Measure & Integration
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Lecture – 25 A
Computation of product Measure – I

We have been studying the properties of the product measure spaces. We define; what is the product of two measure spaces and then we started looking at the problem of how to compute the measure of an element E in the product sigma algebra. So, we will continue that study in the today's lecture. So, today's lecture the main aim going to be computing the product measure.

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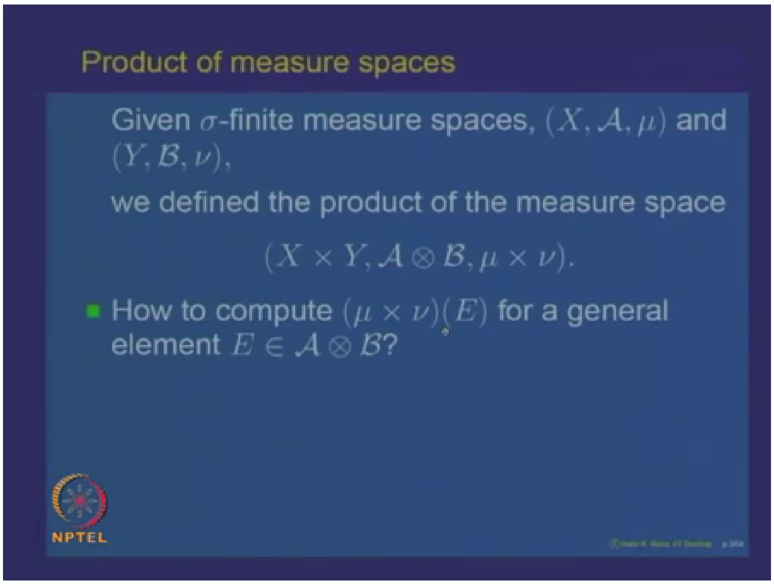

So, let us just recall that given sigma finite measure spaces X A mu and Y B nu, we had defined the product measure space namely X cross Y and the sigma algebra on X cross Y is A times B, which is the sigma algebra generated by all measurable rectangles that is sets of the type E cross F, where E belongs to A and F belongs to B and mu cross nu is extension of the measure which is defined on rectangles by the property that mu cross nu of a rectangle E cross F is mu E times nu of F and then by outer measures we extend that to the sigma algebra A cross B.

So, the problem we wanted to analyze was that given a set E in the product sigma algebra A times B. How do we compute the product measure mu cross nu of E; because at present, we only know that mu cross nu is defined on the sigma algebra A times B via the extension theory; so, to do that we had said that we should try to look at what are call the sections of the set E.

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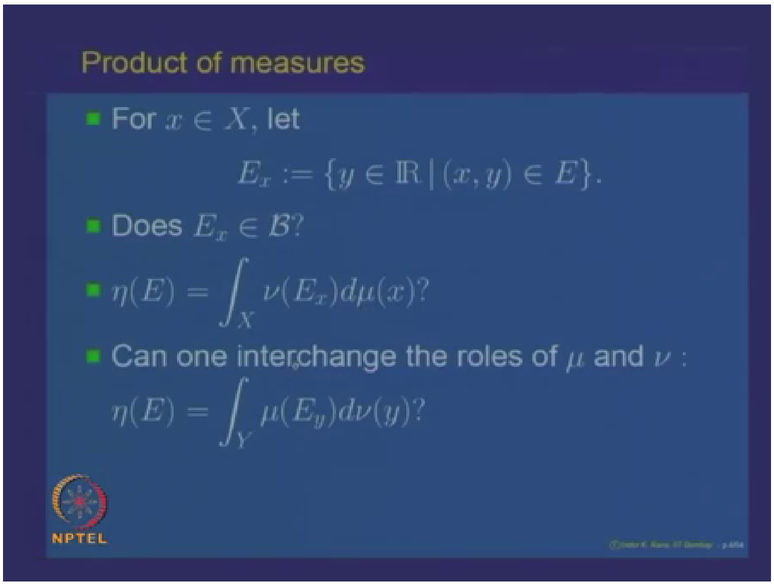

So, let us recall; what is called the section; So, for a point element X in E. E lower X. So, this is the notation used for the set of all points Y. This is not R, this should be in Y; so, all the points Y in Y, such that X comma Y belongs to E.

