**Measure & Integration**

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**Lecture - 23 A**
**Product Measure, an Introduction**

In the previous lectures we had done the basic Theory of a Measures and Integration. And today we will start with a measure the notion of Product Measures and Integration on product measure spaces. So, to start with we will have the notion of a product a sigma algebras.

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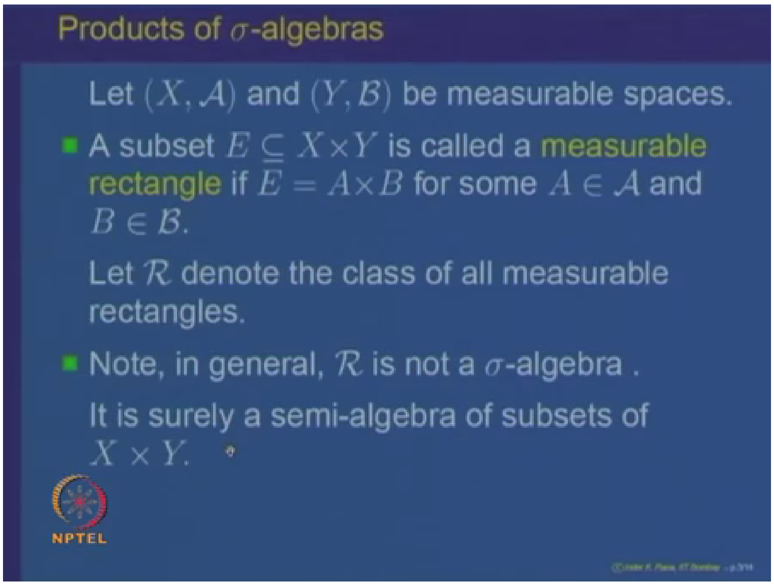


So, let us start with the 2 measurable spaces XA and YB be measurable spaces. Then a subset of the product space X cross Y a subset e is called a measurable rectangle if it looks like A cross B where A belongs to the sigma algebra A and B belongs to the sigma algebra B. The collection of all measurable rectangles or just called as a rectangles will be denoted by the set R. So, the set R denotes the class of all a measurable rectangles which are subsets of the set X cross Y and each subset is of the type A cross B where A is in the sigma algebra A and B is in the sigma algebra B.

In general, we had already observed while discussing the notion of a semi algebras and sigma algebras that sets of the type A cross B where A comes from a sigma algebra and B comes from a other sigma algebra this collection of rectangles in general need not form a sigma algebra. In fact, it does not even it need not even form it need not be even a algebra now surely A and B are being sigma algebras there also semi algebras and then we had shown that the rectangles A of a type A cross B surely form a semi algebra.

So, thus a set of all measurable rectangles they surely form a semi algebra of subsets of X cross Y.

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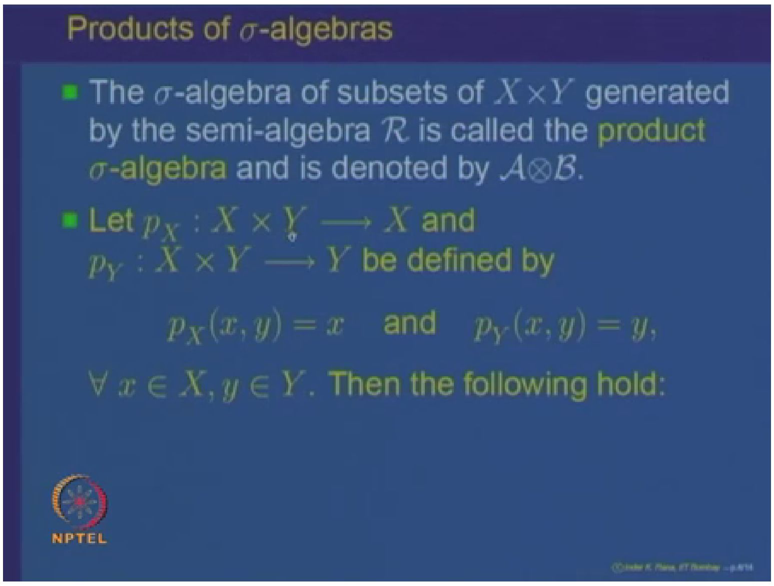


So, this being a semi algebra of subsets of set X cross Y and in general may not be a sigma algebra we can generate a sigma algebra B Y this rectangle the sigma algebras generated by this rectangle is denoted by A times B and A times B here is the a special symbol cross with a circle. So, A times B we will denote the sigma algebra generated by the rectangles R.

