Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 23 B
Product Measure, an Introduction

However would like to find some conditions under which we can ensure these two are equal and that is our going to be our next theorem.

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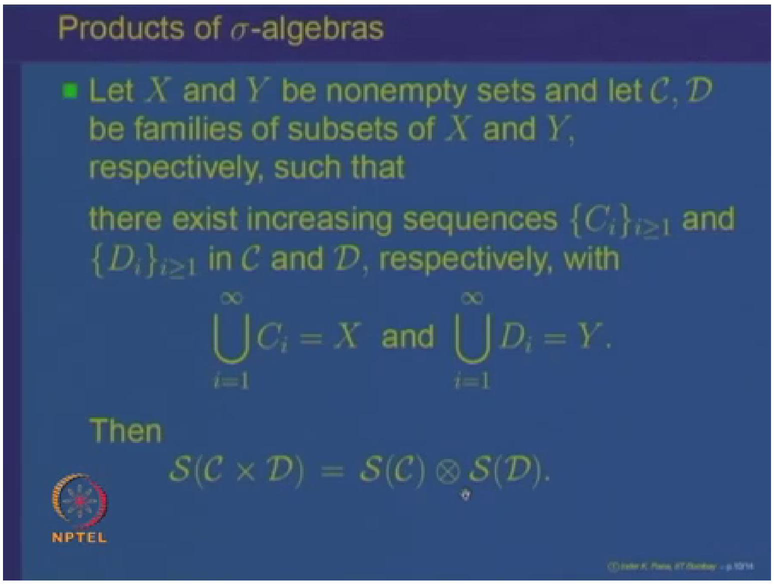

So, theorem says let X and Y be nonempty sets and C and D be families of subsets of X and Y, such that the whole space X can be represented as a union of elements from that collection C. And the space Y also can be represented as union of elements from that collection D. So, we are putting this condition; this collection C and D as such that there is a sequence of elements of C, which gives you the whole space X and there is a collection of elements from collection of a sequence D i; in the collection D says that its union is again equal to Y.

Under this condition, we are going to show that the sigma algebra generated by C cross D; is same as the sigma algebra generated by C times the sigma algebra generated by D. So, this equality holds.

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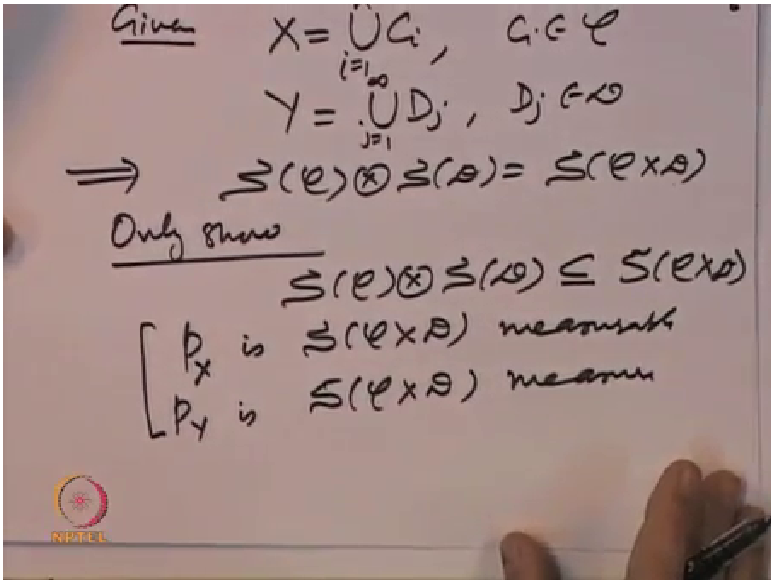

Of course we have already proved that S of C cross D is a subset of the product sigma algebra S of C times S of D. We only have to prove the other way around in equality; so, what we have to show is the following namely. So, we are given; so, this is the fact which is given that X can be written as union of C i; i equal to 1 to n; where C i belongs to C and Y also can be written as a union of elements D j; j equal to 1 to infinity, where D j belong to D. And these two conditions, we want to show imply that S of C; S of C times S of D is equal to S of the sigma algebra generated by C cross D.