And similarly, we can define the section with respect to a point Y in Y. So, the main questions we had formulated in the previous lecture where can we say that this section is a element in the sigma algebra B. This is subset of Y. So, is it an element in the sigma algebra B, if yes then we can define nu of E X which depends on x. So, we get a function X going to nu of X. So, the question is that a measurable function as a function of X. If it is the measurable, we can define it is integral with respect to mu and then ask whether that is equal to the product measure eta of E.

So, and similarly can we interchange the role of X and Y. So, these are the questions we want to analyze. If this turns out to be true, then this will give us the way of computing the product measure of a set E by looking at the sections taking the new measure and then adding up or integrating and similarly, the other way around. So, let us start analyzing these problems one by one.

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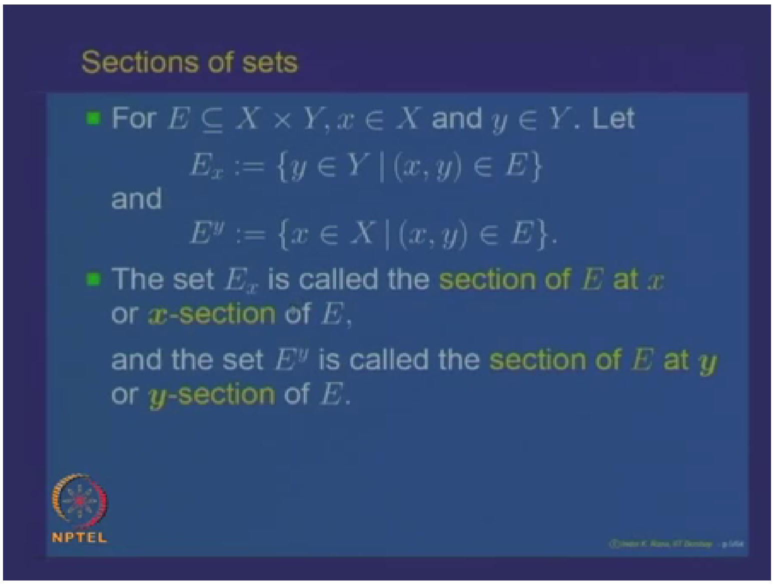

So, for the sake of once again clarity, let us note down E for any element E in X cross Y. E is a subset of X cross Y. X an element in the set X and Y. An element Y the section E lower X is defined as all points Y in Y, say that X comma Y belongs to E.

And similarly, the set E upper Y; so, whenever the point is coming from Y. We will write it on the bottom E lower script X and whenever it is coming from the set Y will write as E superscript Y on the right side as all points X in X, such that X comma Y belongs to E. So, these will be called as the sections. So, E lower X is called the X section of E at the point X and it is a subset of Y and similarly, the Y section of E or the section of E at Y is a subset of X. So, that is E upper Y.

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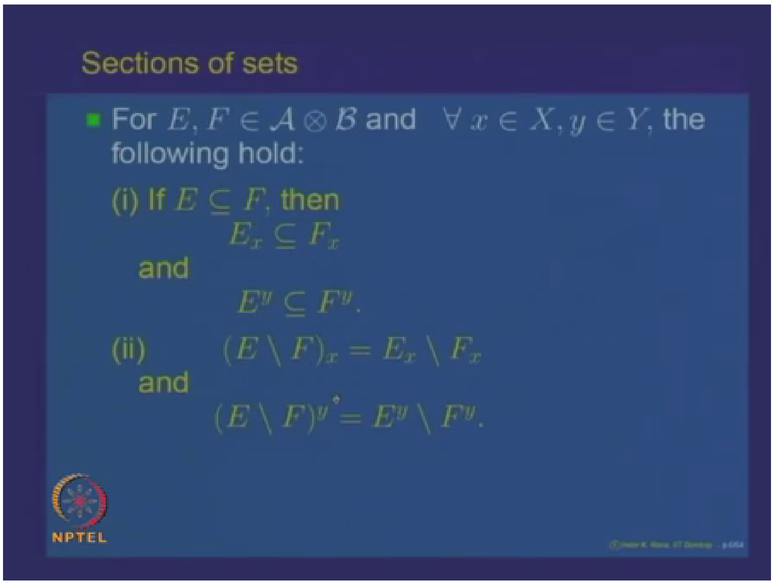

So, the questions we will like to analyze are the following and or we want to claim that the following holds. So, let us look at the some general properties of this sections the first is if E and F are subsets, such that E is a subset of F, then the section of E at X is a subset of the section of F at X and the section of E at Y is a subset of the section of F at Y. So, let us prove these properties before going further. So, let us verify namely.

