Measure & Integration
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Lecture - 24 A
Construction of Product Measures

In the previous lectures, we had started looking at Measure and Integration on Product Spaces. In the previous lecture, we defined the notion of products sigma algebra. And today we will define the notion of product measure. So, let us recall.

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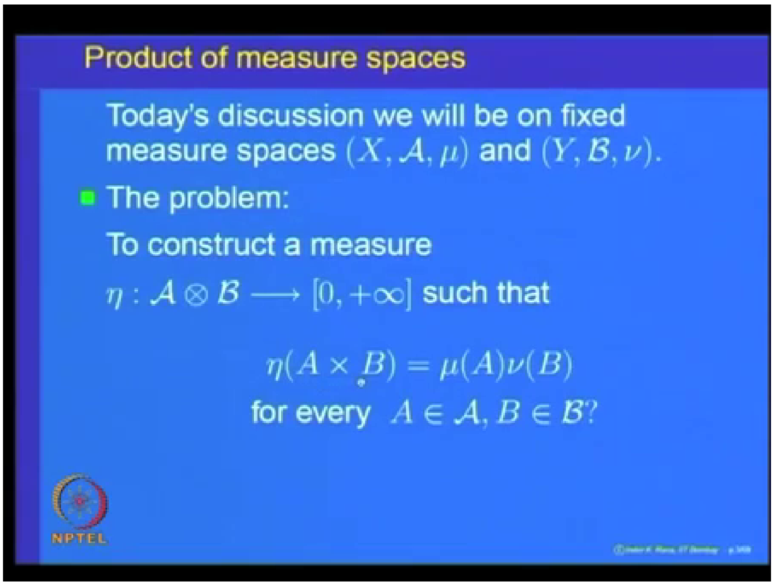

We will fix for today's discussion two measure spaces X, A mu and Y, B nu. So, X is a set; A is a sigma algebra substrates of x; mu is measured defined on the sigma algebra A. And similarly for the measure space Y, B, nu; B is a sigma algebra of subsets of Y and nu is a measure on the sigma algebra B. So, we have already defined the notion of the product measure namely A cross B.

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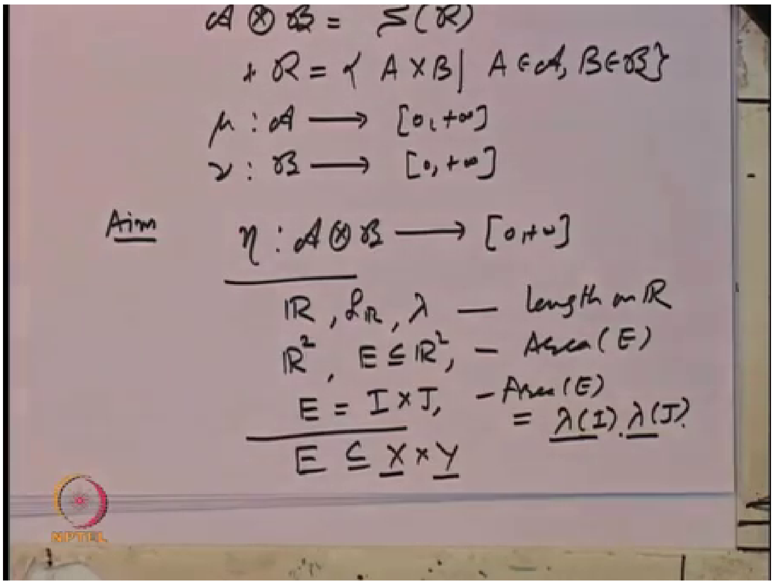

So, if you recall, so we defined the notion of A times B, so this is the sigma algebra generated by all rectangles and rectangles were defined as the sets A times B, where A belongs to the sigma algebra A and B belongs to the sigma algebra B. So, now we are given measure mu on the sigma algebra A and given a measure nu on the sigma algebra on the sigma algebra B, so that is a measure on B. So, our aim or the problem is to define a measure eta on the product sigma algebra A times B using the measure on A. and using the measure nu on B why such things are important.