So, we have already shown that the sigma algebra generated by C cross D is subset of this. So, only to show; so, we have only show that left hand side; that S of C times, S of D is a subset of S of C cross D C cross D, so this is what we have to show. To show this; we will follow the previous proposition which said that; the product sigma algebra is a smallest sigma algebra with respect to which the projection maps are measurable. So, suppose we are able to show that the projection map p is S of C cross D measurable and p of Y is also S of C cross D measurable.

So, if you show this then what do we mean? So, px and py are both measurable with respect to this sigma algebra S of C cross D. So, it must include the product sigma algebra namely S of C cross S of D. So, let us show that these to maps are measurable with respect to the sigma algebra S of C cross D. So, for that; so, what we have to show is the following. So, let us look at the case of the projection map px.

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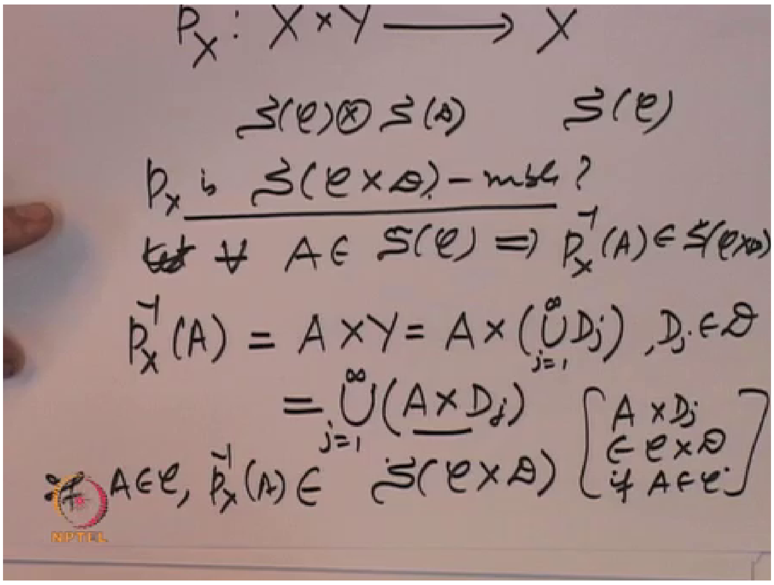

So, px is a map from X cross Y to X and here we have the product sigma algebra S of C times S of D; that is a product sigma algebra. On X we have the sigma algebra S of C; so, what we want show is that px is in fact, measurable with respect to the sigma algebra S of C times S of D. So, px is measurable this is what we want show.

So, to show that let us take a set let; so, to show this we have to show that for every set say A; belonging to S of C should imply that px inverse of A belongs to S of C cross D. So, that is what we have to show. So, let us first observe what is px inverse of A; if a is a subset of X and belonging to; so, let us not to bother at present where it belong. So, let us look at px inverse of a that is going to be equal to a cross Y; by definition. Because A is a subset of x; so, the projection lies in A; that means, the inverse images A cross Y.

And now by the given condition; the set Y is representable as a union. So, this is a cross union of D j's; j belonging to 1 to infinity where each D j belongs to the collection D. So; that means, I can write this is union j equal to 1 to infinity of A cross D j.

So, this implies that px inverse of A is written as union of rectangles which look like a cross D j. Now Dj's are inside the sigma algebra; inside the collection D and if A belongs to C; then this will belong to C cross D. So, and the union; so, the union will belong to the sigma algebra generated by C cross D; why? Because A cross D j will belongs to C cross D; if A belongs to C. So, if A belongs to C then px inverse of A; just now represented as S belongs to C cross d.

So, but what we want show is not only for C for S of C also this property is true. So, will you apply the usual sigma algebra technique to prove this?

