Measure & Integration
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Lecture – 24 B
Construction of Product Measures

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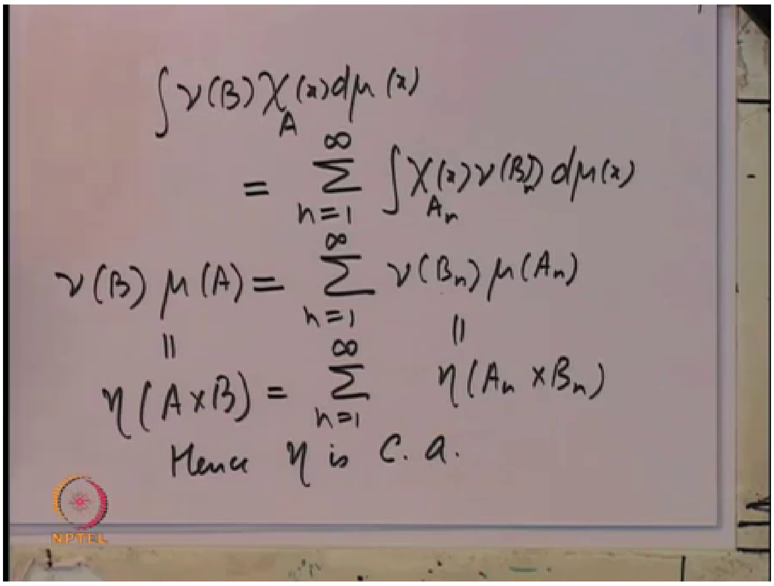

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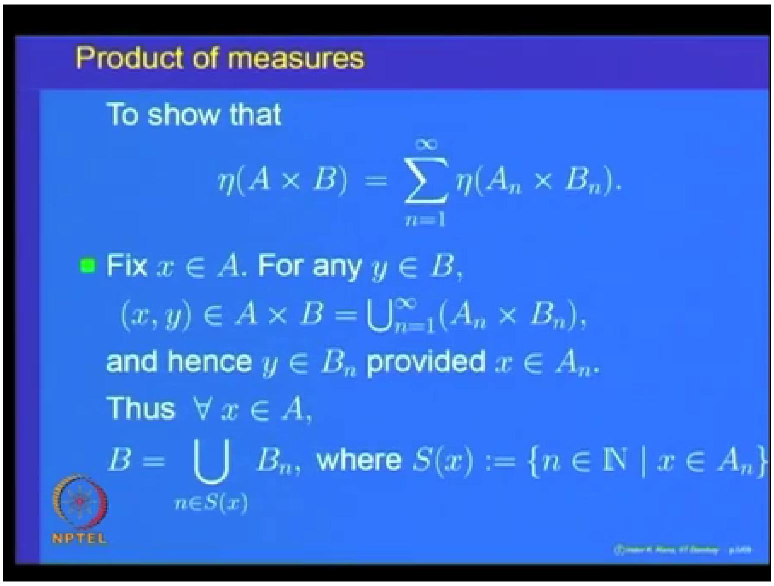

So this is how one proves that eta which is defined.

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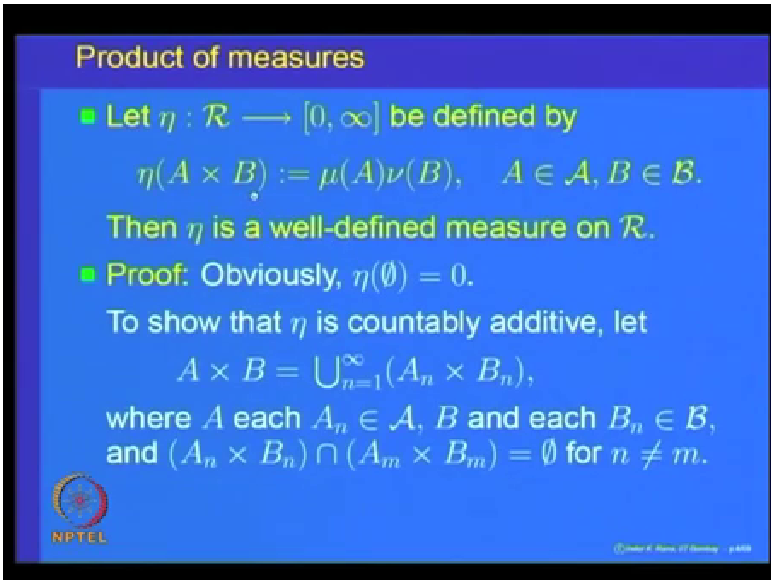

So eta which is defined as eta of rectangle equal to a rectangle A cross B to be mu of A into nu of B is countable additive. So, let me slowly go through the proof once again that is only one small idea involved in it, and other is rest is straight forward applications of the earlier results. So, let us write A cross B so I am going through the proof once again that A cross B is written as a countable does not union of rectangles A n cross B n, and what we want to show that eta of the rectangle A cross B, is equal to summation of the measures of the rectangle each rectangle. So, summation over n of eta A n cross B n.

