Measure & Integration
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Lecture - 22 A
Lebesgue Integral and it is Properties

Welcome to lecture number 22 on measure and integration. If you recall in the previous lecture we had started looking at the properties of Levesque measure and Lebesgue of Lebesgue integrable functions. And we started looking at analyzing when does a function which is Lévesque integrable on the interval a b and it is relation with Riemann integral of the functions on the interval a b. So, let us we had starting looking at the proof of the theorem namely that if f is a function defined on a interval a b to R which is Riemann integrable, then we wanted to show that it is also Lévesque integrable. And Riemann integral is same as the Levesque integral.

So, we will continue the proof of that theorem, and then go on to analyze some more properties of the space of Lévesque integrable functions on the interval a b. So, today's lecture is going to be mainly concerned with Levesque integral and it is properties.

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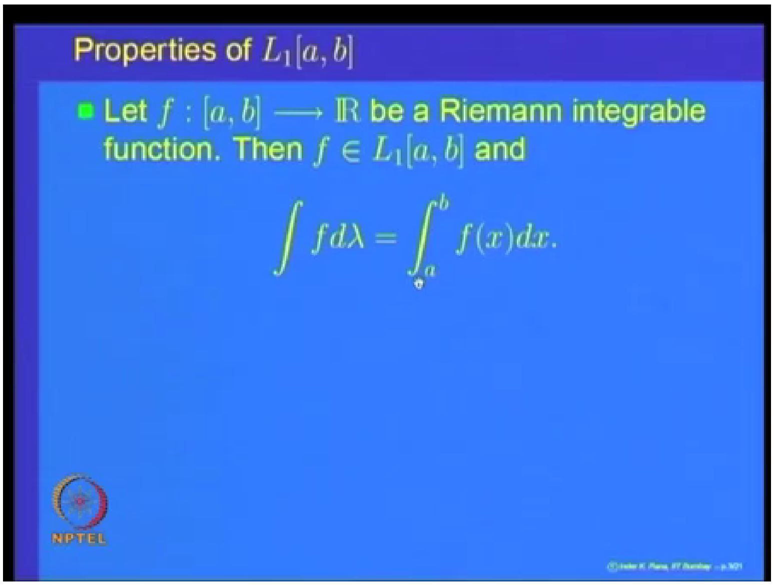

So, the theorem we wanted to prove was that if f is a function define on a interval a b to R and it is Riemann integrable, then it is also Levesque integrable and the Riemann integral of the function is same as it is Levesque integral.

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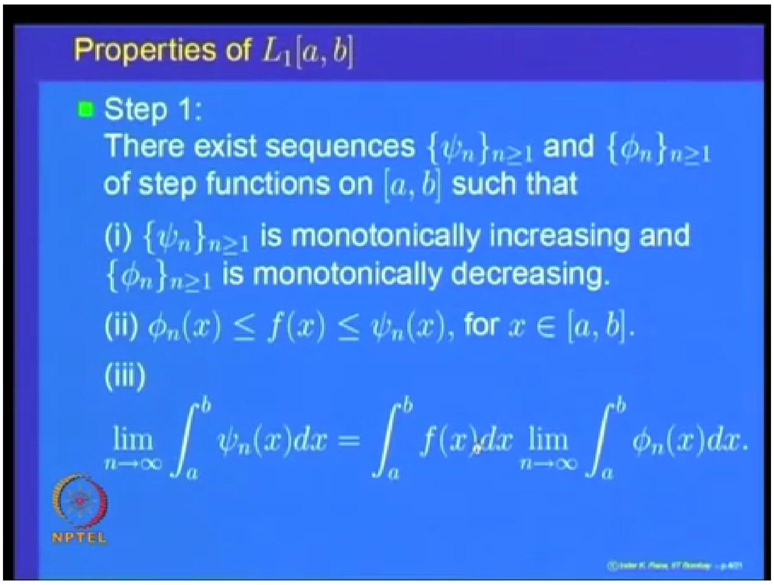

So, to prove this theorem we started with those idea that since f is Riemann integrable, there exist sequences psi n and phi n of step functions on the interval a b, such that this sequence psi n is monotonically increasing. And is sequence phi n is monotonically decreasing. And the function f is between these 2 sequences phi n and psi n for all points x belonging to a b, and the Riemann integral of psi n's converges to the same value as the Riemann Riemann integral of f and that is the same as the limit of the integrals Riemann integrals of the step function phi n.