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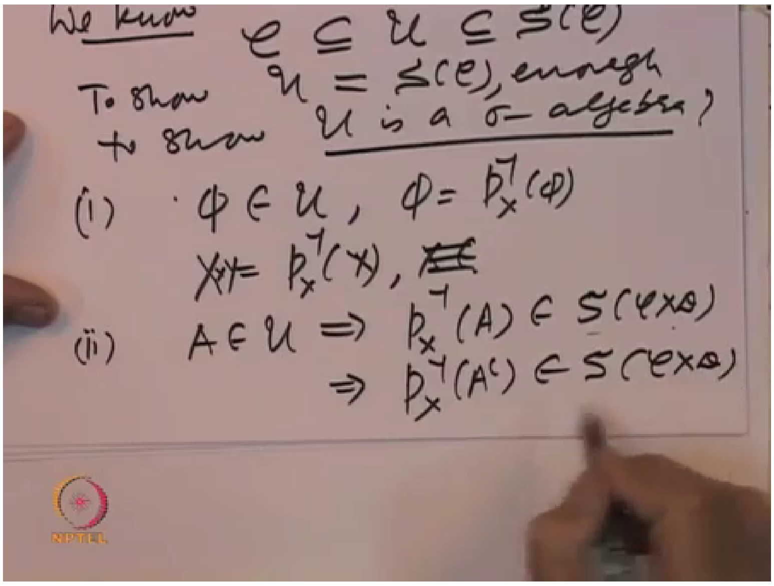

So, let us write U to be the collection of all the subsets A; belonging to S of C such that px inverse of a belongs to the S of C times; so, belongs to S of C times D.

So, let us look at this collection. So, what we have proved just now; so, we know that C is contained in U and U is a collection of subsets of S of C. So, that is contained in S of C; but what we want for every set A in S of C; px inverse of a belongs to S of C; that means, we want to show that this collection U is actually equal to S of C. And U is a subset of S of C; when C is inside U to show I am that U is equal to S of C; it is enough to show that the collection U is A. So, to show that U is equal to S of C; it is enough to show that U is a sigma algebra because once you use the sigma algebra that includes C; so the smallest one that is S of C will come inside. So, everything will become equal.

So, we have to only show that this is a sigma algebra. So, let us look at empty set belongs to U because it is just look; at it is just empty set is equal to px inverse of empty set. So, empty set belongs to U and what about X? X is equal to px inverse of Y and Y belongs to S of C. So, this also belongs to; so, because Y belongs to sorry px, we want to show that the whole space A belongs to S of C.

So, what we want to show is that X belongs to; so, if you look at px inverse of X that is X cross Y and that belongs to S of C cross D. So, that implies X belongs to this collection. So, X is also inside it; so, empty set and the whole space belong to it obviously.

Let us look at the second condition that if a belongs to U; then that me implies px inverse of A belongs S of C cross D. So, that is given to s, but S of C cross D is a sigma algebra. So, it is close under compliment.

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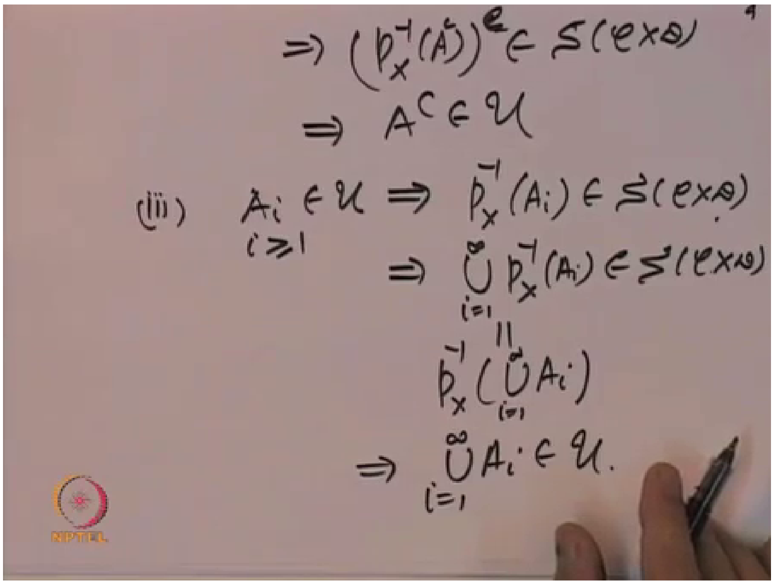

So, that implies that px inverse of A compliment also belongs to S of C cross D. But px inverse of a compliment is nothing, but so, that implies this set is nothing but px inverse of A and compliment of that. So, that belongs to S of C cross D and that implies that A compliment. So, that means that A compliment belongs to U; because oh sorry, it should be other around px inverse of A belongs; that means, this compliment belongs to S C cross D. So, if I said belong it is compliment belongs and this is nothing but the compliment of px inverse of A compliment.

