**Measure & Integration**
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**Lecture - 22 B**
**Lebesgue Integral and it is Properties**

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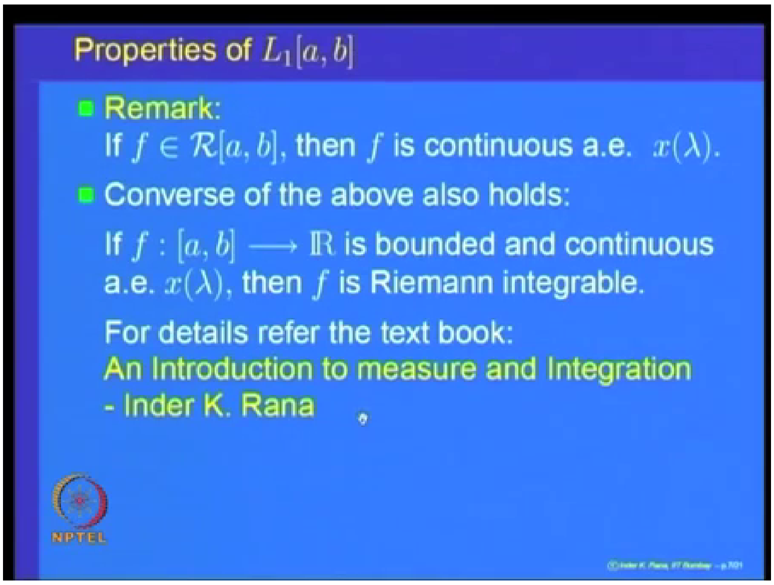


So, as a next step we want to look at so, the space of Riemann integrable functions are inside the class of Lebasque integrable functions. and there here is an observation which one can observed from the proof of this theorem, that if a function is Riemann integrable than it must be continuous almost everywhere. So, to conclude this observation from the proof itself basically what we are to look at is that the function f is the limit of those step functions right. there is the limit of the step functions. So, if you leave aside those partition points. So, we can remember we can have concluded that f is limit of those step functions phi ns and psi ns.

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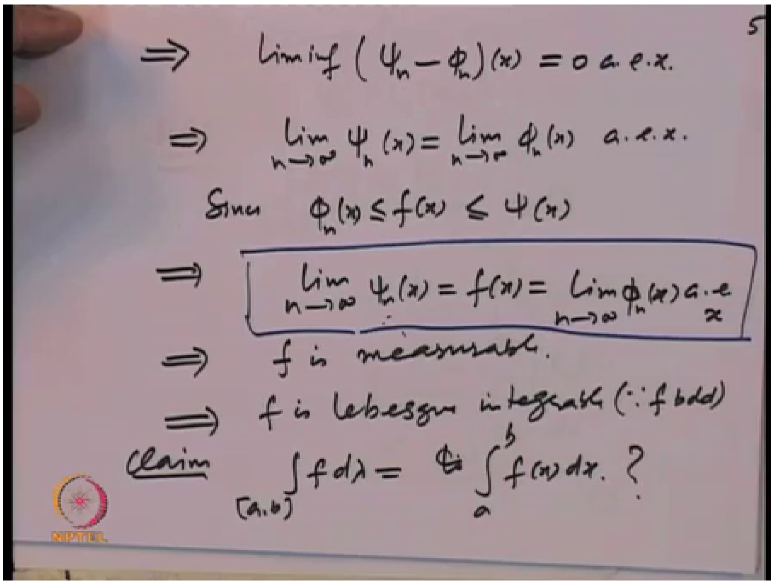


So, we concluded that. So, with that we concluded namely that here, this this is this is the fact that we proved in our theorem that the limit of the step functions phi n and psi n is f of x right. So, that almost everywhere. So, almost everywhere psi f is limit of psi ns and psi ns are piecewise continuous functions they are step functions. So, the points where psi ns may not be continuous are possibly the points of the partition points of pns. So, if you pro pool together all the points partition points they will be at the most countably money and if you remove them along with this almost everywhere set. So, outside a set of measure 0 this function f will become continuous.