So, to prove this what we do is as follows, look at the set A cross B so, fix any element x belonging to A then for any y belonging to B, we know that x comma y belongs to A cross B which is nothing, but union of A n's so; that means, x comma y they will belong to exactly one of them ok, but which one of them. So, x comma y will belong that A n comma B n, whenever this x belongs to that A n, because x comma y belonging to A n cross B n implies x must belong to A n and y must belong to B n; that means, what we are saying is, x comma y belongs to A cross B, if and only x belongs to A n and for that x the y should belong to B n because x is fixed so, that n is fixed, so what are those hence which are fixed, so y belongs to B n provided x belongs to A n.

So, for a fixed x collect together those ends, so confined find the set S of x all those indices n such that x belongs to A n's, see an are not disjoint. So, x can belong to more than one of the A n's. So, look at those if x belongs to A n, but for a fix x it will belong to only one of them. If x belongs to A n then y will belong to B n. So, as x varies, over as x varies over A for every fix x you will get a collection of B n's. So, what are those B n's those B n's are index by n belonging to S of x. So, that x belongs to A n and this union is a disjoint union.

So, for every x fix in A we can decompose B into a disjoint union of B n's over those ns such that n belongs to S of x this being a disjoint union because A n's, B n's are disjoint we get the our first equality that for any fixed x in A nu of B is summation nu of B n's or those n's which belong to S of x.

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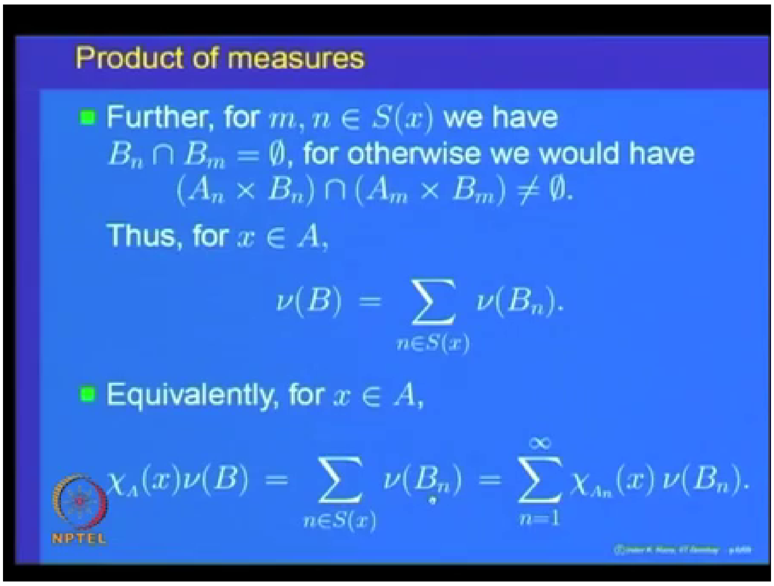

And, now we observed that equivalently this thing like we can write it as, nu of B, I can multiplied by the indicator function of A, right because x belong to A so this will be equal to 1, and this nu of B n, I can multiply if x belongs to A n; that means, n will belong to S of x so, I can multiply here by the indicator function of A n, if n belongs to S of x. And if x does not belong to A n; that means, it cannot belong to any one of the A there this all the remaining terms here will be 0, and all the remaining this side is also equal to 0.

So, what we are saying is for any x in A, I can write this is equal to this, and this equation make sense when our x does not belong to A also, because we have x does not belong to A this side is equal to 0, and that side x does not belong to A, so it does not belong to any of the A n's; so all the terms are 0. This equation first we can write it has indicator function of A times x nu of B, is equal to summation of A n's and now we realize that not only this equation is valid for x belonging to A, this equation is valid for all x in x.

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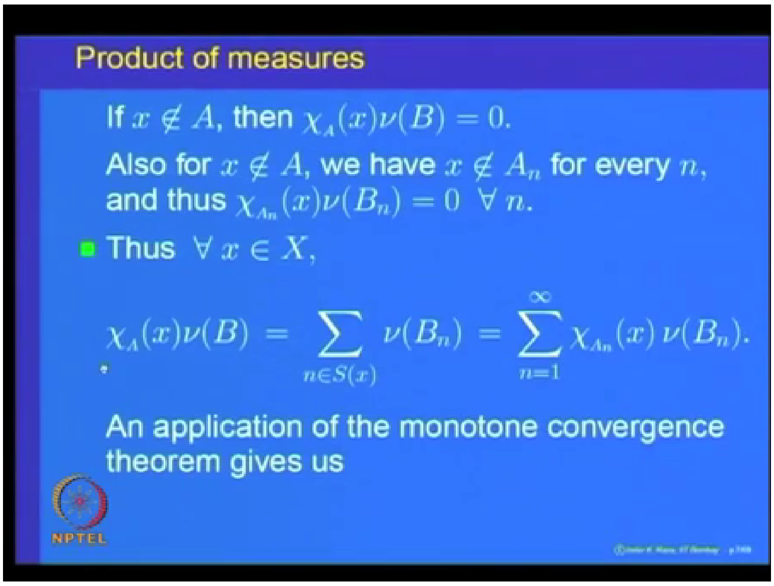

So, once that is observed. That is what is observed here, so what we get is that the equation chi of A x nu of B is equal to the, summation chi of A n nu or B n for all x. And now this is the equation about non negative measurable functions so left hand side is non negative measurable function, which we can realize as a limit of non negative measurable functions, namely the partial sums of the series, and apply monotone convergence theorem, so that will give us that the integral of the left hand side is equal to summation of the so, I can take the integral sign inside by monotone convergence theorem.