So, let us recall this steps which we had proved last time. So, what we are given is f is a function defined on a interval a b to R, and f is Riemann integrable.

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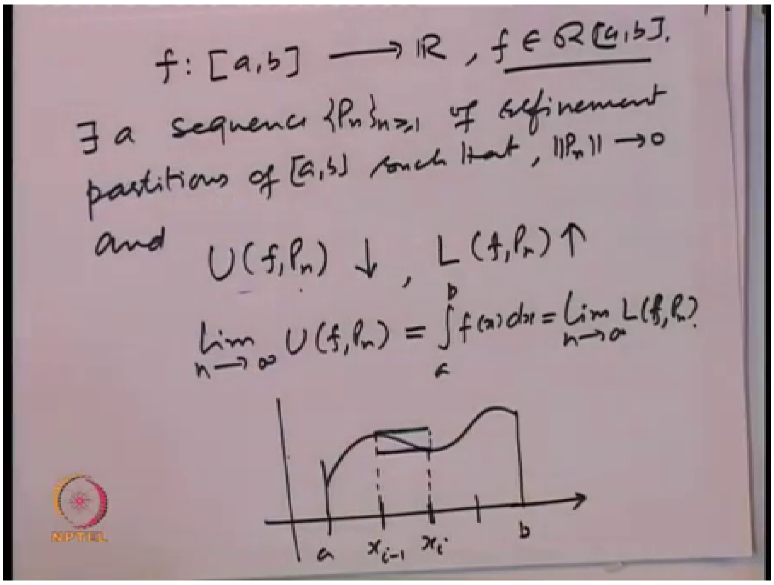

So, what does the Riemann integrability imply? The Riemann integrability of the function implies the following, namely there exist a sequence P n of refinement partitions of the interval a b such that the norm of these partitions goes to 0, and the upper sums of f with respect to these partitions that decreases and the upper sums and the lower sums of f with respect to P n's increases and the common value is the integral.

So, limit n going to infinity of upper sums is same as integral Riemann integral of f, and that is same as the limit of the lower sums. So, this is because f is Riemann integrable. Now let us see what are the upper sums, and what are the lower sums, we need to analyze them a straightly more carefully to look at. So, this is let us draw a picture of the function say the function look like this. So, this is a and this is b and we get the partition.

So, with respect to a partition; so, let us say this is the general interval say x i minus 1 and x i. So, in this interval look at what is the smallest value of the function. So, what is the smallest value of the function? That is this, so look at this height. And look at the largest value of the function in that interval. So, largest value of the function is somewhere here. So, look at that height. So, lower sums consist of the areas of this rectangle with height as the blue line. And the upper sums consist of the sums of all the areas which are the green lines.

So, mathematically what this means is the following. So, let us write mathematically what is means. So, mathematically these things mean the following namely. So, look at considering the function. So, let us write. So, consider let us define.

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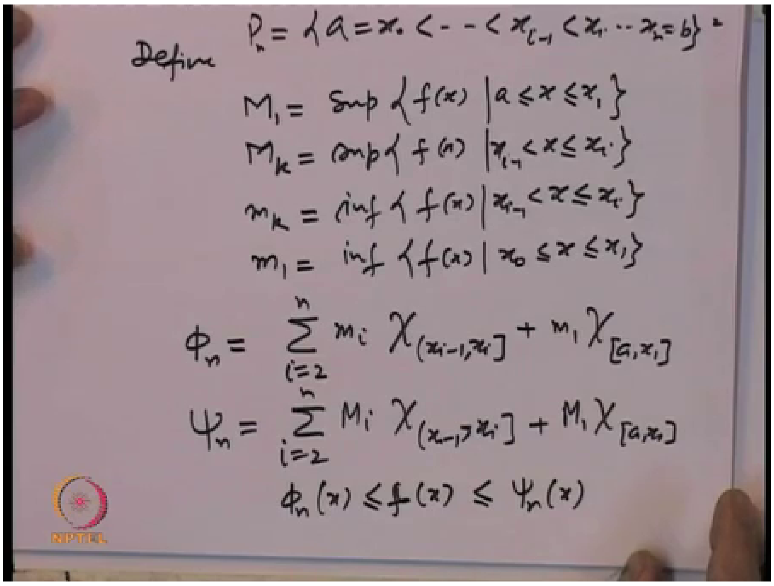