So, I am just indicating the possibility of a proof of that. So, those interested probably I should look into the text book. and in fact, the converse of this theorem is also true namely that if f is a bounded continuous function, which is bounded function which is continuous almost everywhere then f is Riemann integrable so; that means, there is a characterization of Riemann integrabe functions in terms of continuity namely a function f is Riemann integrable if and only if it is continuous almost everywhere.

So, we are not given the proof of this. So, those interested probably should look into the text book as mention above, the in an introduction to measure and integration in which the complete proof is given, but what we have wanted to indicate that the proof of the theorem proof of one part of the theorem namely for a Riemann integrable function it should be continuous almost everywhere is already included in the proof of the fact this now we proved that Riemann integrable implies it is Lebasque inegrable

So, that is a one part of the properties of the space l 1 ab that rab the space of Riemann integrable functions is in lab.

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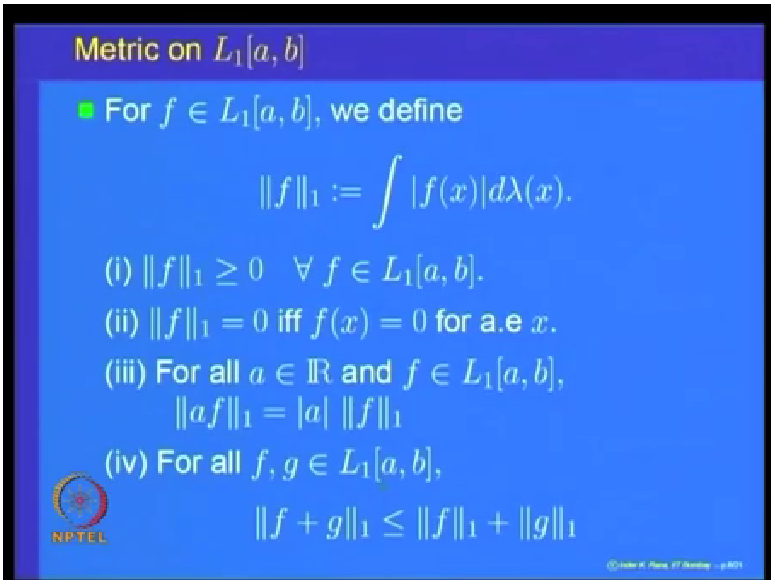


Now let us look at a metric on l 1 of ab. So, recall we have already shown that l 1 of ab a is a vector space, we showed that if f and g are integrable functions then f plus g is integrable f into g is integrable alpha times f is integrable. So, it is a linear space. So, it is a vector space over the field of real numbers.

So, on this we are going to define a notion of a magnitude. So, for a function f in l 1 we define it is what is called the l 1 norm of f to be the integral of the absolute value of f of x d lambda x. So, this is called the l 1 norm of the function f which is in l 1. So, clearly it is a number finite number because f is integrable. So, this right hand side exist and is finite and your integrating a non negative function. So, the first property namely the norm l 1 norm of a function is bigger than or equal to 0 for all functions f in l 1 of ab. So, that is obvious.

The second property namely (Refer Time: 04:46) this function f is 0 almost everywhere then clearly integral of the function will be equal to 0, that we have already observe. So, norms will be equal to 0 and conversely if the l 1 norm is equal to 0 then the function absolute value of f of x being a nonnegative function, it is Lebasque integral 0 that implies that the function must be 0 almost everywhere. So, the second property namely the norm is equal to 0 if and only if the function is 0 almost everywhere.

and the third property which is that if you multiply the function f by alpha the norm of alpha f is equal to the absolute value of alpha times norm of f. and that is obvious because in the definition if you replace f by alpha f then this being a constant the property observe a integral. So, the integral of alpha f is equal to mod alpha times integral mod f. So, that gives you the property that the l 1 norm of alpha f is equal to the absolute value of the constant these which you have multiplying alpha mod alpha times no norm of f 1.