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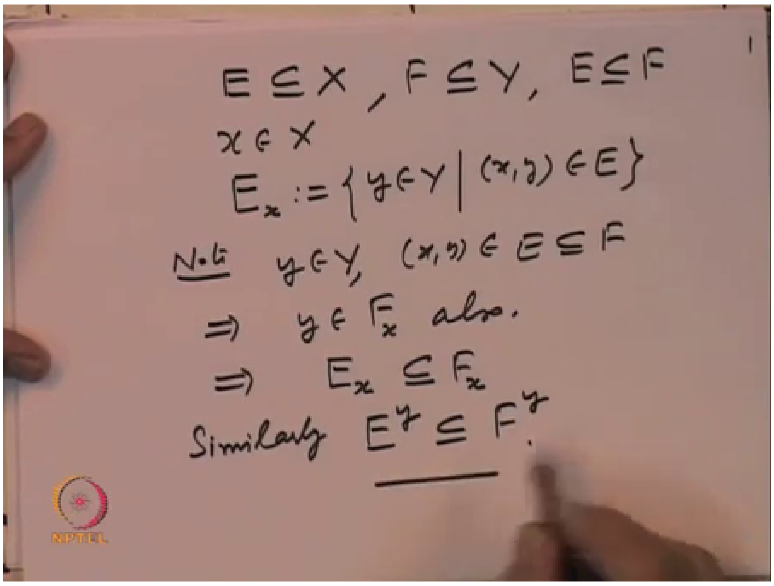

So, we are given E; E is a subset of X and F is a subset of Y. So, for X belonging to X. Let us look at what is E X; and we are also given that E is a subset of F.

So, for any point X in X. Let us look at the section E lower X. So, that is by definition all points Y in Y . So, is that X comma Y belongs to E right now, but, so, if X comma Y belongs to E and E is a subset of F so; that means, from here. So, note. So, what we are saying is note for Y belonging to Y, we have X, Y belongs to E, that is the property here and E is a subset of F. So, X comma Y belongs to F. So, that implies that Y belongs to F of X also right.

So, this is a definition of E F lower X namely Y in Y. So, that X, y belongs to F. So, this implies that E lower X is a subset of F lower X and the other property. This is similar similarly, we will have that E section at Y is a subset of the section of F at Y. So, these two properties hold. So, that is property one here we do not use the any effect that E cross F. This subsets are of in the sigma A, A cross B. This is true of any subsets the next properties says that, if we look at the difference E difference F and take it section. It is same as first, taking the sections and then taking the references whether at a point X or at a point Y.

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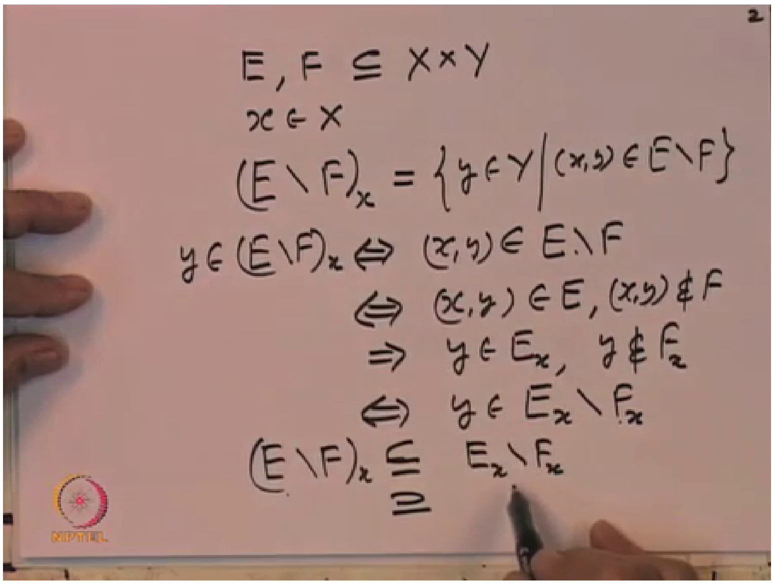

So, let us analyze this namely that if E and F are subsets of X cross Y . So, we are looking to look at the difference and the sections X is a point in X. So, let us look at the section of E minus F at the point X. So, by definition this is all points Y belonging to Y, such that X comma Y belongs to E minus F. So, what does that mean so; that means, Y belonging to E minus F section implies that X, Y belongs to E minus F, but what is the meaning of, saying that X, Y belongs to E. So, the difference F that is same as saying that X, y belongs to E belongs to E, but X, Y does not belong to F that is the meaning of this.