Define, So let us p is the partition say P n is the partition which looks like a equal to x 0 less than xi minus 1 less than xi, xn equal to b. So, let us say that is the partition. So, define let us say M 1 to be the supremum of the function fx x belonging to a that is less than or equal to x 1, x 1 and let us write mk to be the supremum of the function in the general interval. So, f x for x between xi minus 1 and xi. So, keep in mind here I am taking left open and right close here at the end point is both sides close. So, these intervals are disjoint intervals.

So, what I am doing is in the first part I am looking at this then I am looking at left open right close left open right close left open right close. So, I am partitioning the interval a b according to the partition points P n and then looking at the supremums in the respective intervals. Similarly let us write mk to be equal to infimum of fx in x xi minus 1 less than or equal to xi and M 1 to be the infimum in the first interval. So, that is fx x in x 0 x 1. So, what is this value? This M 1 and mks are corresponding to the height which is the green line. So, that is the maximum value or the function in the interval xi minus 1 to xi and the blue ones correspond to small mks.

So, once So, we had done this mathematically let us define the required functions. So, let us define phi n is the function which is summation mi indicator function of xi minus 1 to xi i equal to 2 to n. And in the first one So, let us put that value as M 1 the indicator function of a x 1. And similarly let us write psi n to be sigma i equal to 2 to n capital Mi the maximum value in the interval xi minus 1 to xi. And capital M 1 in the first intervals. So, that is indicator function of a 2 x 1.

So, these are the functions we defined. So, they are corresponding to So, the function which is phi n small phi n it will look like the minimum values like this, and it will look like So, it will look like this. And the capital then psi n's they will look maximum values look like this, and look like this and look like this.

So, quite clearly they are this functions phi n and psi n are step functions, and phi n of x is less than or equal to f of x and less than or equal to psi n of x. So, as a first step the Riemann integrability of the function f over the interval a b gives us a sequence of functions phi n and psi n, where each phi n is a step function each psi n is a step function and phi n is less than or equal to f of x is less than or equal to psi n of x. And since we our requirement was that Riemann integrability implies that is a sequence P n of partitions is a sequence of refinement partitions. So, that implies that the sequence psi n will be a decreasing sequence and phi n will be an increasing sequence.

So, let us write that observation namely that so, we have phi n's.

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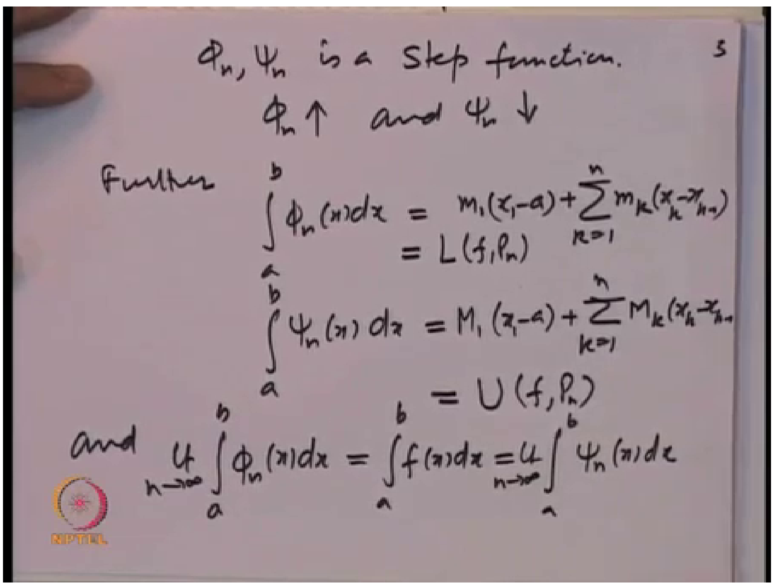