And finally, the triangle in equality property namely if f and g are integrable functions then we have already shown that f into g is also a integrable function. and integral of the absolute value of f plus g will be less than or equal to integral of absolute value of f plus integral of absolute value of g by the triangle in equality of numbers. So, that will give us that the l 1 norm of f plus g is less than or equal to l 1 norm of f plus l 1 norm of g.

So, these are the properties of this magnitude or this norm of f 1 at this stage I just want to point out that this is very much, similar to the Euclidean metric or Euclidean norm on rn see on rn.

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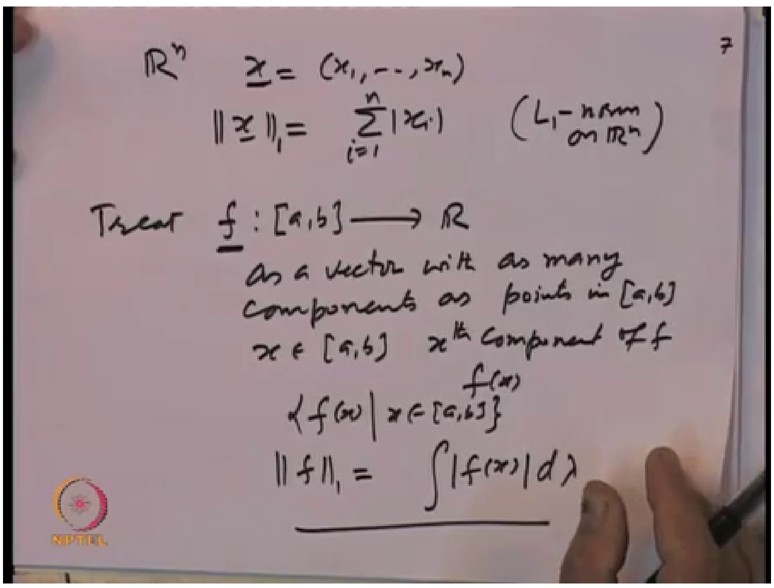

For a vector x with components x one xn we normally define the norm to be equal to you can take the norm to be equal to sigma mod of xi i equal to 1 to n. So, this is what is called the l 1 norm on rn. and now so what you are a saying is that if x is a vector with n components then this must be the norm l 1 norm.

Now, treat a function f defined on a interval ab to r. So, treat this treat as a vector with components with as many components as points in ab. So, I want to treat this f as a vector with components as many components as points in ab. So, for a point x in ab what is the xth component of f is nothing, but f of x. So, you can treat f of x x belong into ab as a vector. So, if you treat that way then what is the l 1 norm of f you would like to define. So, keeping in mind. So, take the absolute values of the components. So, take the absolute value of the components. So, this is component take it is absolute value and you want to sum it and the summation is nothing, but integral d lambda.

So, this is in in that sense this is the perfect generalization of the ordinary l 1 norm on rn. So, the only problem in this is that you do not have the property that if l 1 norm is equal to 0 if and only if f is equal to 0. So, only we have got that the l 1 norm is equal to 0 if and only if f of x is equal to 0 almost everywhere, but let us keep in mind that the l 1 norm does not change, if we change the function on in null set on a set of measure Lebasque measure 0. So, with with that observation in mind we from now onwards what we do is in l 1 of ab we identify functions which are equal almost everywhere.

So, if 2 functions are equal almost everywhere.