Let us give a another characterization of these sigma algebras the product sigma algebra in terms of what are called the projection maps. So, let us look at the map PX P lower X which is defined from X cross Y to X as PX of X comma Y is the first coordinate X and similarly PY is a map from X cross Y to Y and is denoted by PY of xy is Y the second coordinate. So, these 2 maps they are called the projection maps the projection of X cross Y on to X and on to Y. So, the claim is that in case we give X cross Y the product sigma algebra then this are a measurable maps.

So, let us prove this. So, X cross Y.

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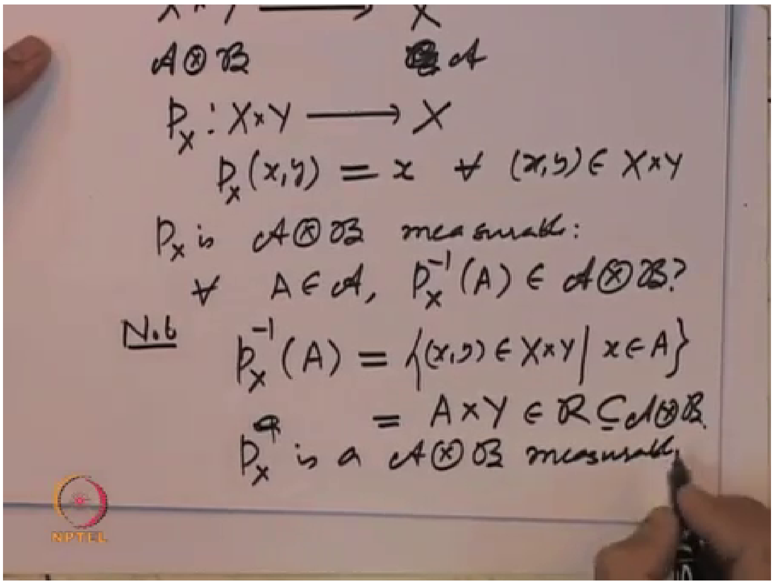

To Y on X we have got a sigma algebra A on b, we have got the sigma algebra B on Y we have got the sigma algebra. So, let us. So, let us look at A. So, X cross Y to X. So, here we have got the product sigma algebra A times B on X cross Y and on X we have got the a sigma algebra A and PX is the map defined on X cross Y to X and which defined as PX X comma Y is equal to the first coordinate X for every X comma Y belonging to X cross Y. So, our claim is that this PX is may a measurable map when we give the product sigma algebra on X cross Y. So, PX is A times B measurable.

So, to show that what we have to show is the following; that means, for every set A belonging to the algebra A if you look at PX inverse of A then that belongs to the sigma algebra product sigma algebra A cross B. So, that is what we have to show. So, let us calculate. So, we note that PX inverse of A. So, it is all X comma Y belonging to X cross Y such that X belongs to A. So, that is a meaning of the set PX inverse of A, but that is same as X belongs to A and Y is independent. So, this is A cross Y just the set A cross Y and a belongs to the sigma algebra A and the set Y belongs to the sigma algebra B.

So, this is actually a rectangle to this belongs to PX inverse of A, actually belongs to A rectangle. So, which is which generates the sigma algebra A times B. So, that shows that the inverse image of every set in the sigma algebra A under PX is in that sigma algebra product sigma algebra A cross B. So, that shows that PX inverse; that means, PX is a A times B, B measurable set A measurable map. So, PX is A times B measurable and similarly we can show that the PY.