But that is same as saying that X, Y belongs to E; that means, Y belongs to EX and X, Y does not belong to F; that means, Y cannot belong to the section of F at X. So, that implies that Y belongs to the section of E at X, but does not belong to the section of F at X so; that means, Y belong to EX difference of F of X. So, that says that E lower X. So, that says that E difference F section X is a subset of E lower X difference F lower X, but notice in this all the arguments are reversible. So, supposing Y belongs here. So, that is same as implying the earlier statement that Y belongs to EX and Y does not belong to F section at X.

But that is same as saying X, Y belongs to E and X, Y does not belong to F and that is same as saying that the earlier statement and that is meaning that is. So, all the arguments are reversible. So, the other way round inequality also holds. So, these two sets are equal. So, it's says that if you take the difference of E with F and then take the section at X that is same as the taking the sections first and then taking the difference.

So, that is at point X. A similar proof will work for the differences at if you take the difference of E with F until the section at Y. The corresponding property says it is first taking the section and then taking the difference or the corresponding section at Y. So, basically we are saying the properties of subsets are preserved under taking sections that is a part one and the properties of taking sections is preserved also under taking differences of sets. So, whether you first take the sections and then take the difference. It is same as first taking the difference and then taking the sections.

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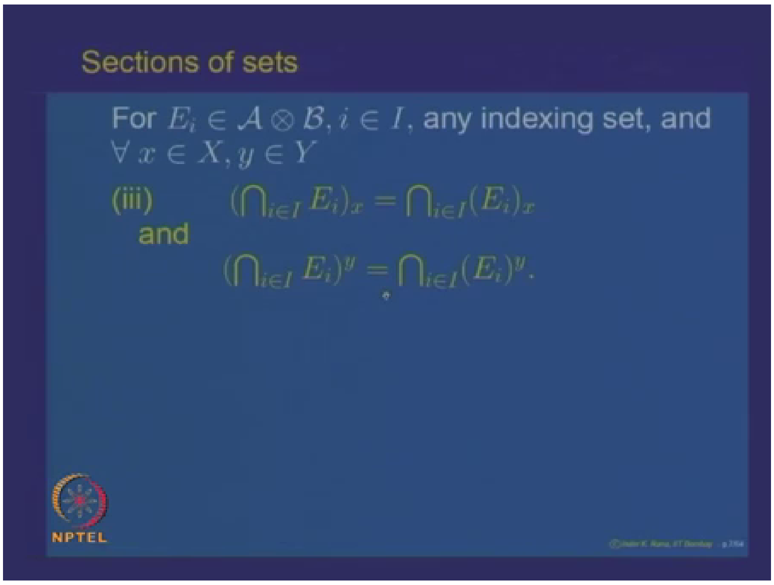

So, 2 elementary properties of the sections, let us look at some more general properties of the sections once again these are true for any sets. Not necessarily sets in a cross B, but we will be using them only for a cross B, but they are true. So, let us look at a sequence of sets not a sequence actually arbitrary family of sets E i s in subsets of X cross Y, where I is any indexing set then if you look at the interactions of the sets E i s and then take the section the claim is it is same as taking the sections first and then taking the intersections and similarly, at the point Y.

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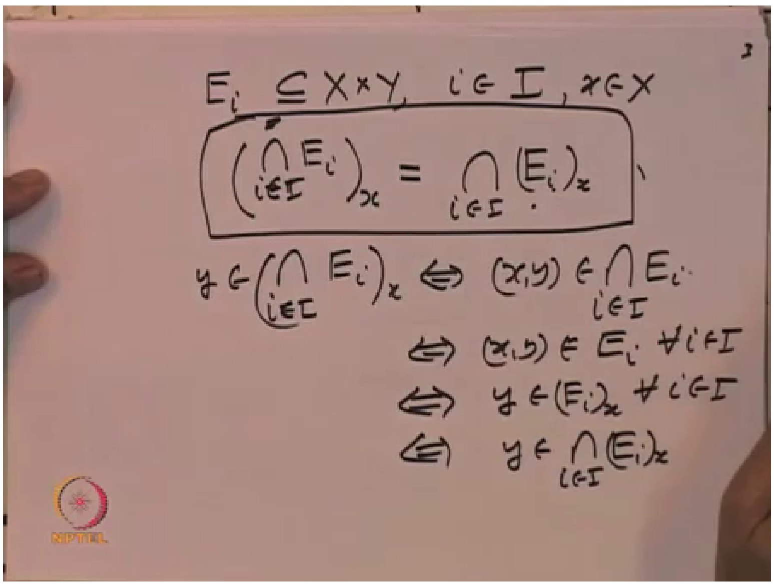

So, let us section whether at X or at Y. So, let us look at this property. So, E i s as subsets of X cross Y. I belonging to some indexing set I. So, we want to look at take the intersections of the sets EI. I A belonging to I look at the interactions of this family Eis and then take it section at point X. So, let us take a point X belonging to X. We want to compute and show that this is same as first take the section of every set E i at X and then take intersection I belonging to I.