So, each phi n psi n is a step function phi n's are increasing and psi n's are decreasing. And moreover further let us look at the Riemann integral of the function phi n x dx a to b. So, because phi n is constant on each sub interval of the partition this is nothing but equal to M 1 times the length of the first interval. So, that is x 1 minus a plus the sums of those rectangles. So, that is small mk times xk minus xk minus 1 k equal to 1 to n. And similarly the Riemann integral of the can psi n x dx is equal to capital M 1 in the first interval times the width of the interval that is x 1 minus a plus k equal to 1 to n the areas of the other rectangles. So, that is capital Mk times xk minus xk minus 1, right.

So, those are the Riemann integrals and by the definition of the upper and the lower sums this is precisely the lower sum of f with respect to P n and this is precisely the upper sum of the function f with respect to the partition P n. And saying that the function is Riemann integrable implies that the upper sums and the lower sums converge to the same value. So, this and the Riemann integrability implies that the Riemann integral a to b of phi n x dx limit n going to infinity is same as the Riemann integral a to b fx dx and that is same as the lebesgue same as the Riemann a limit of the upper sums. So, that is limit n going to infinity the Riemann integral a to b of psi n x dx.

So, that proves our first step. So, as a consequence of the Riemann integrability of the function f we have constructed 2 sequences of step functions psi n and phi n, where psi n is monotonically increasing and phi n is monotonically decreasing. And limit are both the integrals of both of them converge to the given integral of f. Now let us observe at this stage. So, in some sense the step functions and the Riemann integrals are the building blocks for the Riemann integral.

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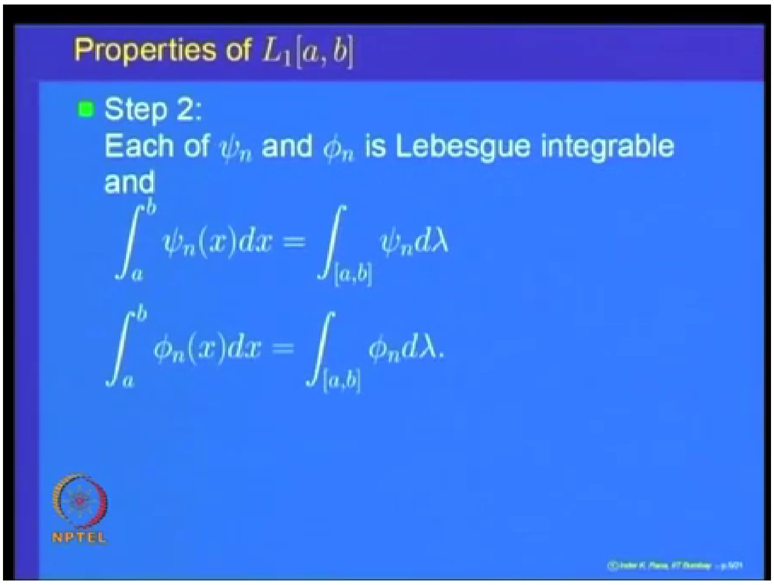

Now, let us observe that the lower sum of the function f with respect to the partition which was the Riemann sum, which was the Riemann integral of phi n is also nothing but the Lévesque integral of the function phi n with respect to the Lévesque integral. Because this is this function phi n is a simple function simple measurable function and it is integral is nothing but this integral. So, the observation is that for every n integral a to b of phi n x dx the Riemann integral is same as the Lévesque integral of the function phi n with respect to the Lévesque integral over the interval a b.