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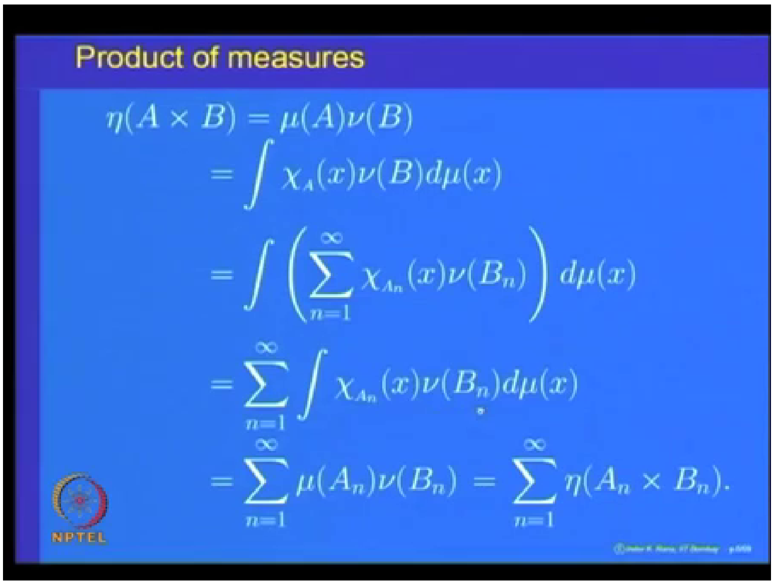

And say that integral of chi A x nu B d mu y is nothing, but integral of this summation and so, here is the application of the monotone convergence theorem, I can take this integral inside. So, that is equal to summation of integral of indicatory functions, and now you just matter of writing down the values of this nu of B n is a constant, so goes out. This integral is nothing, but mu of A n so, and that nu of B n and the left hand side this was integral of chi of A nu of B nu of B is a constant. So, that is integral of mu of chi of A, with respect to mu so that is mu of A.

So, that gives us that eta of A cross B is equal to summation eta of A n cross B n, whenever A cross B is a disjoint union of rectangles.

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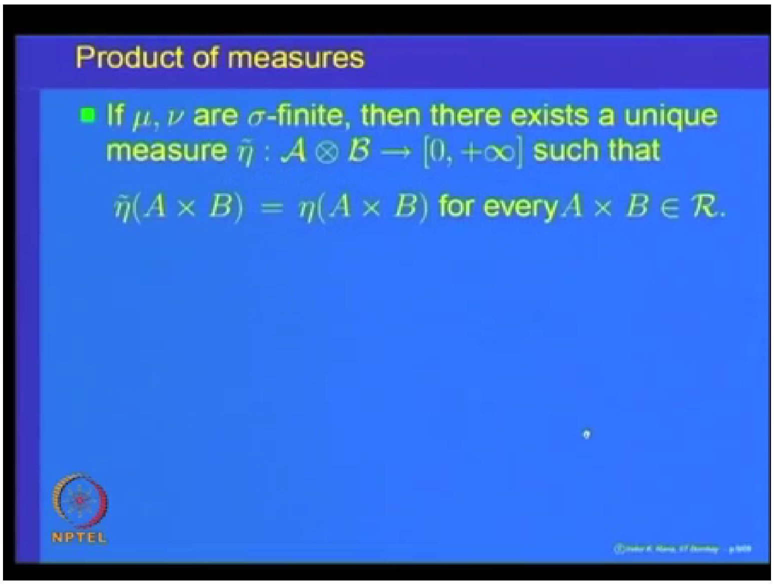

So, that proves, that this is eta is a countable additive function. So, what we have gotten is eta is a countably additive function.

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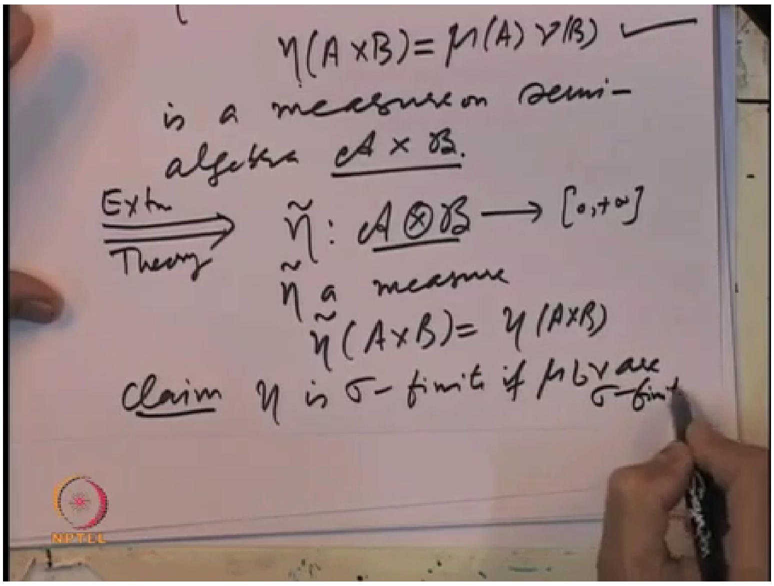

So, let us just observe, so what we got is we got eta defined on A cross B, 0 to infinity by eta of A cross B equal to mu of A nu of B is a measure, we got that this is a measure on the semi algebra; so, this is important on the semi algebra A times B.