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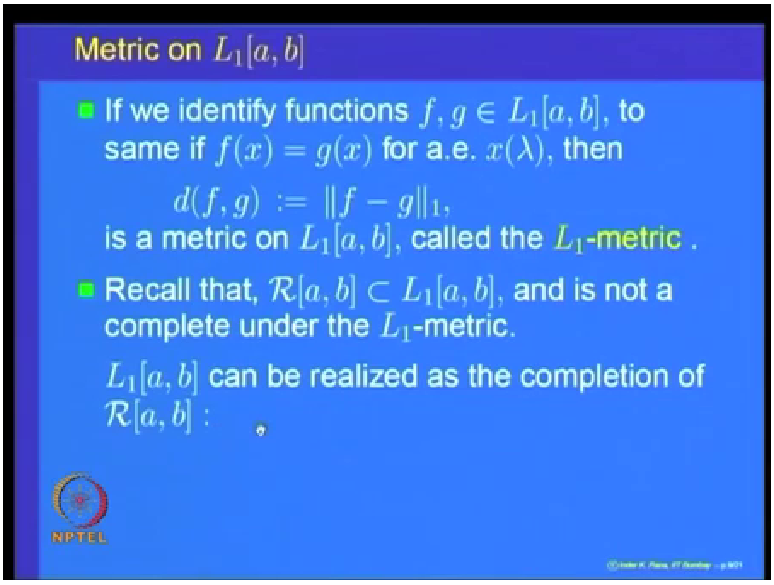


If f and g in l 1 are same are equal almost everywhere lambda, then we treat these functions to be as same. So, with that understanding this becomes a metric dfg which is equal to norm of f minus g l 1 norm of f minus g, becomes a metric on l 1 of ab and is called the l 1 metric.

And the fact we wanted to the observation, I want to point out is rab is a subset of l 1 of ab and. So, rab as I sub space with l 1 metric is not complete in the l 1 metrics. So, that is an observation which you should read from that text book already mentioned. So, and this was one of the defects of Riemann integral that motivated the development of Lebasque integral namely the space rab under the l 1 metric is not a complete metric. So, so this is not a complete metric under the l 1 metric so, but there is the theorem namely every a metric space can be a complete it.

So, there is an abstract theorem in metric spaces that if every metric space can be completed. for example, the you look at the set of rational numbers that is not complete under the usual distance of absolute absolute value of the distance and it is completion is the real numbers and that is the important property of real numbers that it is complete under that metric.

So, similarly rab under the l 1 metric is not complete, and there is a abstract theorem that rab can be completed it can be put inside a complete metrics space what we are going to show is that that completion is nothing, but l 1 of ab. So, we are going to prove that l 1 of ab the space of all Lebasque integrable functions on ab can be realized as the completion of the space rab under the l 1 metric and this will be doing in 2 steps one.

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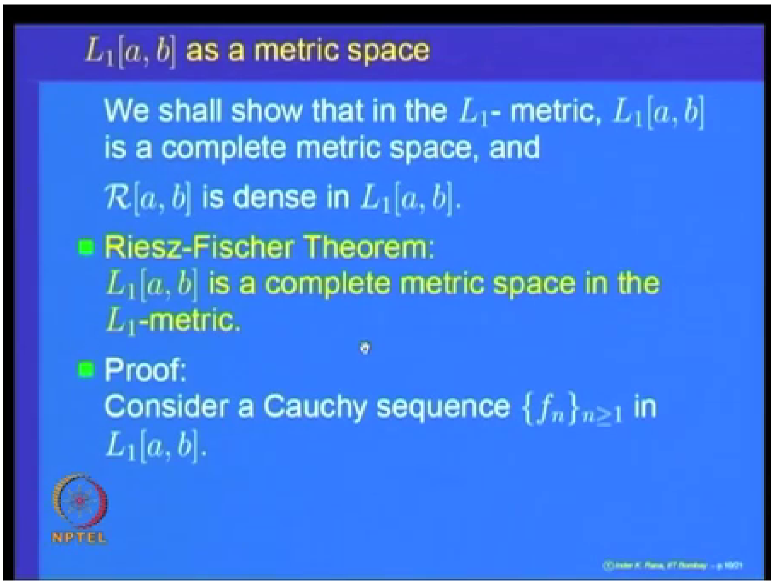


Who will show that in the l 1 metric the space l 1 ab is complete. that is one and it to be a completion of rab will show that rab sits inside l 1 of ab as dense sub set. So, rab sits inside l 1 of ab as dense sub set in the l 1 metric and l 1 is complete. So, that will show that l 1 of ab is a realization of the completion of the space rab like rationales are dense in the real line re reals den real are the field of real number is complete that. So, I say that real number is the completion of the field of rational numbers.