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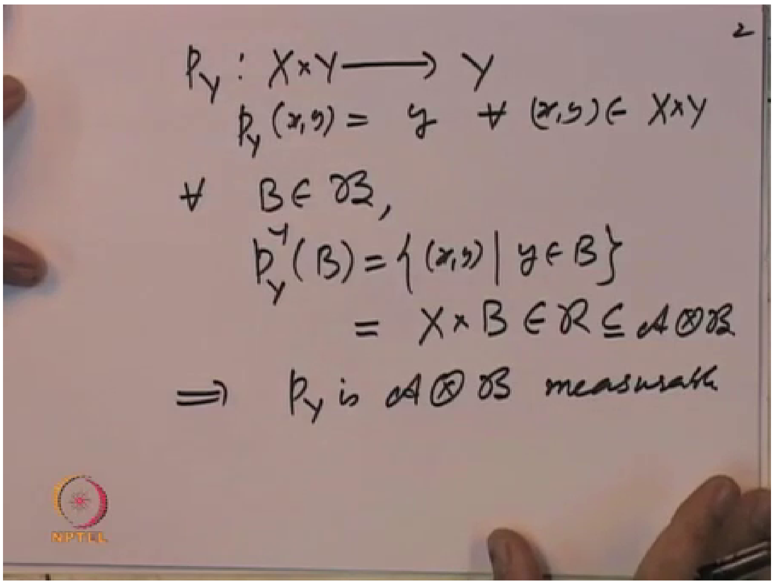


So, PY which is a map from X cross Y to Y where PX, PY of X comma Y is Y for every X comma Y belonging to X cross Y.

So, then for every set B in the sigma algebra on Y that is B if we calculate PY inverse of B that is all X comma Y such that all X comma Y such that PY. So, Y belongs to b and that is same as X cross B which belongs which is a rectangle. So, which is a rectangle and hence is in the sigma algebra A times B. So, for every set in the sigma algebra B PY inverse of B is in the product sigma algebra A times B. So, that implies that PY is A times B measurable. So, this is a measurable map.

So, what we are saying is that the product sigma algebra A cross B the product sigma algebra A cross B is a sigma is a sigma algebra on the product space X cross Y which makes both the projection maps PX and PY measurable.

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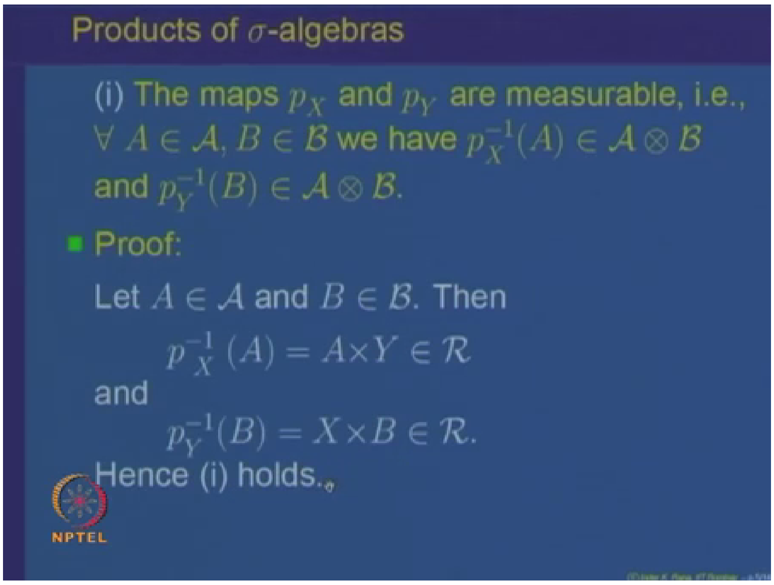


So, that is the property we have just now proved. In fact, something more can be said one can even show. So, this is a proof we just let us go through the proof again if a belongs to A and B belongs to b then PX inverse of A is just A cross Y which is a rectangle and PY inverse of B is again a rectangle and hence they both belong to the product sigma algebra and in PX and PY are measurable.