So, implies by our general extension theory, via outer measures and so on. We can extent, we can define eta tilde on a times B define eta a measure; eta tilde a measure, and eta tilde of A cross B to be equal to eta of A cross B; that means, this eta can be extended way outer measures to the sigma algebra generated by A cross B the semi algebra A cross B. And if we recall we had said that this extension will be unique provided this eta is a sigma finite measure. So, we claim eta is sigma finite if mu and nu are sigma finite. So, we want to show n x that if A n B if mu and nu are sigma finite.

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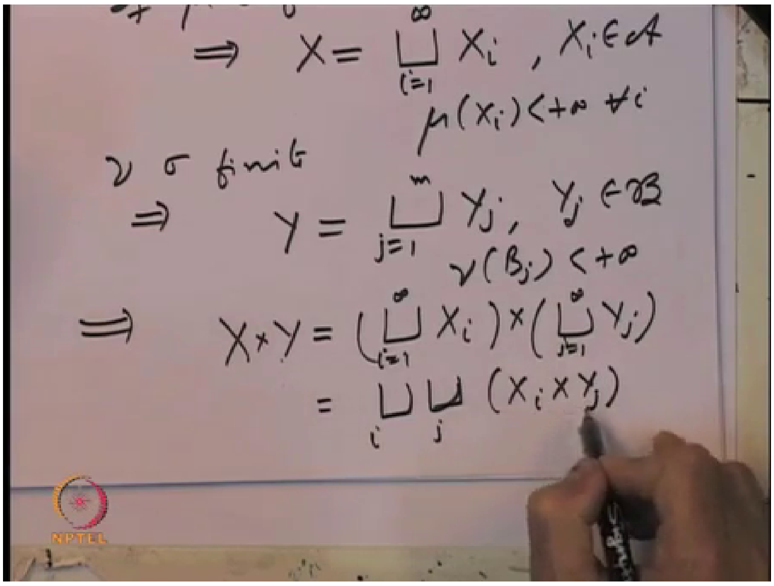

So, let us assume, if mu sigma finite. So, that implies I can write x as a disjoint union of sets X i want infinity each X i in the sigma algebra A. And mu of X i finite for every i, and similarly nu sigma finite implies, I can write Y as disjoint union of sets of Y j where each Y j is an element in the sigma algebra B and nu B j is finite, but then this implies we can write X cross Y, as disjoint union of x is cross disjoint union of Y j's. And, now it is a just a simple method to set the in quality equality namely, this is same as the unions over i unions over j of rectangles X i cross Y j, right because if X comma Y belongs here; that means, X belongs to the union X i's and Y belongs union Y j so; that means, X belong to only one of X i, and only to one of Y j. So, it belongs here and conversely.

This is a disjoint union and now we only have to observe the fact that so, X cross Y has been decomposed has been decomposed into a disjoint union of sets X i cross Y j and we only note.

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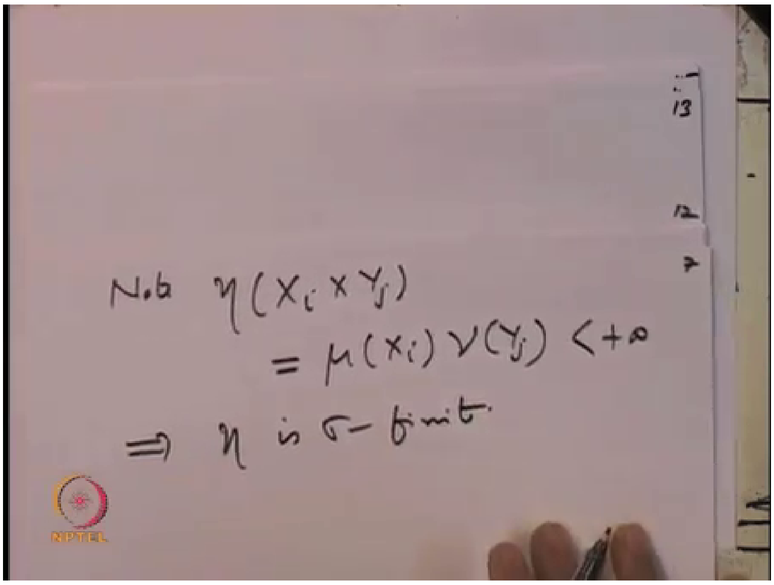

That eta of X i crosses Y j it is a rectangle. So, it is measure is mu of X i times nu of Y j and both of them being finite so, this is a finite quantity, so X cross Yis written as a disjoint union of sets X i cross Y j and each piece as got finite measure. So, that implies eta is sigma finite.

So, the measure eta is sigma finite on the rectangles and hence has A, a unique extension to the sigma algebra so, this is what we wanted to prove, that eta that extension is also sigma finite.