So, in the similar way we want to prove that l 1 of ab is the completion of the space of rab. so; that means, we want to prove that l 1 ab in the l 1 metric is complete and rab is dense in it. So, the completion at part is called riesz Fischer theorem. So, the riesz Fischer theorem says that l 1 of ab is a complete metric space in the l 1 metric. So, I let us look at a proof of this. So, the proof, that to prove that it is a complete metric space what we have to show we have to show that every Cauchy sequence fn in l 1 of ab converges to a value in l 1 of ab every Cauchy sequence in l 1 of ab is convergent and convergent to a point in l 1 of ab. So, that is what we have to show.

So, for that. So, let I will to show that. So, this is what we have to show.

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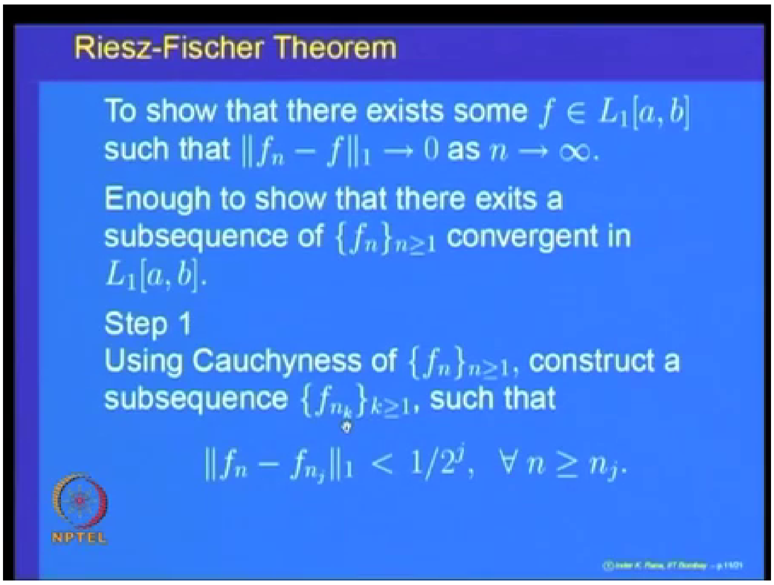


that there exist some function f in l 1 of ab say that fn minus f l 1 norm converges to 0 as n goes to infinity. So, this to prove this fact here is an observation what we will show is it is enough to show that there exists a subsequence of fn which is convergent in l 1 norm. So, what will show is to you show that fn is convergent in lone metric it is enough fn is Cauchy sequence to show that a Cauchy sequence is convergent, it is enough to show that there exist a subsequence of the sequence Cauchy sequence which is convergent that will prove that the sequence is a convergent. So, to do that. So, we have to produce a subsequence of fn which is convergent to a function in l 1.

So, let us look at the first step of our construction to construct that subsequence we are going to use the cauchyness of cauchyness of the sequence fn. So, fn is Cauchy. So, the first step is that saying that the sequence fn is Cauchy implies, that I can pick up a subsequence fnk of fn says that the norm of fn minus fn ks is less than 1 or 2 to the power k for all n bigger than or equal to nk.

So, to do that we start with. So, what is cauchyness means. So, saying that the sequence fn.

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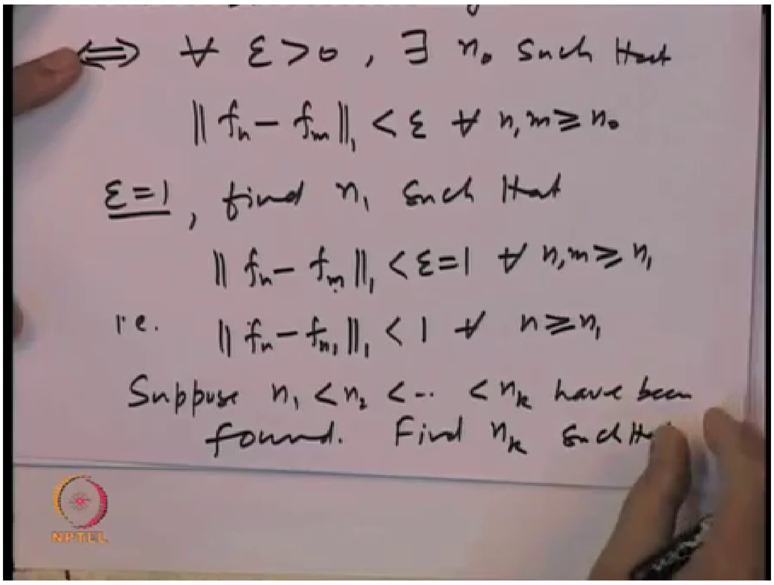