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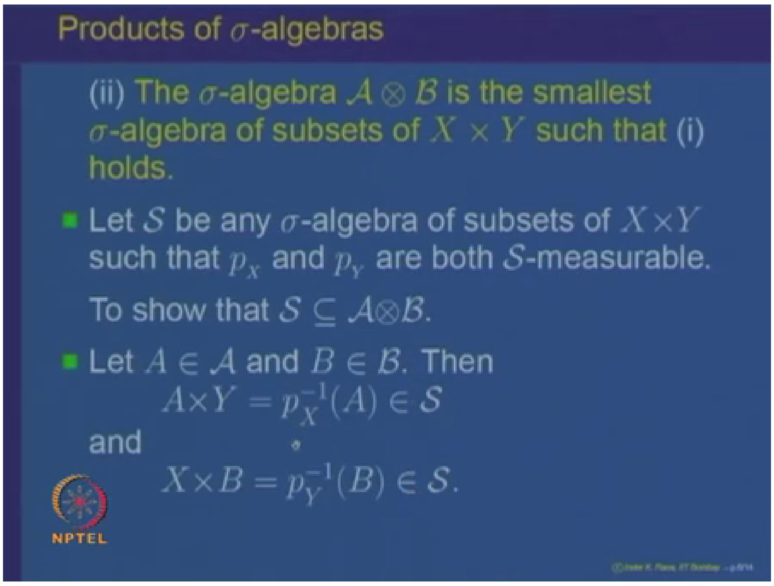


So, what we want to show actually is that A cross B on X cross Y is the smallest sigma algebra of subsets of X cross Y such that the earlier property holds namely this is the smallest sigma algebra of subsets of X cross Y such that PX and PY are both measurable. So, let us look at a proof of that. So, let us assume. So, let S be a sigma algebra of subsets of X cross Y such that both PX and PY from X cross Y.

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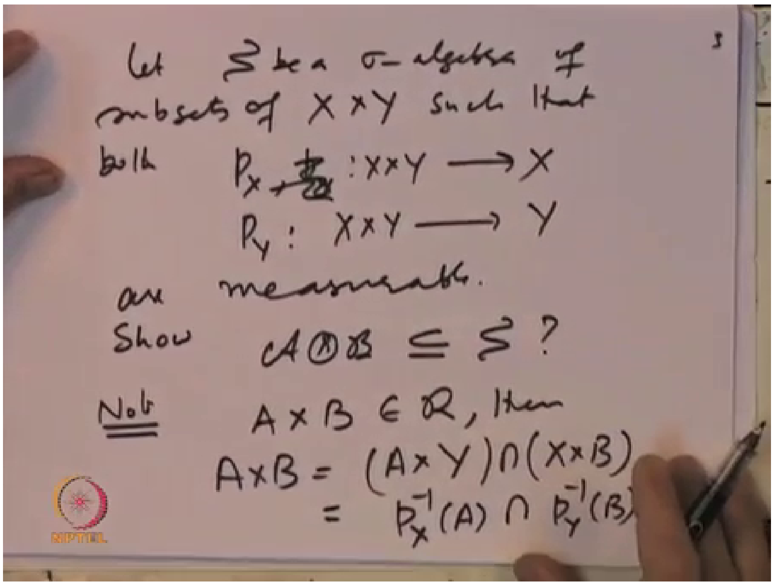


So, PX will be. So, let us write PX will be in X and PY which will be X cross Y to Y are such that both these maps are measurable, we want to show that A times B A times B is also a sigma algebra of X cross Y with respect to which both PX and PY are measurable. And we want to show this is the smallest; that means, if S is a any other sigma algebra. So, that PX and PY are measurable.

You want to show that it must be including A times B; so A times B is inside S. So, let us to prove this let us take a set. So, note if you take a rectangle. So, if you take a set A cross B which is a rectangle then this rectangle A cross B, I can write it as A cross B can be written as A cross Y intersection with X cross B. So, this is a simple (Refer Time: 10:59) fact that A cross B, I can write it as A cross Y intersection with X cross B because the first component A cross X will give me a and the second component Y intersection B will give me b and this set A cross Y just now we saw it is nothing, but PX inverse of A and the second set A X cross B is PY inverse of the set B.

So, the set A times B A cross B can be written as PX inverse of A intersection with PY inverse of B. And we are given that the sigma algebra S has the property that both PX and PY are measurable. So, as a consequence of this for every set A in the sigma algebra

A PX inverse A will belong to it belongs to the sigma algebra S and PY inverse also belongs to PY nverse of B also belongs to S because PX and PY are both measurable. So, this belongs to S. So, what we had shown is that if S is a sigma algebra with respect which both PX and PY are measurable then S must include all rectangles.