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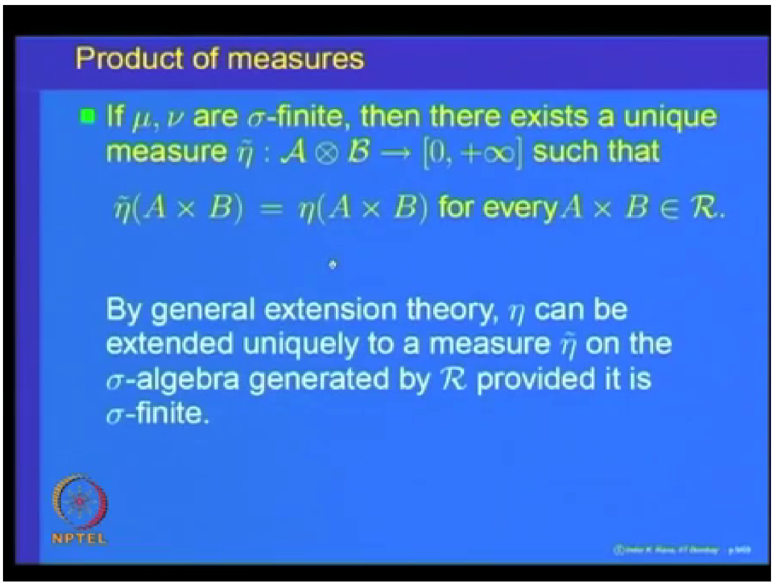

So, general extension theory gives me a unique.

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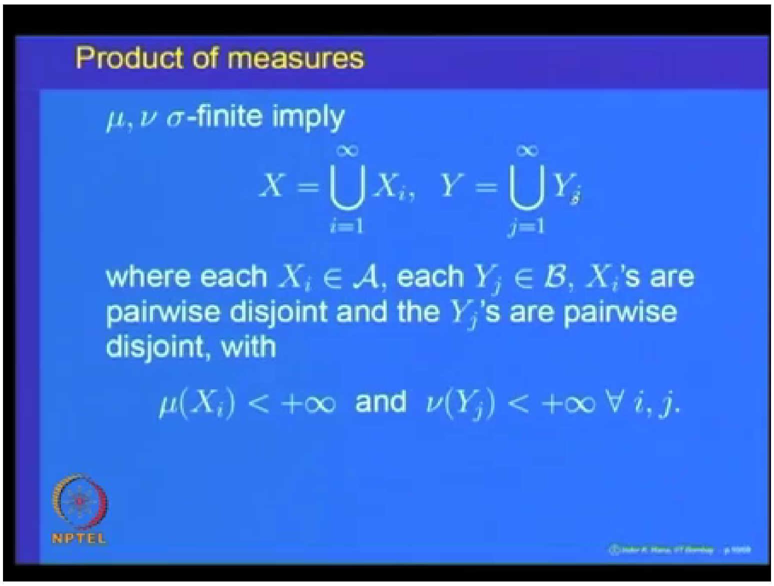

So, that mu and nu are sigma finite, so that implies X is disjoint union Y is disjoint union. So, we can write X cross Y as a disjoint union of your rectangles X i cross Y j that is what I just now illustrated and each piece has got a finite measure.

So, by that process we get a eta is sigma finite on rectangles. So, by extension theory eta can be extended uniquely. So, that is the important thing, eta can be extended uniquely to a measure on a on the product sigma algebra. So, that for rectangles it is the product.

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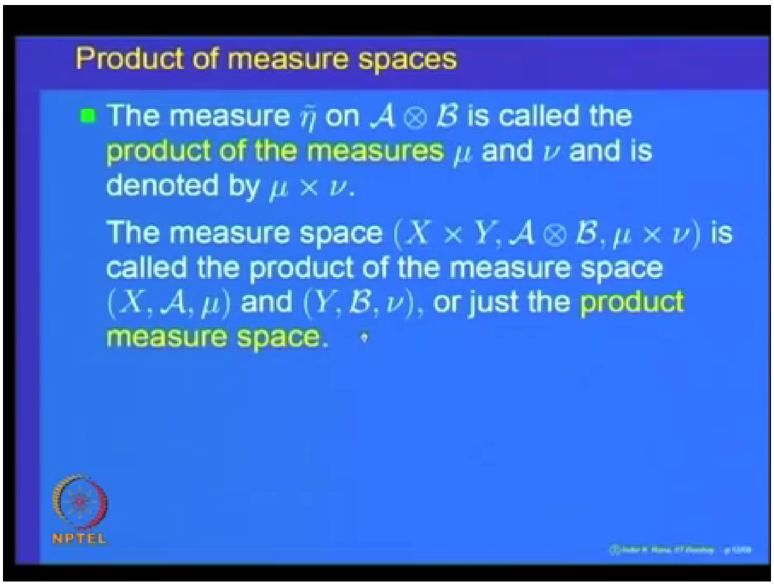

So, this is the measure eta which is defined on a times b on the product sigma algebra is called the product measure and is normally denoted by mu cross nu.