Is Cauchy saying that the sequence fn is Cauchy; that means, it is same as for every epsilon bigger than 0, there exist some stage and not such that norm of fn minus fm is less than epsilon for every n and m greater than or equal to n naught. So, that is cauchyness.

So, to our construction. So, so start. So, take epsilon equal to 1. and find n 1 such that norm of fn minus fm is less than epsilon equal to 1 for every n and m bigger than n 1. So, in particular when m is equal to n 1 that will give me. So, that is fn minus fn 1 will be less than one for every n bigger than or equal to n 1. So, that is the first stage ok.

So, now, suppose in. So, suppose n 1 less than n 2 less than nk have been constructed have been found. So, now, use cauchyness. So, find nk such that. So, how do I find nk.

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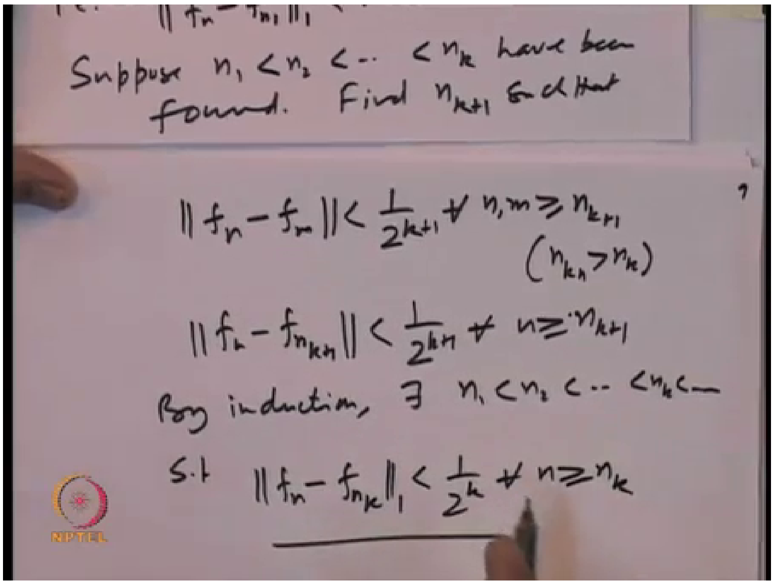


So, I will find nk such that norm of fn minus fm will be less than 1 over 2 to the power k for every n and m less than n and m bigger than or equal to find nk plus 1. So, there a next one you have to construct nk plus 1. So, I will find that nk plus 1 such that nk plus 1 is greater than mk and this property holds. So, then for m equal to nk plus 1, I will have fn minus fn k plus 1 will be less than 1 over 2 to the power. So, I i think put it to k plus 1 also that does not matter k plus 1 for every n bigger than or equal to n k plus 1.

So, by induction there exist n 1 less than n 2 less than nk less than. So, on such that norm of fn minus fn k is less than 1 over 2 to the power k for every n bigger than or equal to nk. So, that is step one we do that. So, once that is done we go to step 2.