So, our analysis shows. So, implies.

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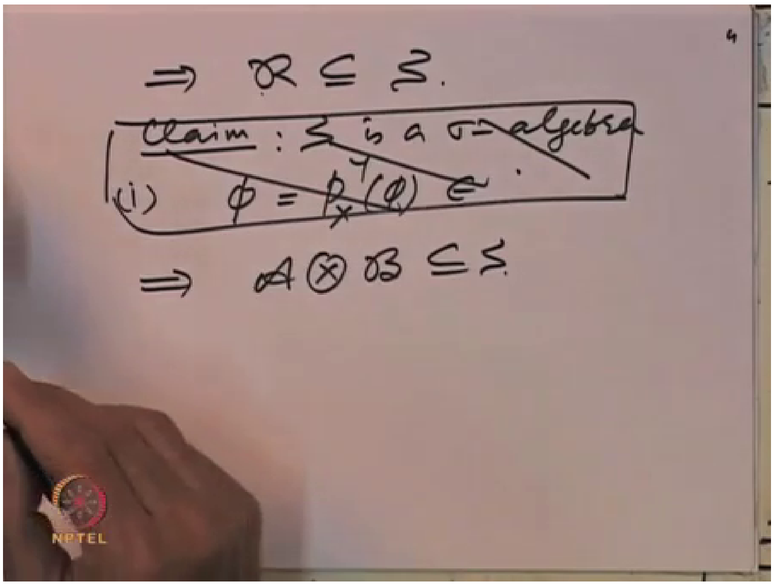


That all rectangles are inside the sigma algebra S, and we wanted to show that the product the sigma algebra is inside S. So, it is enough to show that S is a sigma algebra. So, let us claim and try to prove that S is a sigma algebra S is a sigma algebra. So, what we have to show that first, if we look at empty set then I can write a empty set as equal to either PX inverse of empty set or PY inverse and hence this belongs to and so P, we want to show that S is a sigma algebra so; that means, oh sorry S is already a given to be a sigma algebra sorry we do not have to prove this.

So, what we have shown all the rectangles are inside S and S is a sigma algebra. So, that implies that A times B is also inside S. So, A times B is a smaller sigma algebra with respect to which both the projection maps are measurable. So, so then let us go through the proof again. So, if S is any other sigma algebra of subsets of X cross Y, say that both PX and PY are measurable in that case we want to show that S X is this is a typo we should show that S includes A times B. So, let us take a set A in A and B in B then A cross Y is PX inverse of A, as we observed and X cross B is PY inverse of B and both belong to S because PX and PY are measurable.

So, A cross Y and X cross B are both sets in the sigma algebra S.

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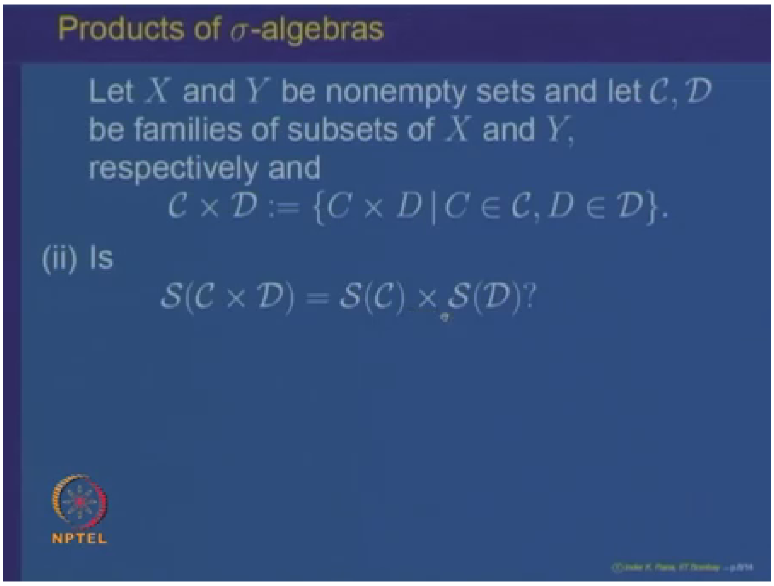


So, their intersection must belong to the sigma algebra that means, A times B A cross which is A cross Y intersects with X cross B belongs. So, all the rectangles are inside S and S is a sigma algebra. So, all the sets in the product sigma algebra that is a smaller sigma algebra including rectangles must also come inside. So, that shows that the product sigma algebra is inside S. So, product sigma algebra is a smallest sigma algebra or subsets of X cross Y with respect to which both the maps PX and PY are measurable let us look at some more properties of a generating sigma algebras on products basis.