So, whenever A has the property that px inverse of A belongs to the sigma algebra; px inverse of A compliment also belongs to the sigma algebra as cross C cross D; that means, A compliment belong. So, U is close under compliments and finally, we will look at suppose A i's belong to U; for a sequence i bigger than are equal to 1; belongs to U ; that means, px inverse of A i belong to the sigma algebra S of C cross D.

So, these are usual techniques for proving sigma algebra. So, that implies; this is the sigma algebra S of C cross D and px inverse of A i's belong to it. So, union of them px inverse of A i's their union; 1 to infinity also belongs to S of C cross D. But this union of the inverse image is inverse image of the union. So, that is px inverse of union; i 1 to infinity. So, that belongs to S of C cross D; that means, we are shown whenever px inverse of A i's belong to the sigma algebra, the px inverse of the union also belong; so; that means, implies that the union A i's 1 to infinity also belong to U . So, that proves that U is a sigma algebra of subsets of X. So, U is a sigma algebra include C; So, U is equal to S of C; that means, px inverse of A; is measurable.

So, what we have shown is that if you look at the map p X; X cross Y to X then it is S of C cross the measurable. And the product sigma algebra S of C cross S of D is a smallest one with respect to this is measurable. So, that will prove that S of C cross D includes the sigma algebra S of C times S of D. So, that is how we prove that whenever the families C and D of the property that you can write.

So, whenever X is a union of C i's and Y is a union of D j's where Ci's where C and D j's in C wherever this property is true. Then whether you take the class is C and D and take the rectangles and generate the sigma algebra; that is going to be same as generating the sigma algebras first and then taking the product sigma algebra. So, this is a useful theorem which gives us a dividend namely the following.

So, we look at consequence of this which says that.

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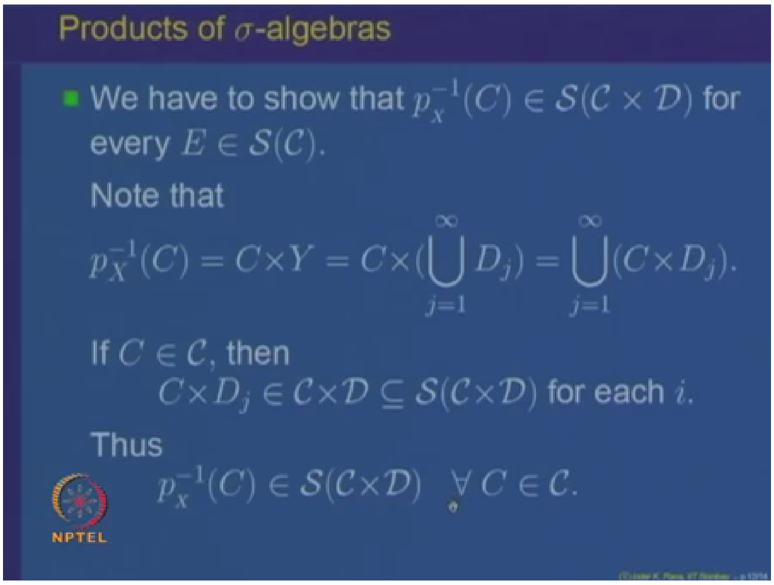

So, this is just a repetition of what the ideas should I have said to show that px inverse is measurable, we have to show this and that can be written as px inverse of C. Because Y is a countable union; so, you write this as a union. So, in the union splits into this union of C cross D j's and if C belongs to C. So, then this belongs to C cross D; which is in missed as of D C cross D.