So, let us just recall that on the real line and the Lebesgue measurable sets, we had defined the notion of the Lebesgue measure. So, that extended the notion of length. So, that extended the notion of length on R for in the subsets of R which are not necessarily intervals. So, we do want to do the corresponding thing on R 2. So, on R 2 given a set E, we would like to define the notion of area of E, and if E is an nice set for example, if E looks like a rectangle I cross J then we know that it is area. So, let us call it as area of E is defined as length of I times length of J.

So, this motivates the notion of the product. So, if we have sets in X cross Y, so in general if E is a set in X cross Y, and we have notion of size here and notion of size here, then we will like to know define the notion of size for sub sets E in X cross Y. And for sets which are nice which has very simple to describe, we would like to put it as a product of the length into breadth. So, the abstract question the problem abstract problem is the following that we want given two measures spaces X, A mu, and Y, B nu, we want to construct a measure let us call it as eta on the product sigma algebra A times B such that for sets which are rectangles. So, what are the rectangles on abstract measures spaces, they are the sets of the type A times B where A belongs to the algebra sigma algebra A and B belongs to the sigma algebra B.

So, for such sets, we want that the notion of the size for subsets namely, so our notion size is the measure. So, the measure of a set A times B should look like mu of A, so some ca something like length of A into nu of B length, length of the set B. So, this is the abstract problem given two measure spaces X, A, mu, and Y, B, nu, how to define a measure in a nice way on the product sigma algebra such that on rectangles it looks like the product of the corresponding measures. So, eta times A, eta of A times B should look like mu of A times nu of B.

In fact, this requirement that eta of A cross B is mu of A times nu of B itself says a way of doing this so that means, this fixes the notion of the measure for rectangles which are of the type A cross B. So, if we can show that this set function can eta which is defined by this equation for measurable rectangles A times B by this equation, and if you can show that is a measure it is countably additive then we know that measurable rectangles form semi algebra and they generate the sigma algebra A times B. So, we can take advantage of our extension theory and then extend this eta, if it is a measure on the semi algebra of all rectangles to the sigma algebra A times B, so that is roughly the route two we want to follow.

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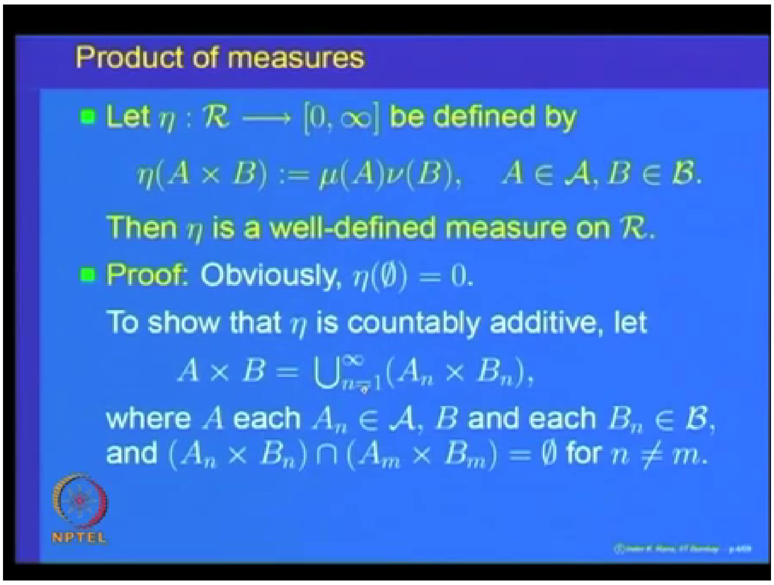

So, to do that to implement this possibility, so let us write that eta defined on rectangles. So, we are defining first eta on rectangles A times B. So, eta of A cross B is defined as mu of a times mu of b; obviously, it is a well-defined set function and we want to claim that this is actually a measure. So, eta or empty set is zero that is because if A or B are empty set then this is 0. So, we want to show that it is a measure on R that means, we have to show that eta is a countably additive set function.

So, to show that let us take a rectangle A cross B, A times B, and suppose it can be represented as a union of rectangles A n cross B n which are pair wise disjoint. So, A cross B is written as union n equal to one to infinity of rectangles A n cross B n where all the sets a A n's are all in the sigma algebra A, the sets B and B n's are all in the sigma algebra B and this rectangles are pairwise disjoint. That means, A n cross B n intersection with some A m cross B m is empty whenever n is not equal to empty n not equal to m.