So, let us look at this problem let us look at 2 sets X and Y of course, non empty sets and let us look at 2 families of subsets of X and Y. So, C is a family of subsets of X and D is a family of subsets of Y, then we can form a rectangles by elements of this families. So, let us denote C times D, to be the collection of all sets of the type C cross D where C is in the collection C and D is in the collection D. And now So, this is a collection of a subsets of X cross Y and we can generate a sigma algebra out of out of it. So, on the other hand we can first generate a sigma algebra out of the collection C and then generate the sigma algebra out of the collection D.

So, there are 2 ways of generating sigma algebra of subsets of X cross Y the first is look at C cross D, that is a collection of subsets of X cross Y and look at the sigma algebra of subsets of X cross Y generated by them on the other hand look at the sigma algebra generated by C. So, call it as S of C that is a sigma algebra of subsets of a X and generate the sigma algebra B Y the collection D. So, call that as S of D and then take the rectangles generated by these 2 a sigma algebras. So, that is sc cross S of D. So, the question is are these 2 things equal. So, let us observe that since a C cross D is already contained in S of C cross D. So, the sigma algebra generated by them will be inside it.

So, the first observation is that in general S of C cross D will be inside S of C cross S of D. So, that follows from the effect. So, this follows from the effect. So, we want to. So, we have got a set X, we have got a set Y.

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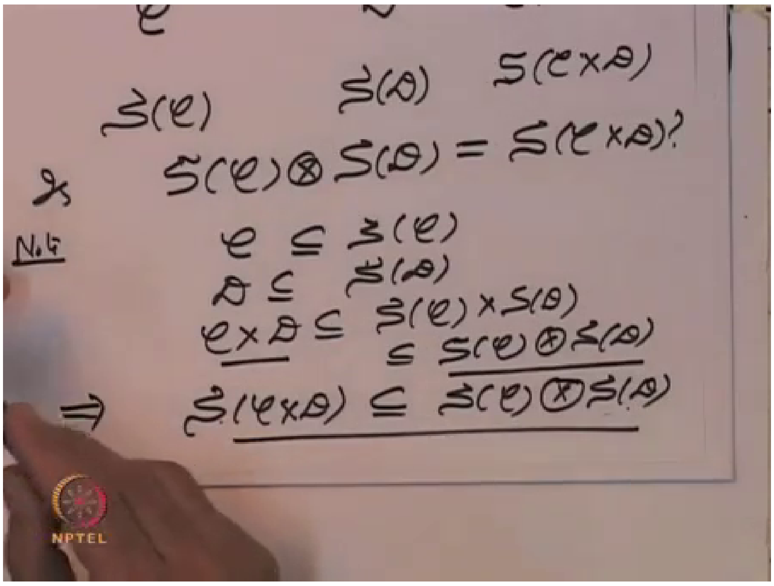

We have a collection of subsets of X we have a collection of subsets of Y that is D and we have a collect. So, we form a collection of subsets of X cross Y that is C cross D. So, we can generate the sigma algebra B Y, C we can generate the sigma algebra B Y D and also we can generate the sigma algebra B Y C, C cross D. So, the question we are analyzing is S of C cross S of D equal to S of C cross D.

So, the first observation is we are going to note is the following. So, C is contained in S of. So, the question is whether this product is equal to this. So, the question first observes that C is subset of S of C, and D is contained in S of D. So, C cross D is going to be subsets of S of C cross S of D. And we can generate the sigma algebra B Y these rectangles. So, that will be a subset of S of C times S of D. So, C cross D is always in the product sigma algebra product of the sigma algebras S of C times S of D. And so that implies the smallest one that is the sigma algebra generated by C times D must be inside S of C times S of D.