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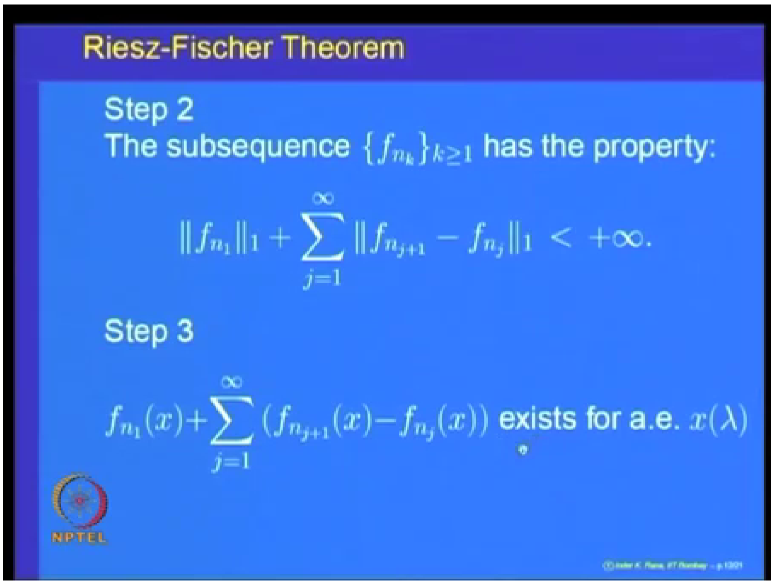


And the step 2 is showing that this subsequence, that we have constructed has the property that the some of the l 1 norms of fn 1 plus the l 1 norm of one to infinity of this is finite. So, that the sums of the l 1 norms of the fn 1 plus the l 1 norms of fn 1 plus the l 1 norms of fnj plus 1 minus fnj j one to infinity that is all finite and that term is that follows from the effect that just know we looked at this so; that means, norm of fn k plus 1 minus fnk is less than 1 over 2 to the power k.

So, if especially is n to be equal to nk plus 1 then I have I have I have this property for every k. So, that implies that if I look at the summation. So, norm of fn 1 plus sigma norm of fnk plus 1 minus fnk k equal to 1 to infinity will be less than or equal to norm of fn one plus sigma 1 to infinity 1 over 2 to the power j and that is finite. So, that will prove that the required property step 2 namely sums of norms of this quantities fn one and this is finite once that we we have that, now I can apply my series from of the Lebasque dominated convergence theorem.

So, as a step 3 as a next step, I want to conclude that if I look at the corresponding functions fn one x plus summation j one to infinity fnj plus 1 x minus fnjx then this series is convergent almost everywhere. So, that is preciously follows from the series from of the Lebasque dominated convergence theorem. So, if I look at this series then I know that l 1 norms of this series is finite. So, that is preciously the perfect situation where the series from of the Lebasque dominated convergence theorem applies and says that this functions they must this series must converge almost everywhere. So, this sum must exist almost everywhere and if you denote the sum by f of x then f of x equal to this exist almost everywhere.

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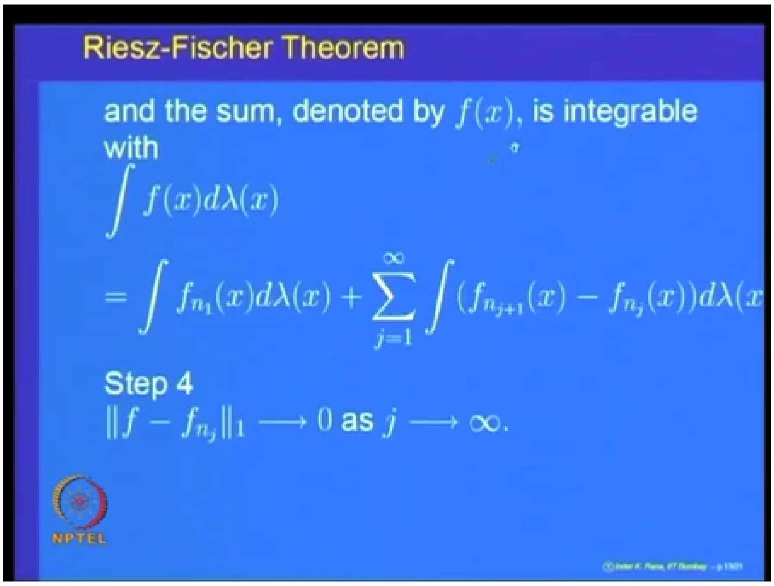


and it also says that that function is actually a integrable function and integral of f is equal to integral of fn 1 plus integral of the sums of the corresponding series. So, this integrals of f is equal to integral of fn 1 plus this series.