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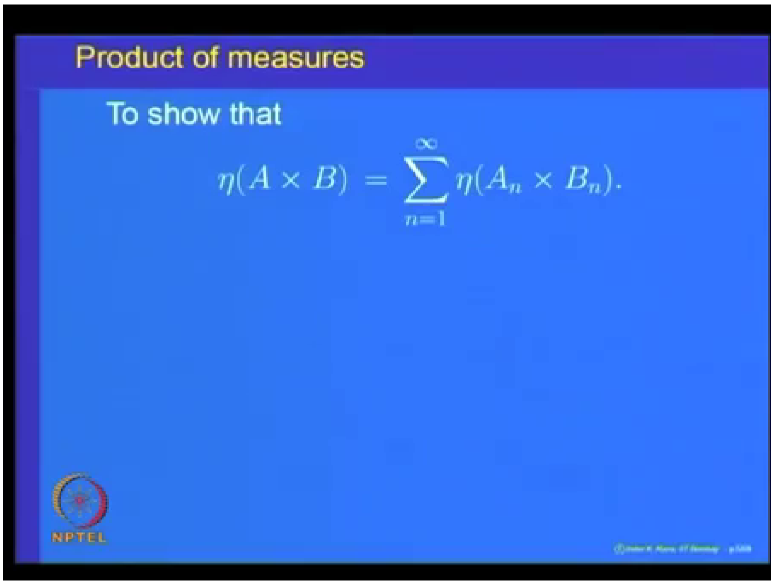

So, if a rectangle is written as a countable disjoint union of rectangles then we want to show that eta of A cross B is equal to summation n equal to 1 to infinity of eta A n cross B n, so this what we have to show. So, to show that let us proceed as follows.

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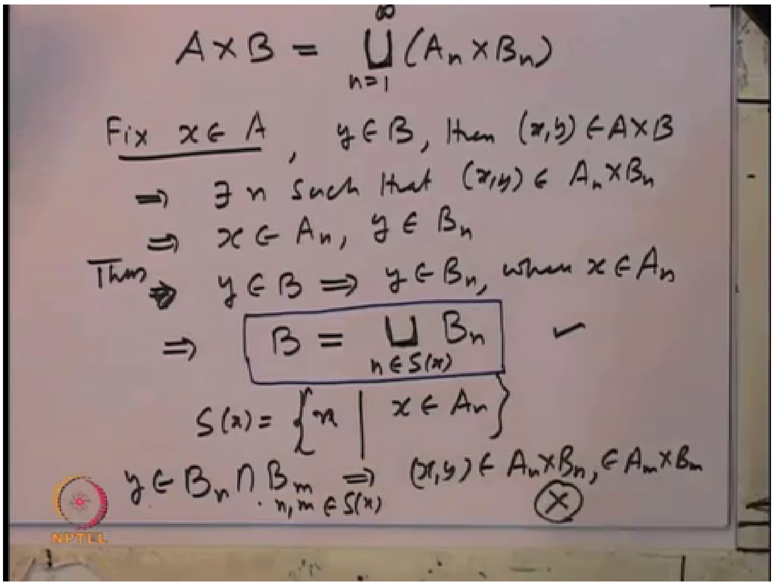

So, let us write. So, here is a rectangle A cross B which is written as a union of rectangles A n cross B n, n equal to 1 to infinity and these rectangles are pairwise disjoint. So, the disjointness will represent it by putting a square cup. So, the notion of union instead of putting is the U, you will put it at this square just to indicate that. So, we do not have to right every time that they are pairwise disjoint. The symbol instead of indicates that they are pairwise disjoints. So, we want to compute eta of A cross B, and show it is equal to summation of eta A n cross B n.

So, to do that let us proceed as follows let us fix any point x belonging to A. So, for any x belonging to A, if you look at y belonging to B, then x comma y belongs to A cross B. We have fixed A, and take any point y in B then the ordered pair x comma y belongs to A cross B, so that is A cross B. So, it will belong to one of the sets here, so which set it will belong it will belong to some A n and B n. So, the next curves y belongs, so this implies there exist a n such that x comma y will belong to A n cross B n, so that is a possibility.