So, this is always true that the sigma algebra. So, first take the product of the families C and D and generate the sigma algebra out of out of it, and that is always a subset of first generate the sigma algebras S of C and S of D and then they take their a product S of C cross S of D. So, the question is the other way round equality true, and we will show by an example that the in general the inequality.

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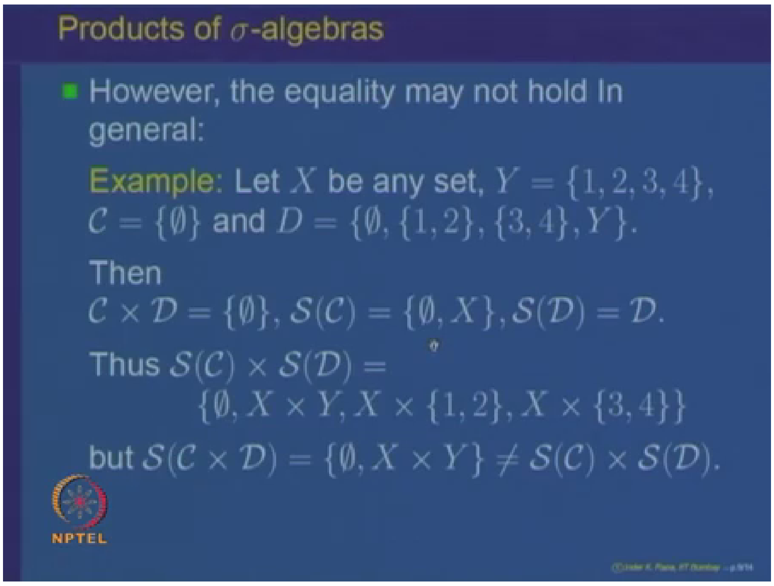


The other way round equality may not hold namely that S of C times S of D may not be a subset of S of C cross D. So, for that a simple example works. So, let us take X to be any set and Y to be the set consisting of 4 elements 1 2 3 and 4.

Let us look at C the collection which consists of just the empty set and the collection D which consists of the empty set the set 1 and 2 and the set 3 4 and the whole set Y. So, the collection D consists of 4 sets the empty set the whole space the subset with 2 elements one and 2 the subsets with 2 elements 3 and 4. So, note if we generate the sigma algebra B Y C that is same as the algebra generated by C. So, that will be just empty set and the whole space. So, S of C is empty set and the whole space X and the sigma algebra generated by D is equal to D itself because D itself is a algebra, the complement of one and 2 that is 3 and 4 that is here compliment of 3 and 4 in the set Y that is here 5 and empty set and the whole space are there.

So, D actually itself is a algebra. So, the sigma algebra generated by D is equal to because it is a finite collection. So, the sigma algebra is a algebra itself that is equal to D and if you look at the product the rectangle is related by C and D. So, the first component is always going to be empty set; that means, C and C cross D is as the empty set. So, the sigma algebra generated by S of the sigma algebra generated by the collection C cross D will be the empty set and the is complement that is the whole space that is a X cross Y. So, S of C cross D consists of just 2 elements namely empty set and the whole space X cross Y on the other end if we look at S of C cross S of D, then it consists of the empty set the whole space of course, and then it will consist of sets of the type X cross the set in Y that is one cross 2 and of course, the set X cross 3 cross a 3 comma 4.

So, there are at least the sets empty set the whole space and the sets of the type X cross the 2 elements set 1 2 and X cross the 2 elements at 3 and 4. And of course, this is not going to be equal to S of C cross D. So, even S of C cross D is not equal to even the rectangles. So, it can not be actually equal to the whole of S of C times S of D also. So, in general we cannot expect that given. So, in general we cannot expect that if you take 2 classes of subsets a one of X and one of Y. So, C is a collection of subsets of X and Y is a D is a collection of subsets of Y.

So, we can take the rectangles from C and D formed by taking elements from C and D. So, that is the sets denoted by C cross D and then we generate the sigma algebra out of this collection, so that will be the sigma algebra S of C cross D that may not be always equal to the sigma algebra generated by C times the sigma algebra generated by D. However, we would like to find some conditions under which we can ensure, these 2 are equal and that is our going to be out next theorem.