So, we are look at this function f and now it is only to claim that the difference f minus fnj goes to 0 that thus effects the limit of that subsequence. So, let us look at.

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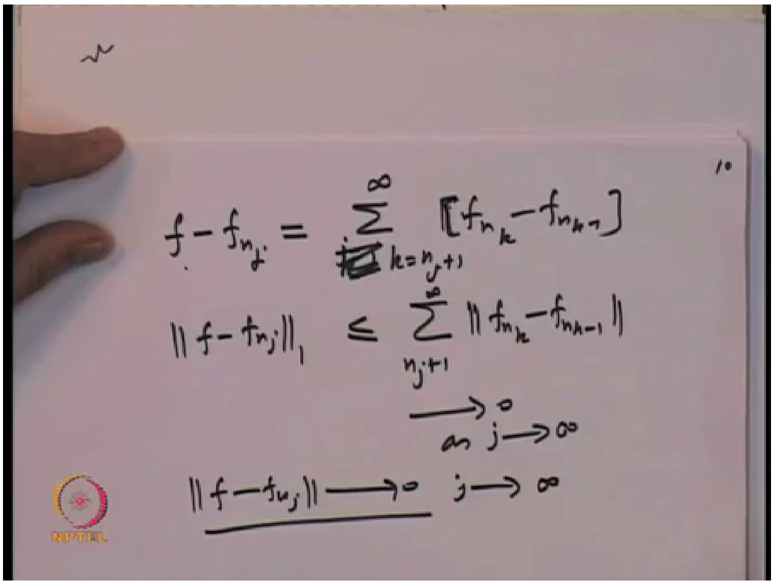

So, will note that if you look at f minus fnj that is preciously equal to summation j equal to k equal to j sorry let me nj so; that means, k equal to nj plus 1 of f nk minus fnk minus 1 1. So, the difference between f and fnj is nothing, but the tail of the series with the terms of fnk minus fnk minus 1.

So, norm of f minus fnj l 1 norm is going to be less than or equal to summation norm of fn k minus fn k minus 1 from k, from the stage nj plus 1 onwards and that is the tail of the convergent series geometric series 1 over 2 to the power j. So, that goes to 0 as j goes to infinity. So, that will complete the proof that hence the f minus fnj goes to 0 as j goes to infinity. So, that will prove that the subsequence fnj is convergent.

So, let us just go through go back to the proof once again basically the proof requires one observation namely to show that a sequence a Cauchy sequence is convergent. it is enough to produce a subsequence which is convergent and that subsequence is produce in such a way that by using the cauchyness we produce or subsequence with the property that the norms of the consecutive terms of that subsequence are less than 1 over 2 to the power j. once that is done we apply the series form of the Lebasque dominated convergence to conclude that the sum of fn one x plus summation fnj plus 1 x minus fnj that exist almost everywhere that is a integrable function and the integrals converge and once we have that it is obvious that norm of f minus fnj goes to 0, because once we subtract that the remaining thing is the tail of the series 1 over the 2 to the power j and that must go to 0.

So, with that we prove the riesz Fischer theorem. So, today we analyze 2 important things namely one that the space of Riemann integrable functions is inside the space of Lebasque integrable functions Lebasque integrable on the interval ab and the notion of Lebasque integral extends the notion of Riemann integrable. So, that was one property and the second property you have observed that the space of Lebasque integrable functions on the interval ab under the l 1 metric is a complete metric space. So, and we will con continue this analysis tomorrow. And we will show that the space of Riemann integrable functions which is inside the space of Lebasque integrable functions sits inside as a dense subset hence that will prove that l 1 of ab is the completion of the space of Riemann integrable functions.

Thank you.