But now if this weapons, so that implies that x must belongs to A n, and y must belong to B n; if the ordered pair x comma y belongs to an cross B n then x belongs to A n and y belongs to B n. But the what does that imply that implies that if y belongs to B so that means, implies, so thus let us write what we have y belonging to B implies y belongs to B n, here what this n where x belongs to A n. So, whenever x belongs to A n, y will belong to some B n. So, that implies that I can write the set B as union of over sets B n where n belongs to let me write S of x.

So, what is S of x? S of x is the set of all those indices n such that x which is fixed belongs to A n out of the indices 1, 2, 3 and so on look at those n for which x belongs to A n. So, if x belongs to A n and then why will belong to some B n, that means, y belongs to those B n's, so is that x belongs to A n, so this is what we want to claim. And not only that you want to claim that this B n's which are involved here they are pairwise disjoint. That means this union is a pairwise disjoint union. Why is that because if this is not disjoint, that means, if B n if point y belongs to B n interaction B m, where both n and m are in S x x that means, see that will imply that x y belongs to A n cross B n and also it belongs to A m cross B m where n and m are in the set of S of x. So, n and m belong to S of x.

Suppose, so I want to show these two unions is disjoint. So, take two elements here B n and B m that means, for m and n belonging to S of x look at the intersection suppose there is a y in that intersection that will meant what that x belongs to A n, and y belongs to B n. And similarly, x belongs to A m and y belongs to B m; that means, x y belongs to both of this which is not which is a contradiction because A n and B m are disjoint. So, what we have saying is the following namely we are saying the following that for any x fixed, I can write my set B as a disjoint union of sets. So, this is what the conclusion is I can write the set B as a disjoint union of sets B n’s which are coming in this union, but what which n’s, those n’s such that x belongs to An.

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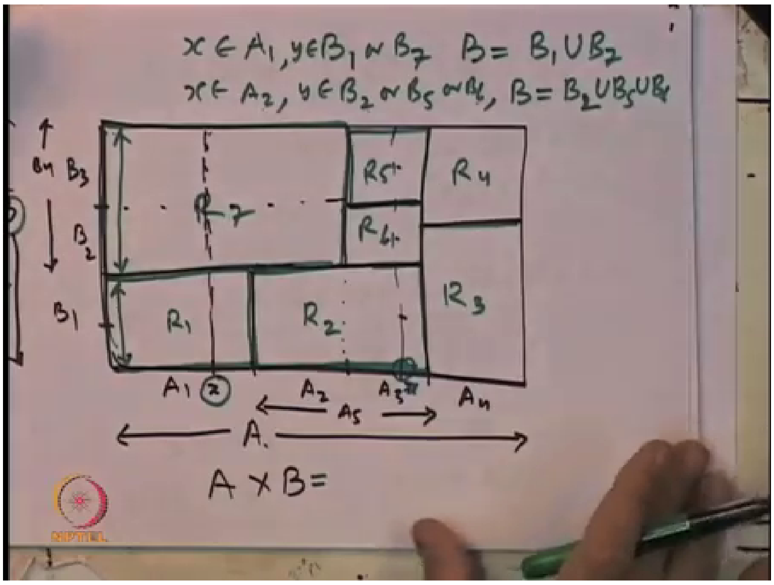

So, to look at a pictorial view of this, let me just take a very a simple example illustration of this. So, this is the set A, this is the set A, and this is the set B. So, we have got a rectangle A cross B, we have got a rectangle A cross B. And it is written as a disjoint union of rectangles. So, here are the disjoint union of rectangles. What are those rectangles one is this rectangle, the second rectangle is this, the third rectangle is this, the fourth rectangle is this, the fifth is this and the sixth rectangle is this. So, this A cross B.

So, the set A cross B is written as a disjoint union of sixth rectangles. So, let me call this first rectangle R 1, this is R 2, this is rectangle R 3, this is a rectangle R 4, R 5, R 6, and R 7. So, these seven rectangles are. And their sides of each one of them we can write down A 1 cross B 1. So, this rectangle this side is A 1 and this side is B 1. For this rectangle this side is this portion, and this is the width and so on.

