take a set A which belongs to S of C.

So, that implies that A belongs to S every oh sorry, this is a sigma algebra. So, we are looking at the case of sigma algebras right. So, S is the we are taking the intersection of all sigma algebras which includes C right. So, if A is inside the class S of C, then A belongs to S and S is the sigma algebra that implies at a compliment belongs to S for every S and that implies that a compliment belongs to the intersection of all this collection.

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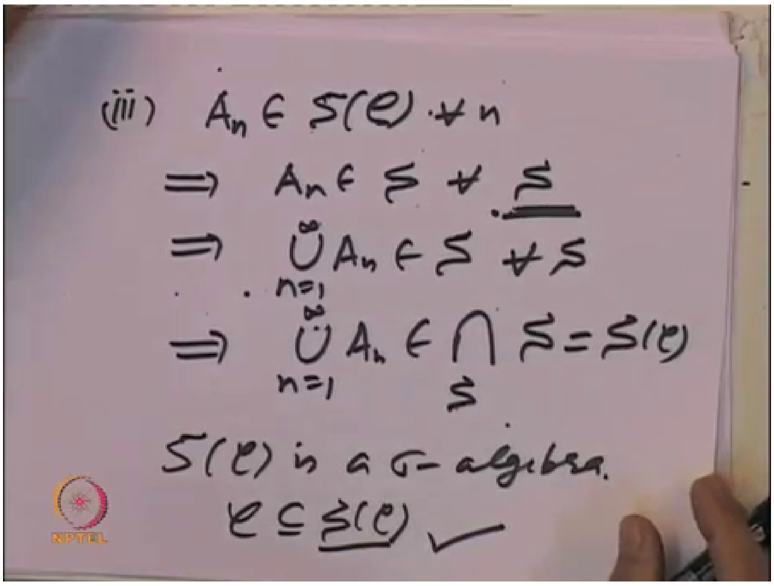

So, hence a compliment belongs to which is nothing, but S of C and similarly, let us take the collection, the third property. Let us take a sequence A n which belongs to S of C. So, that implies, write A n for every n. So, that implies A n belongs to S for every S and that implies, because this is a sigma algebra union of A n's n equal to 1 to infinity also belongs to S and that implies for every S and that implies that union n equal to 1 to infinity an also belongs to the intersection of all this algebras S over S and that is nothing, but S of C.

So, what we have shown is that if we look at S of C the intersection of all sigma algebras which include C, then they them self form a sigma algebra. So, that shown is s of c is a sigma algebra that c is contained in s of c is once again obvious because S of C is a intersection of all sigma algebras which include C. So, the intersection also will include. So, that also is a obvious property and why it is the smallest, the smallest property also is true once again by the very fact that S of C is the intersection of all the algebras. All the sigma algebras which include C S of C being the intersection is the smallest anyway.

So, that proves the fact that S of C is a sigma algebra of subsets of X C is inside S of C and if S is any other algebra which in of subsets which include C, then C must be then S of C must include must be inside S, because S of C is a intersection of all. So, intersection is inside every element.

So, what we have shown is given a collection of subsets of set X C A collection of subset it may not be an algebra, but we can put it inside a sigma algebra of subsets of X denoted by S of C and such a thing exists, because of this construction. So, such a thing is called the sigma algebra generated by the class C.

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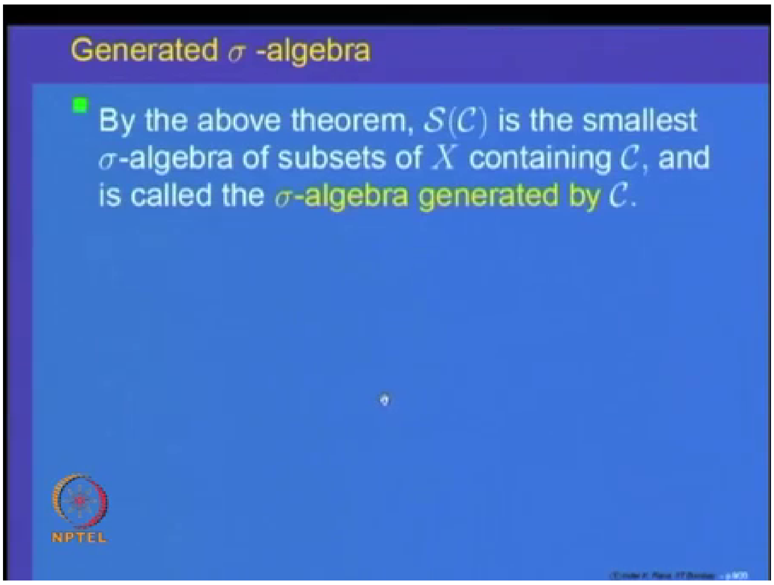

So, S of C, we are going to call it as the sigma algebra generated by the class.