So, px inverse of C is an element in S of C cross D, whenever C belongs to C. So, px inverse of C is an S of C cross D for every C. So, to prove it is for all S of element of S of C; this property is true.

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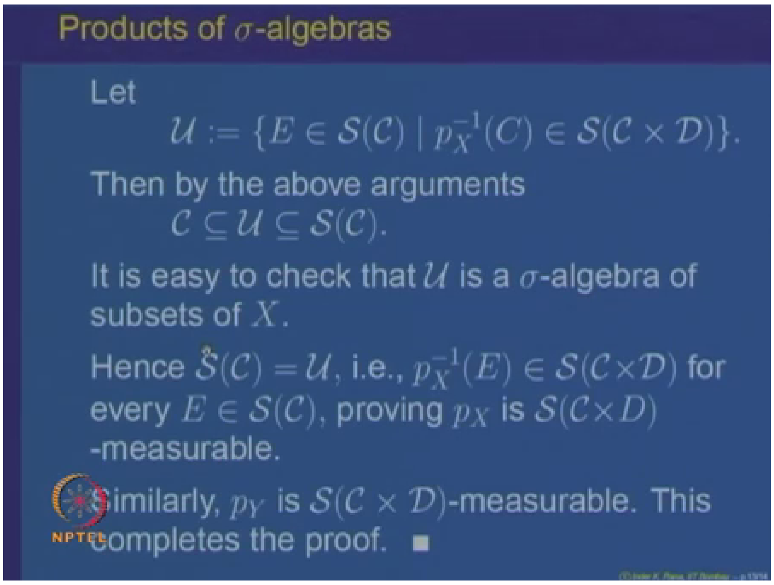

We use the sigma algebra technique; namely look at the set U which is collection of the all the sets in S of C which have this property. Then show that we already know that C is inside U and U is inside S of C. So, to prove the equality one just that show that U is a sigma algebra. So, that we have just now shown it is sigma algebra. So, that proves the fact that; whenever C and D are there two classes of subset of X cross Y then S of C cross D is same as S of C times S of D.

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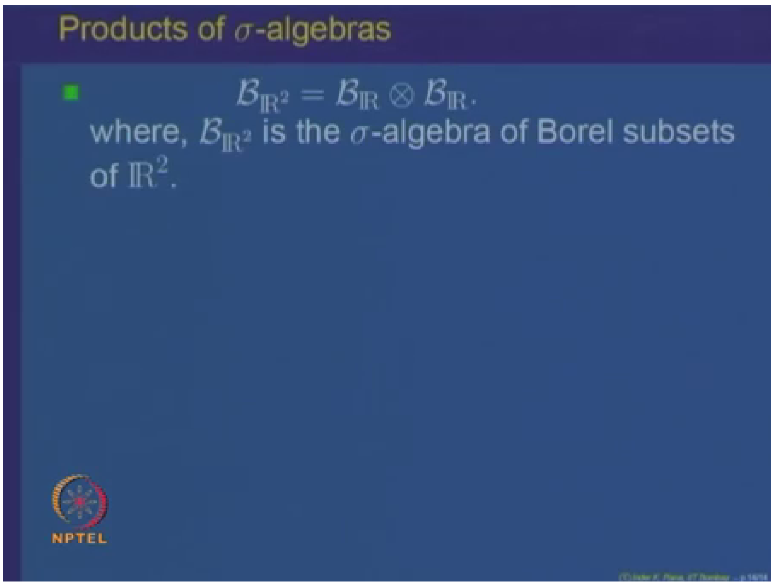

As a consequence of this; let us prove the fact that on the plane B R 2; that is a sigma algebra generated by Borel subsets of R 2; is equal to the Borel sigma algebra R times, the Borel sigma algebra of R. So, to prove this fact let us just observe the following namely.