And now I wanted to illustrate that point. So, let us take a point x belonging to A fixed. So, when x is fixed, what are the points y in B n. So, to find those, let us go vertically. So, if we go vertically, so for any y belonging to this set B, y belonging to B, either y will belong here or y will belong here. So, if so that means, this set B can be written as a disjoint union of this portion, and this portion of B, so that will be B 1 and this will be B 7. So, if x belongs to A 1 then y can belong to B 1 or B 7 or it can belong to B 7, so that means, B will be equal to B 1 union B seven. So, B 1 and union B 7, B 1 is the width height of R 1 and B 7 is the height.

For example I take a point x here this is a point x let us take a point x here which belongs to A 2 . So, when I go above for any if I fix this then how does the if y is split. So, if x belongs to A 2, if that is fixed then to be inside the rectangle, I can be here, I can be here, I can be here, so it will belong to B 2, B 6 and B 5. So, y can belong to B 2 or B 5 or B 6 so that means, in that case B will be equal to B 1 B 2 union B 5 union B 6. So, what we are saying, so depending on where the point x is right the set n x will be 1 and 7; if x is in A 2 then it will be n x will be 2, 5 and 6. So, B in either case B is a disjoint union of rectangles right some of the B i's. So, this is the important thing which I want to convey.

So, this is what is the conclusion of this argument that if A cross B is a disjoint union of rectangles A n cross B n and I fix any point x belonging to A and analyze the points y in B then x comma y belongs to A cross B. So, it will belong to some A n cross B n. So, y will belong to some B n's. So, which B n's it will belong it will belong to only those B n's for which x belongs to A n. So, B can be written as a disjoint union of B n's where n belongs to S of x. So, this is a disjoint union, so that was the first important observation. Once we have, so once this is a disjoint union and all the B i's the set the set B and the set B i's B n's are all in the sigma algebra where nu is defined. So, this being disjoint that implies.

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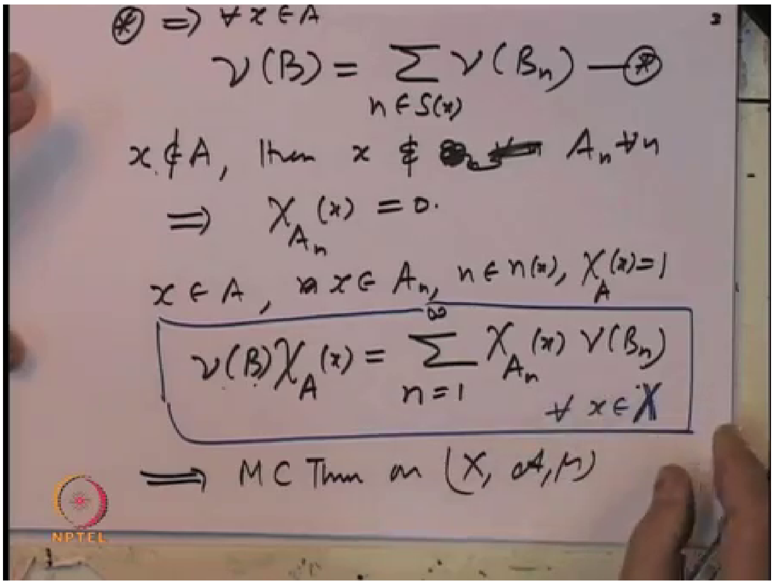

So, what we what we get is that nu of so this star. So, star implies namely that nu of B is equal to summation nu of B n's where n belong to S of x. So, now I would like to transform this equation slightly. So, what was x, x was a point in A, it was right. When x is in A, so for every x fixed; that means, for every x fixed in A, we had this. So, if now suppose x does not belong to A, if x does not belong to A, then obviously, x does not belong to B n's x does not belong to B n for every n sorry then if x does not belong to A then x does not belong to A n for every n. So, that implies that chi A n of x will be equal to 0.

So, if x belongs to A then x will belong then A n then it will belong to some of the A ns and in that case for those n it will be equal to 1. So, if x does not belong to A, and if x belongs to A and x; that means, x will actually belong to some A n and that means, this n will belong to n x and; that means, we will have chi A of x will be equal to 1. So, what I am saying is this equation nu of B equal to this, I can write it as nu of B times the indicator function of A x is equal to summation over all n equal to 1 to infinity chi of A n x times nu of B n.