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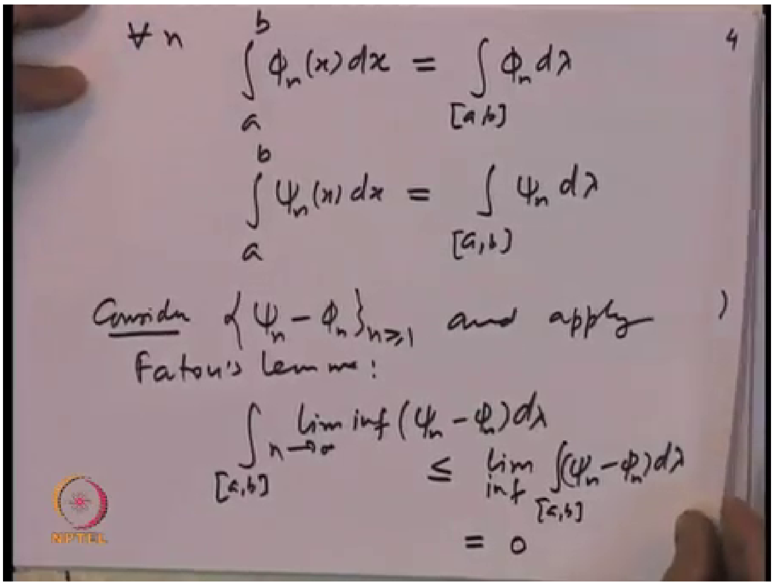

And this observation is simply by the fact that phi n is a step function. So, hence it is a simple measurable function that is the integral is nothing but the value times the Lévesque measure of the portion on which that value is taken, and that bring here sub intervals. So, it is same as the Riemann integral. And similarly the Riemann integral a to b of psi n x dx is equal to integral over a b of psi n d lambda. So, this is the observation which is going to play a important role for us.

So now let us define let us consider the because we have got the observation that the function fn the function phi n is always dominated by fx is less than or equal to psi n. So, let us look at the function psi n minus phi n. So, look at the function. So, consider the sequence psi n minus phi n. So, look at the sequence of these step functions. These are step functions as well as they are simple measurable functions and they are non negative. So, they are non negative simple measurable functions and by Fatou's lemma. So, we are going to apply Fatou's lemma to this. So, this consider the sequence and apply Fatou's lemma. So, what will that give me? So, that will give me that limit inferior of psi n minus phi n d lambda limit n going to infinity integral over a b.

So, the Lévesque integral of the limit inferior is less than or equal to limit inferior of the integral psi n minus phi n d lambda. So, that is by Fatou's lemma, but let us observe here that integral of psi n over a b minus integral of phi n over a b, the limit inferior of that is equal to 0, because integrals the Levesque integrals of psi n are same as the Riemann integrals of psi n and Levesque integrals of phi n are same as the Riemann integrals of phi n and those Riemann integrals converge to the Riemann integral of f. So, this right hand side is equal to. So, this is equal to 0. So that means, that the limit inferior of psi n minus phi n that function is a non negative function and it is integral Lévesque integral is equal to 0. So, that we know implies that the function must be 0 almost everywhere.

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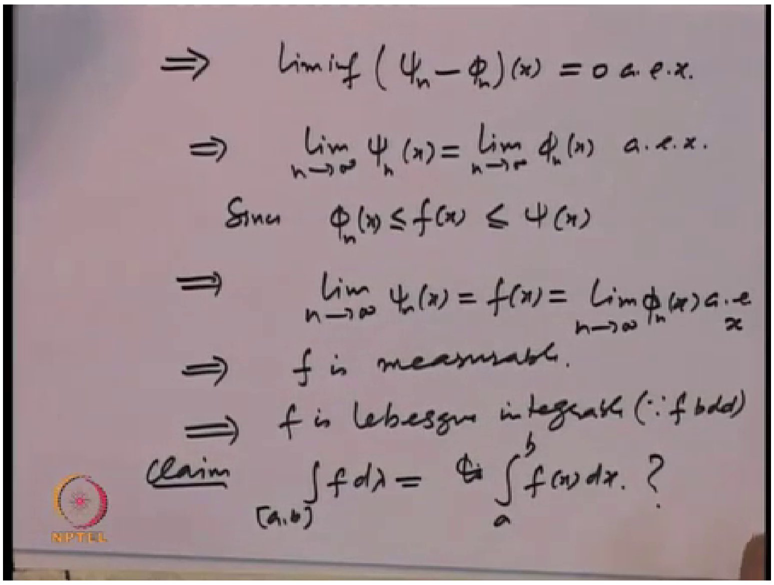