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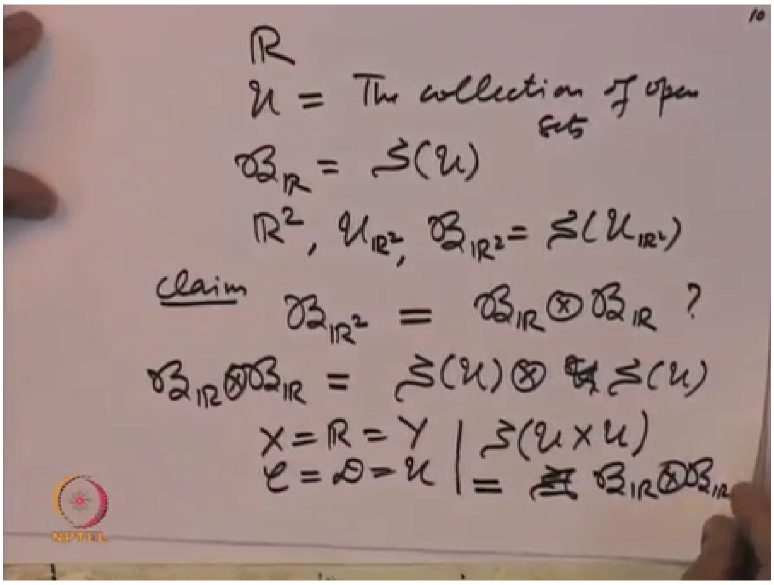

So, on the real line we have got U; the collection of open sets and B R 2; B of R the Borel sigma algebra of R is nothing but the sigma algebra generated by open subsets of real line.

So, let us look at the set R 2; on R 2 we have got the collection of open sets. And the sigma algebra generated by them. So, let us write U of R 2 the collection of open subsets of R 2 and generate the sigma algebra. So, B of R 2 is the sigma algebra generated by open subsets of R 2. So, we are claiming is the following; look at the Borel sigma algebra of R and look at the product of this with Borel sigma algebra of R. So, that gives you the sigma algebra of subset of R 2 and on the other hand you got the sigma algebra of subsets of R 2 called the Borel sigma algebra subset of R 2 and what we want show is that these two are equal.

So, note on the left hand side; so B of R cross B of R is nothing but the sigma algebra generated by open sets; times the sigma algebra; again the same sigma algebra S of u. So, it is the product of the same sigma algebra with itself, the Borel sigma algebra with itself and the Borel sigma algebra is generated by open subsets of the real line. So, this is a perfect chatting for applying our previous theorem; so, we have got X equal to R equal to Y and C equal to D equal to open sets.

So, if you look at; so that implies. So, our previous theorem will imply that if you look at C cross D. So, if you look at U cross U and then look at the sigma algebra generated by at; that must be equal to the sigma algebra B R cross BR.

So, that is from our previous theorem but what we want to observe here is; that if you look at U cross u.

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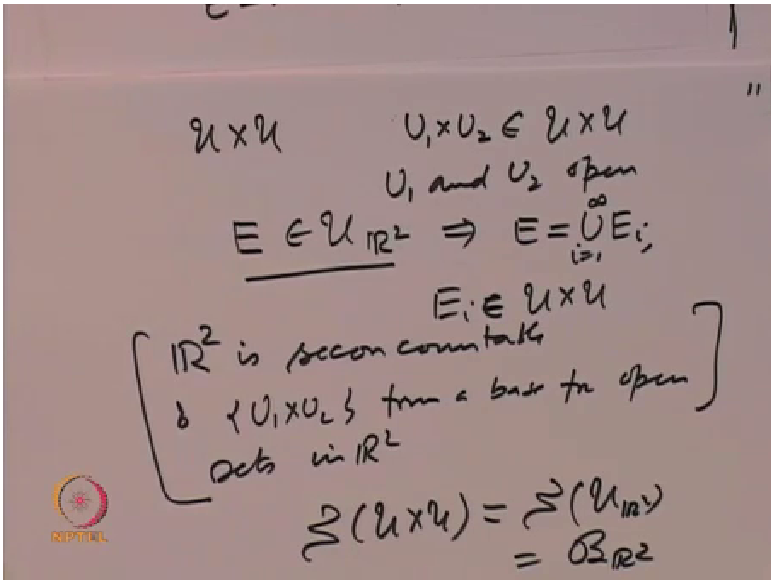

So, if you look at the sets of type U cross U; then these are sets of the type. So, what is the set in the type U cross u. So, that is the sets of the type were open set a open set U 1 cross an open set U 2. So, these are the type of set which belongs to U cross. So, U 1 and U 1 both open and now if you take any open set say a E; any open set in R 2.