So, let us understand this once again that y is this so. So, if x belongs to A, then the left hand is this value of the indicator function is one. So, left hand inside is nu of B, so that is here. And if belongs to A, then it will belong to some A n right. So, if x belongs to A n this values 1, so the right hand term is nu of B n. If x does not belong to A n, then this value is going to be 0; if x does not belongs to an then this value. So, if x belongs to A, and x belongs to A n this value is 1; otherwise this value is 0. So, on the right hand in this summation only those terms will be non zero for which x belongs to A n and that means, the value of the value of the right hand term will be indicator function will be 1 and nu of B n. So, this will be this question otherwise both sides are equal to 0. So, that holds.

So, what we are saying is that. So, from the earlier equation, we have come down to the second inclusion namely this holds for every x belonging to A and so for every x belonging to x, so for every x this equation holds, so that what we have proved. Because when x belongs to a it is the earlier equations R when x does not belong to A both sides are equal to 0, so this equation holds. So, this is the second crucial step in the arguments namely if A times B is a rectangle which is written as a countable disjoint union of rectangles A n and B n, then indicator function of A times x into nu of B is summation over n indicator function of A n times nu of B n.

And now this is the equation involving nonnegative this is the equation involving nonnegative, simple nonnegative measurable functions. So, own the measures space X, A mu. So, this is a sequence. So, now I can apply monotone convergence theorem. So, this apply monotone convergence theorem on X, A mu. So, this is a nonnegative function which is a limit of so the sum means it is a limit of the partial sums. So, it is a limit of nonnegative measurable functions. So, monotone convergence theorem will give me there the integral of this is equal to limit of integrals of this.

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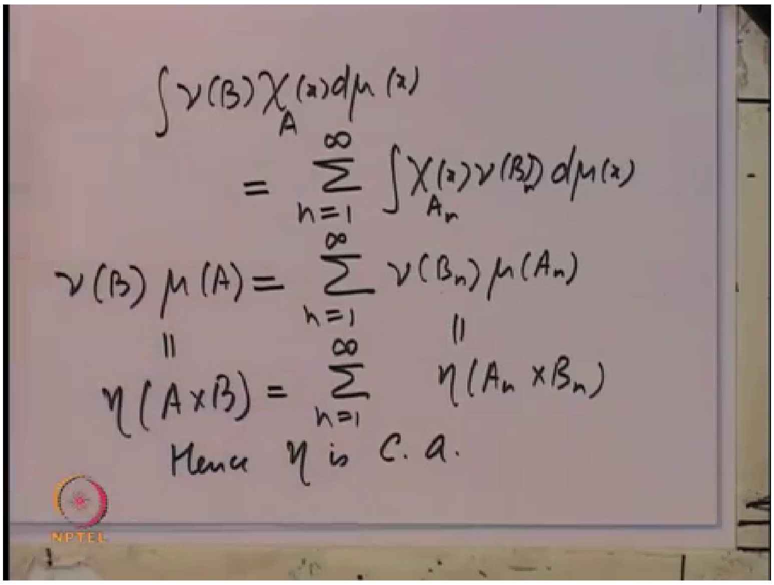

So, an application of monotone convergence theorem to this equations star gives me that integral chi of B indicator function of A, x, d mu x can be written as summation n equal to 1 to infinity integral of indicator function of the E A n nu of B n d mu x. So, this is a straight forward application of monotone convergence left hand side is nonnegative miserable function, which is limit of nonnegative measurable functions on this major space. So, integral of the left hand side with respect to mu must converge to the integral is equal to the limit of the integrals on the right hand side. So, this is an application of a monotone convergence theorem.

And now let us compute the right hand side the left hand side the integral of nu of B nu of B is a constant. So, that goes out integral of the indicator function of A with respect to mu. So, that is mu of A is equal to summation n equal 1 to infinity and this integral again nu of B n is constant. So, nu of B n goes out of the integral sign and internal of A n with respect to mu, so that is mu of A n. So, what we have gotten is nu of B into mu of A is summation nu of B n into mu of A n but this thing is nothing but eta of A cross B, and this is nothing but each term is nothing but eta of A n cross B n. So, what we get is eta of A cross B is equal to summation n equal to 1 to infinity eta of A n cross B n and that proves that so hence eta is countably additive.

So, that proves that the eta is a measure.