So, this implies that limit inferior of the sequence which is psi n minus phi n x must be equal to 0 almost everywhere x. But we know that psi n's are increasing psi n's are decreasing and psi n's are increasing. So, this limit inferior exists that is that is. So, that implies that limit n going to infinity of psi n's is equal to limit n going to infinity of phi n x almost everywhere x, but we know since f of x is between phi n and psi n of x this implies along with the earlier effect this implies that limit n going to infinity of psi n x must be actually equal to f of x actually equal to limit n going to infinity of psi n of phi n of x for almost all x.

So, this must happen, but that implies because each psi n is a measurable function each phi n is a measurable function this implies f is a measurable. But recall f was Riemann integrable function, so obviously, it is a bounded function. So, implies that f is Lévesque integrable because f bounded. So, by boundedness of f every bounded measurable function and all this is define on a b which is a finite measure space. So, that must be integrable that we had observed earlier. So, f is Riemann integrable.

So, only we have to prove now; so, claim. So, that integral fd lambda over a b is equal to integral a to b of fx dx. So, this is the only thing left to be shown. So, we have what we have shown till now is f is Lévesque integrable. So, this integral exist and now let us observe one thing not only f is Lévesque integrable it is a limit of a sequence of functions phi n's phi n's converge to f of x. So, the sequence phi n is converge into f of x. And psi n's are decrease into f of x. So, whichever fact we require you can use. So, let use the fact that psi n's are limit of psi n's converge to f of x they are decreasing and each psi n is a integrable function. So, we can apply Lévesque dominated convergence theorem to conclude; so, by dominated convergence theorem.

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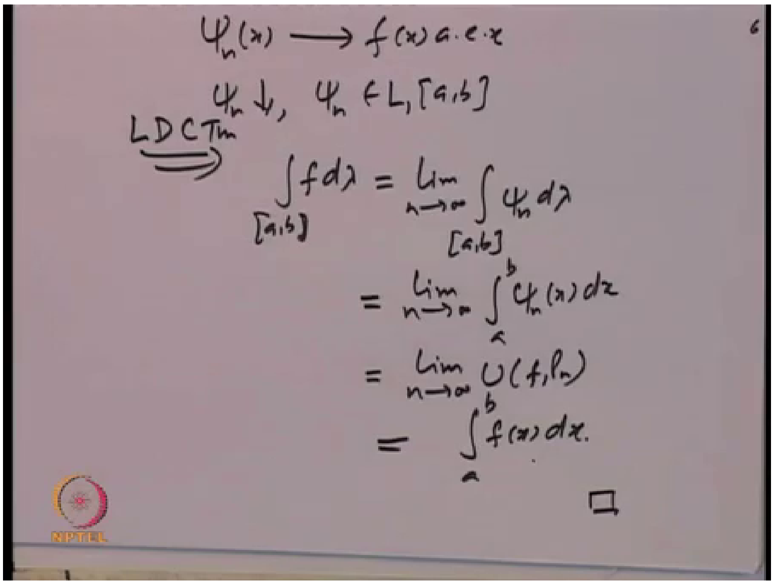

So, since psi n x converge is to f of x almost everywhere x and psi n's are decreasing and integrable psi n's decreasing psi n's integrable. So, implies by Lévesque dominated convergence theorem that integral of f d lambda over a b must be equal to limit n going to infinity of the labesque integral of psi n d lambda. So, that I end that is the observation of a already made that the Labesque integral of the step function psi n is same as the Riemann integral. So, n going to infinity. So, that is a to b of psi n x dx.