Then this is effect from matrix basis that the open sets in R 2; the sets of the type U 1 cross U 2; form a base for the topology of open sets; for the topology of R 2. So, what we are saying is; if is then this implies that E can be written as a countable union of sets E i's going to infinity; where each E i is a set belongs to U cross U.

So, this fact is from basic topology namely R 2 is second countable and the sets; U 1 cross U 2 from a base for open sets in R 2. So, this together imply that every open set in R 2 can be written as a countable union of sets E i and these E i;s are open rectangles you can call them. Each E i is a open set cross another open set. So, by that fact that will follow that the sigma algebra S of U cross U is same as the sigma algebra on R 2 generated by open sets and that is B of R 2.

So, that will prove that the Borel sigma algebra in R 2; so, that will prove that the Borel sigma algebra in R 2 is same as. So, if you want generate Borel subsets in R 2; what you can do is you can generate Borel subset in their line and then take the product sigma algebra; Borel subset cross Borel subsets and that will give you the product of the sigma algebra of Borel subsets in R 2.

So, with that we come to the conclusion of the description of sigma algebras got a product of sigma algebras on R 2. So, the main thing is the two remember is the following namely given X and given Y.

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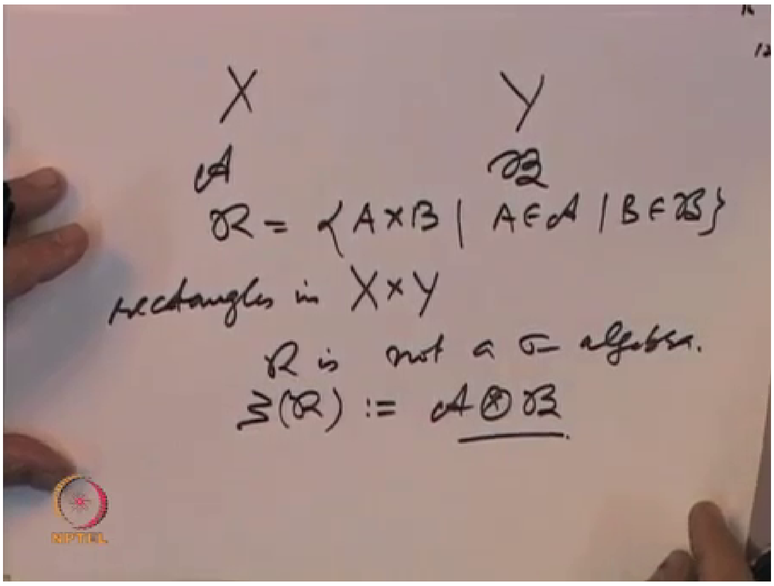

We can have the product set X cross Y; here we got sigma algebra a here we have got sigma algebra b. So, we take sets A in A and B in B. So, that gives the sets of type A cross B; so, these kind of sets are called measurable rectangles. So, sets R equal to A cross B; where A belongs to A and B belongs to B; give you subsets of X cross Y. So, there called rectangles in X cross Y; and in general this rectangles do not form.

So, R is not A sigma algebra; in general it is not a sigma algebra. So, you can generate; so generate the sigma algebra out of this rectangles and that is denoted by A times B. So, this is called the product sigma algebra. So, product sigma algebra is the sigma algebra generated by all rectangles. And another way of generating product sigma algebras is by generating the sigma algebras. So, that we looked at; so, here is X; here is Y there is the collection C here.