So, let us summarize what we have done we started with the 2 measures space is X, A mu and Y, B, nu and for the product set X cross Y we first define the rectangles namely sets or the type A cross B where A belongs to the sigma algebra A and Y belongs to the sigma algebra B. So, that gives us sets of subsets of X cross Y called measurable rectangles, they only form a semi algebra. So, we extend we generate the sigma algebra by this semi algebra of rectangles and call that as the product sigma algebra denoted by A with circle cross A times B. And now given measures mu on the sigma algebra A and measure nu on the sigma algebra B, we want to define a measure on the product sigma algebra. So, that is done by defining the product for a defining this is the new measure first on rectangles, so eta of the rectangle A cross B is defined as the product of mu of A and nu of B and we show that this is a measure.

So, this becomes a measure on the semi algebra of rectangles and if it is sigma finite; that means, if you assume that the given measures mu and nu are sigma finite, then this extends uniquely to a measure on the product sigma algebra A cross B. And that measure is called the product of the measures mu and nu. So, given two measures space is X, A, mu and Y, B, nu which has sigma finite we get the product measures space X cross Y, the sigma algebra A cross B generated by the rectangles and the product measure mu cross nu go obtained via the extension theory. So, this is the product measure space constructed as just now said.

So, now the next problem we want to analyze is the following namely, this product measure mu cross nu that we have gotten, is obtained Y extension theory, but it does not tell us how does one compute the product measure mu cross nu of a set in A cross B. So, that is not indicated because we are making use of the extension theory. So, next problem that we want to analyze is the following, so namely we have got the product measure space X cross Y the product sigma algebra A times B, and the product measure mu cross nu.

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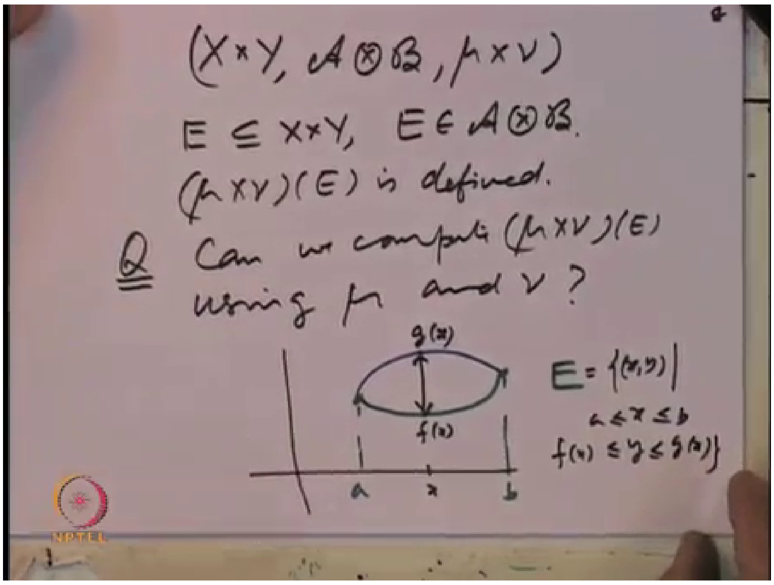

So, let us take a set E contained in X cross Y which is of course, E is an element in a times B. So, mu cross nu of this set E is defined. So, the question is can we compute this quantity mu cross nu of E, using mu and nu. So, that the question, and there let us just recall something from our elementary calculus, supposing in the plain we have got a set which looks like the flowing, it looks like this is a set. So, this is a set E which looks like the following namely here is a point A, and here is a point B.

So, the set E looks like, so let us just write what does E look like, E is equal to all X cross x comma y, say that x belong between a and b, and y so, at E point x if I look at y, this is the portion of y. So, it starts with green boundary, so y is bigger than or equal to some function f of x that is a green curve and less than or equal to here is g of x.

So, this is what we call in calculus or elementary analysis, sets of type 1. And for such sets we can we can find out what is the area.

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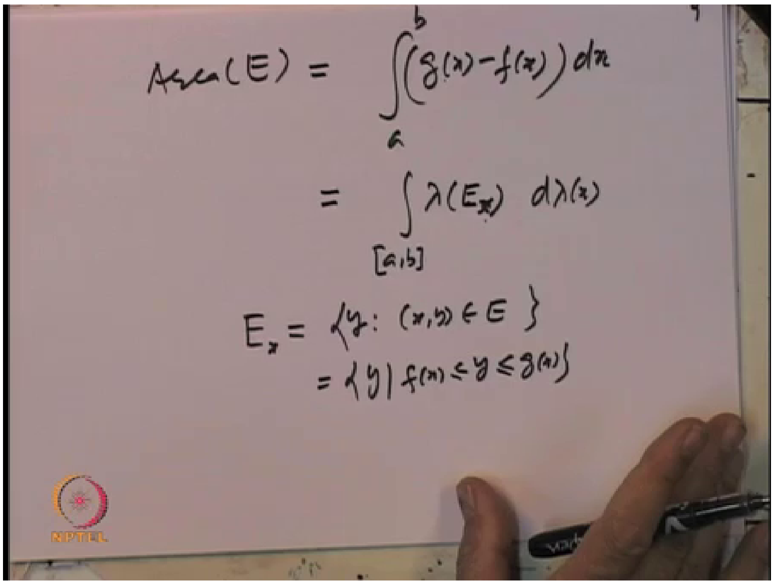

So, area of the set E if you recall from calculus, it can be obtained as you look at this difference height, what is this i, so that is nothing, but g of x, minus f of x, and integrate that from a to b, d x. So, Riemann integral as an application of Riemann integration we do that, we define it to equal to this; but now let us rewrite this; this I can write it as these are Riemann integral; so, Riemann integral I can write integral over a, b of d lambda with respect to Lebesgue measure, and what is g x minus f x, that is precisely the Lebesgue measure of this height.