So, to show this let us take a point. So, Y belonging to intersection i. I in i of E i at X. So, let us take a a point Y in this set on the left hand side. So, that is if and only if the definition says; that means, X, Y belongs to intersection of I in I of the sets EI, but if a point belongs to intersection that is; that means, that the point belongs to each one of the sets. So, it belongs to E i for every i belonging to I and once that is true. So, that is, but the saying that X, Y belongs to E i that is same as saying Y belong to E i x for every i belonging to i. So, Y belonging to the intersection section is same as Y belonging to intersections. So, for every i; that means, Y belongs to intersections of the sections of E i that X I belonging to i.

So, that proves that this property is true. So, what we have shown is that, this property is true namely for any arbitrary family of subsets of X cross Y. If you take the intersections of these sets and then take the section at a point X, it is same as first taking the sections and then taking the interaction of those sections. So, this is at a point X in X. A similar proof will work for the sections at Y namely for every Y in Y. The section of the interaction is same as intersection of the sections.

A corresponding result also is true for the unions. So, let us prove that also once again the proves are similar in all these cases is only a matter of is pure set theory actually. So, what we want to do is E i s.

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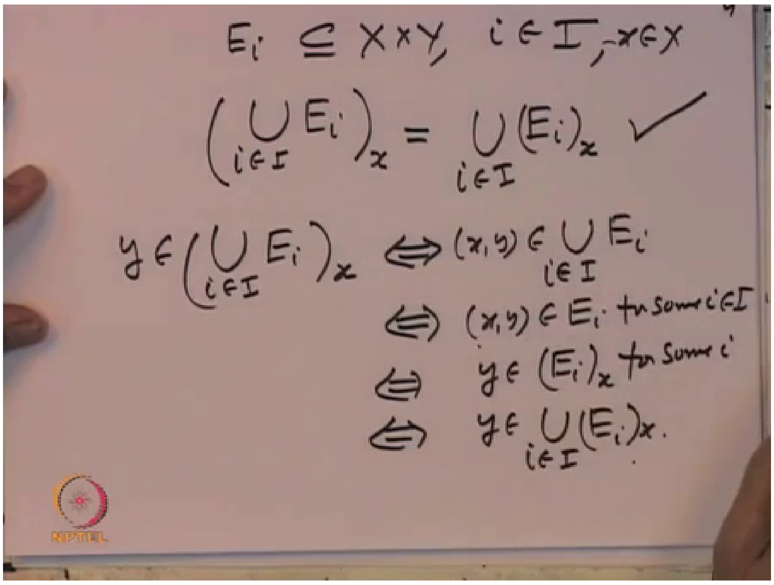

Are subsets of X cross Y, where i belongs to i. So, what we want to do is we want to look at the union of this sets Ei. i belonging to i and then it take it section at a point X. So, let us take a point X in X we want to show. This is same as for each one of the EIS take the section X at a point X, that X we have fix and then take it's union over i belonging to i. So, section of the unions is equal to union of the section. So, that is what we want to prove. So, to prove that, so, note. So, Y belong to the left hand side of the set. So, that is i belonging to i union E i and it section at X.

But Y belong into the section at a point X is same as saying that the point XY belongs to the union of E i s. I belonging to i, but the definition of saying that a point belongs the union means at least it should belong to one of them. So, X, Y belongs to E i for some i belonging to i, but that is same as saying XY belongs to EI; that means, Y belongs to the section E i at X for some i, but that is same as saying it belongs to the union. So, it belongs to at least one of the sections of EIS; that means, Y belongs to union i belonging to i of E i at the point X. So, Y belonging to the section of the union if one only if Y belongs to union of the sections; so, that proves that this property is true namely section of the unions is union of the sections at a point X in X and a similar property holds section of the unions at a point Y in Y. So, basically what we are saying is all the set theoretic operations, behave nicely with respect to taking sections and this is true for all subsets E i s of X cross Y.