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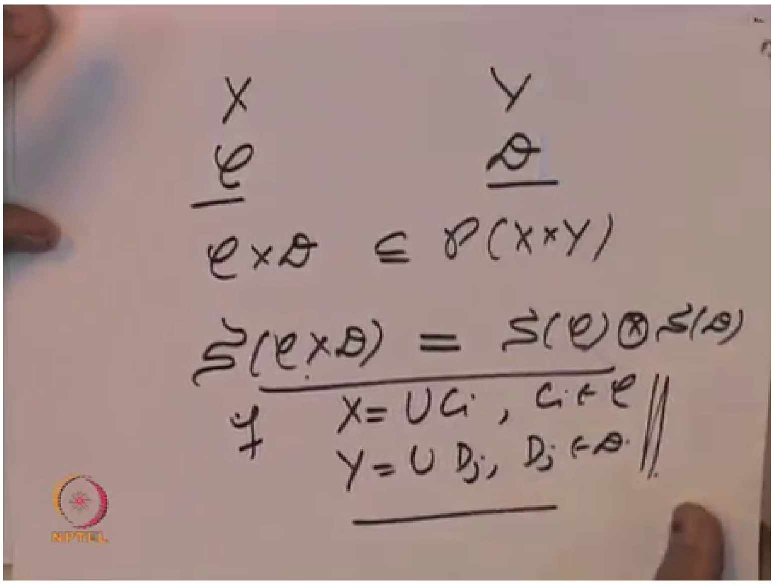

There is collection D here. So, you got C cross D; a collection of subsets is a collection of subsets of X cross Y. So, on one can generate the sigma algebra by this collection or the other hand one can generate first the sigma algebra S of C here and generate the sigma algebra by this S of D here; and then look at the product of them. So, these two are equal if you can write X as union of Ci's; Ci's belonging to C and Y can be written as union of D j's; D j's belonging to D.

So, under this conditions these 2 are equal and if this conditions is not true then only you can say it is the left hand side is a subset of the a right hand side. So, these are the product sigma algebras, so what we want to do in the next lecture is the following.

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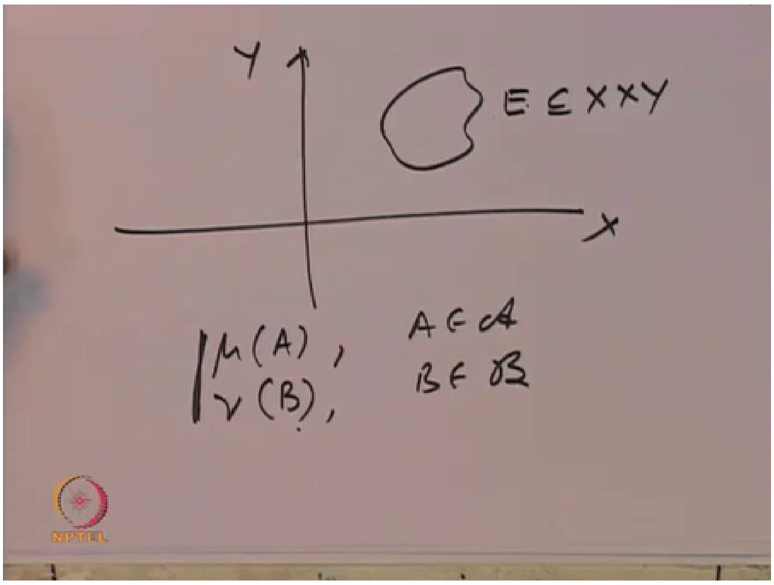

So, here is the set X; here is the set Y. So, this is a subset E in the product sigma algebra. So, what we want do is that for. So, this if we have got a notion of size of sets A in the sigma algebra; A belonging to A and in notion of size for sets B; B belonging to B, we want to know what could be what are the sets for which E contained in X cross Y for which we can define the notion of size.

So, in a sense what we trying to do is that will try to construct a measure on the product of the sigma algebras a cross Y. By looking at to the measures on X and on Y; so we will do that in the next lecture. So, in the next lecture we will look at measures on the product spaces, how to construct given a measure on the space X on a sigma algebra A and a measure new on the sigma algebra B of subsets of Y; how to construct a measure in an actual way on the product sigma algebra A times B. That will also generalize the notion of areas in R 2 and volume in R 3 and so, on; so, will continue the study of construction of measures on product spaces in our next lecture.

Thank you.