So, Lebesgue measure of let me write as E x, what is E x, E x is equal to all y such that x comma y belongs to E, which is same as all y such that y is between f x and g x right. So, that is set I am writing it as follows, so I am writing as Lebesgue measure of a notation called E x. So, you can think of that look at this set x, let look at that set E to find its area, we are just adding up the areas of these small strips. So, I can think it as that way, and that is what this integral seems to indicate.

So, we will like to generalize this in the case of our construction the same idea we want to generalize it. So, here is what we want to do.

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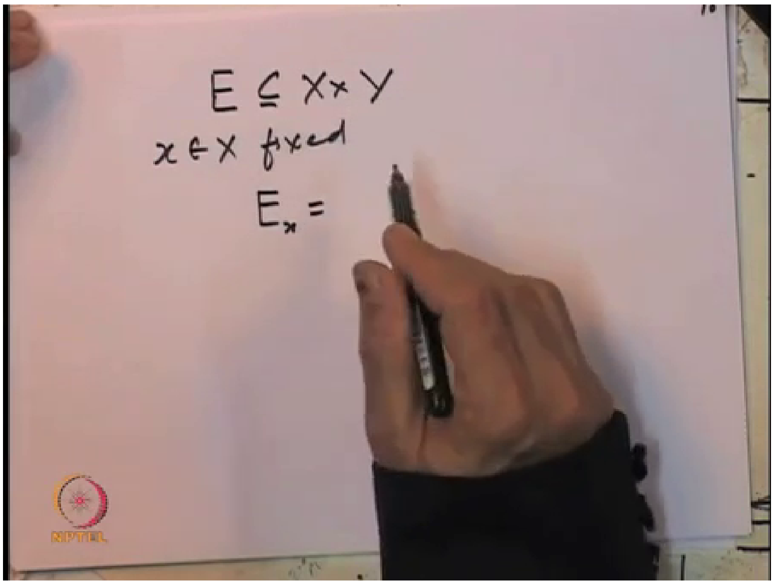

So, given a set E in X cross Y for x belonging to x fix. Let us look at E x that is so, here abstract now, x is abstract set y is a some abstract set.

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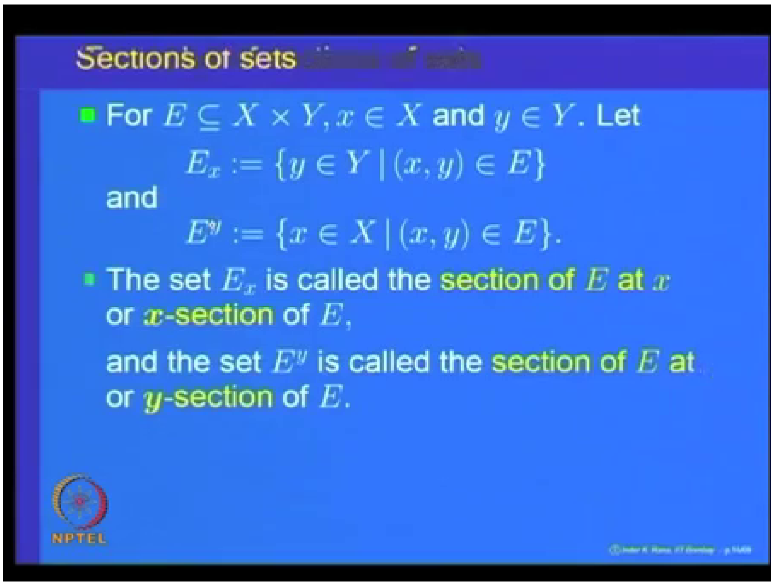

So, look at all those points y belonging to y, section is the part of the horizontal line and the E x section is part of the vertical line.

So, as I said E x is called the section of E at, x or just the x section of E, and similarly E y this set E y is called the section of E at y, or just the y section of E.

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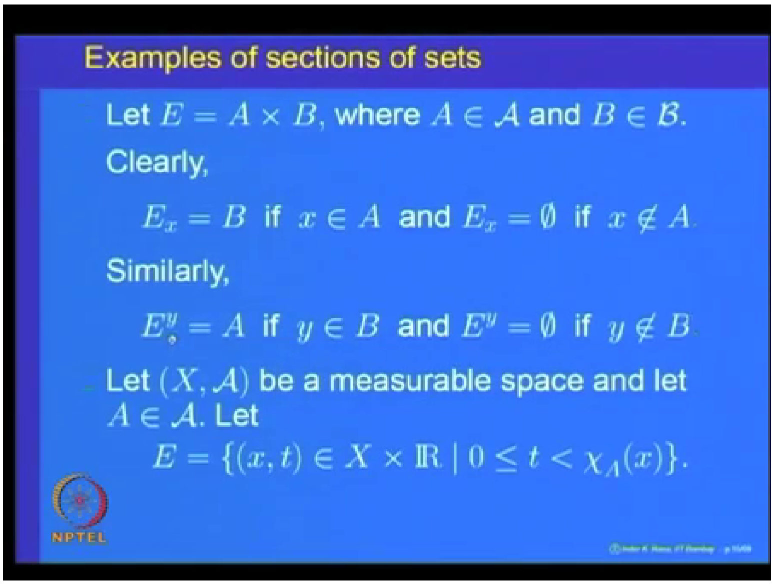

So, here are some simple properties we want to verify for these sections. So, first of all we want to verify, and let us look at some examples, first let us take a set E which actually looks like a rectangle. In this X cross Y let us take a actually rectangle A cross B where A belongs to A and B belong to B.