And this Riemann integral we have observed is the upper sum and which limit is equal to So, this is the limit of the upper sums of f with respect to P n and that converge to integral the a to b fx the dx. So, that will prove that integral Labesque integral of f is same as the Riemann integral of f over d lambda. So, that proves the theorem. So, let us go back and recall the proof the what are the first step as a first step in the proof, using the Riemann integrability of the function we construct 2 sequences of step functions phi n and psi n. So, that f is trapped in between them and the upper sums are nothing but the upper sums are nothing but the Riemann integrals of phi n's of psi n's and the lower, lower sums are nothing but the phi n's So that means, Riemann integral of phi n converge into the Riemann integral of f and which is also equal to So, here is a equality sin equal to the Riemann integral of phi n.

So, that is this construction is purely from the fact that fns that the function f is Riemann integrable. And the second observation is that each phi n and psi n being as step function is also measurable and Lévesque integrable. And the Lévesque integral of phi n is same as the Riemann integral of phi n and the Lévesque integral of psi n is same as the Riemann integral of psi n. So, that is the second observation we will break. So, these 2 integrals Riemann integrals of the step functions are same as the Lévesque integrals.

And now because of the earlier consequence that the Riemann integrals of phi n's and psi n's converge to the Riemann integral of f look at the difference phi n minus psi n.

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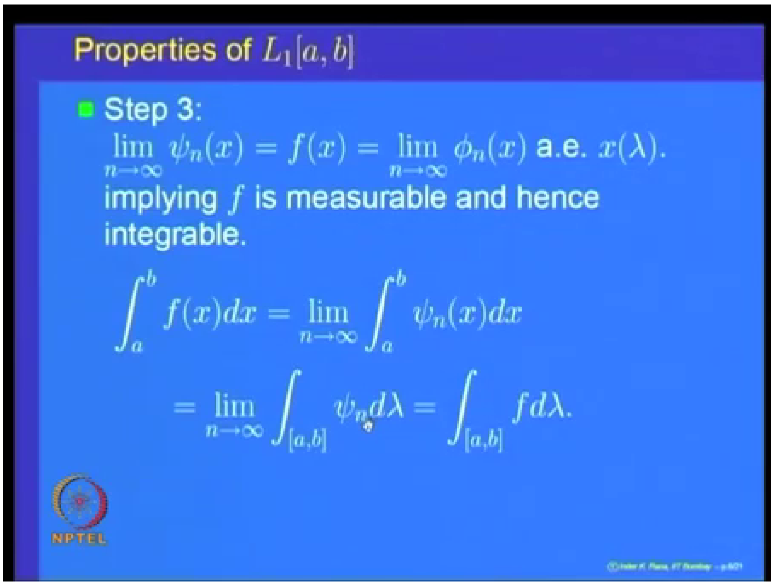

So, look at that sequence of measurable functions phi n minus psi n that is a measurable non negative measurable function, and an application of Fatou’s lemma I will give us that this psi n's and phi n's both must converge to the same value and that is f of x almost everywhere. So, that will imply that f is measurable and being a bounded measurable function on a finite measure space it becomes integrable. And now this psi n's are decrease into f of x. So, an application or Lévesque dominant it convergence theorem gives that the Riemann integral of f is same as the limits of the Riemann integrals of psi n's which are equal to the Lévesque integrals of psi n and that dominated convergence theorem gives you that is the Lévesque integral of f over a b.

So, that proves the theorem a completely namely. So, this is a the step we wanted to this was the beginning of our lectures. Namely we wanted to say that to remove the defects of to remove the defects of Riemann integral namely, that the fundamental theorem of calculus may not hold and that the space of Riemann integrable functions may not be complete under what is called the l one matrix we wanted to extend that the notion of integral from Riemann integrable function to a bigger class.

So, we have constructed here a class of functions on a b which are called the Lévesque integrable functions on a b and we have shown that if the class of Riemann integrable functions is a subset of the class of Levesque integrable functions. And the notion of Levesque integral extends the notion of Riemann integral beyond the class of Riemann integrable functions. So, that was there is a first step in our extention theory. So, Levesque integral is an extension of the Riemann integral.