So, then for any x in A, if for what are the points y says that x comma y will belong to E, that is means y must belong to B. So, E x the x section of E for a rectangle A cross B is nothing, but B the set B itself if x belongs to A, and if x does not belong to A, then the point x comma y is never going to belong to E. The x section is empty set. So, here is a simple observation that for a rectangle A cross B, the x section is equal to the set B if x belongs to A and it is empty set if x does not belong to A.

Similarly the y section of E or the section of y at a point y in y so all x such that x comma y belongs to so, if y belongs to B then for all x in A, x comma y is going to belong to E so; that means, the y section of e is equal to A if y belongs to B, and it is a empty set if y does not belong to E. So, for rectangles it or it these are very easy to compute what are the sections, for a rectangle A cross B the x section for x belonging to a is B otherwise empty it is only the y section equal to A, if y belongs to B otherwise it is empty.

Now, let us look at another example, so let us take a measurable space x A and look at ordered pairs x comma t. So, t belongs to R such that this t lies between the evaluated indicator function of A at x. So, we are looking at the order pairs x comma t such that for every x t lies between 0 and A, and X belongs to X. So, what are the sections of the set E, this is a subset of A cross B ok, and where B is y is the real line. So, it is subset of x cross R we want to find it sections.

So, let us observe that for a point x in A, if x belongs to A, then this indicator function of A the value will be equal to 1. So, t will be between 0 and 1, so if x belongs to a, then t will be between 0 and 1. So, the section is going to be the interval 0 1; 1 not included and if x does not belong to A, if x does not belong to A, then this is going to be 0 so, t is going to be the singleton 0, so the section if x belongs to A. So, the section depends on whether x belongs to A or not. The section of E at a point x is equal to the interval closed interval 0 open at 1 in R if x belongs to A otherwise it is the 0 set or another way of looking at this is the following.

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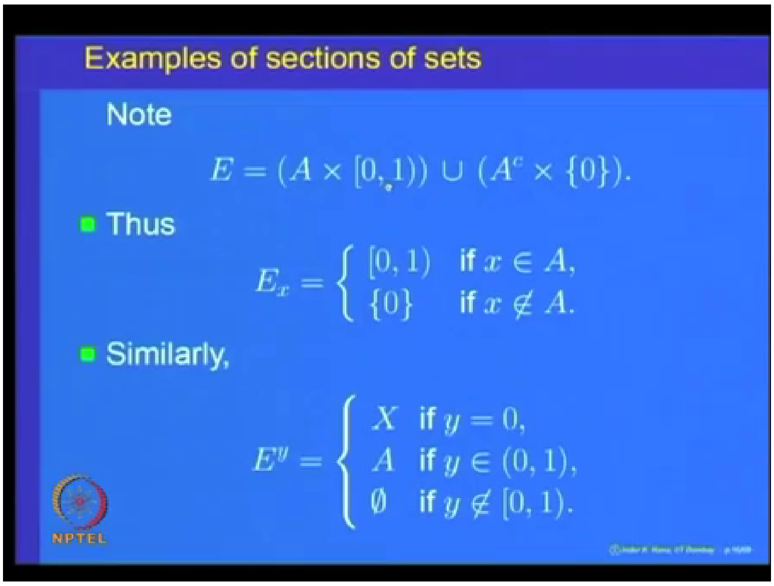

That the set E, I can write it as A cross the interval open closed at 0 and, open at 1, union a compliment of this. A compliment cross was singleton 0, this is another way of writing the same set E as I explained just now, so the section so now, is a union of two disjoint rectangles. So, section in the first case even x belongs to A section is going to be 0 1. And in the second case the section is going to be the single ton 0, if x does not belong to A, so these are x sections we can similarly find the y sections so, for y belonging to 0 to 1; that means, y is the real line. So, for a real number between the closed at 0 at open at interval it is going to be A at 0, so and if y is equal to 0 then, this going to be the whole space x and empty set. So, this is easy competition from this it follows.

So, this is how one computes the sections of these sets. So, these sections are going to play important role, in computing the measure of a set E in the product space. So, in the next lecture we will analyze the x sections, the y sections, various properties of these sections, under compliments intersections, and unions, and then show that each section for a set E in the product sigma algebra each section is again a appropriately measurable set, whose measure can be defined. And then you can take the measure of that and define the functions, and compute the integral of the product set product compute the product measure of the0 set E. So, we will continue the study of sections and their implications for product measures in the next lecture.

Thank you.