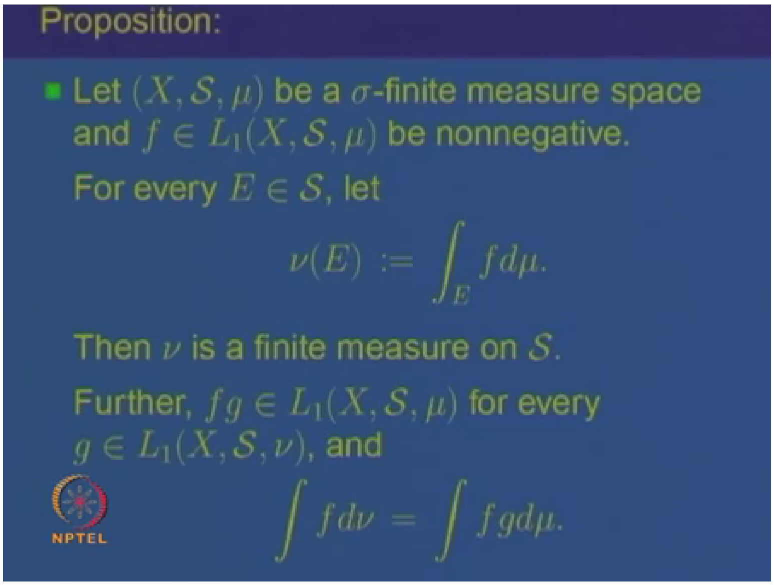
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

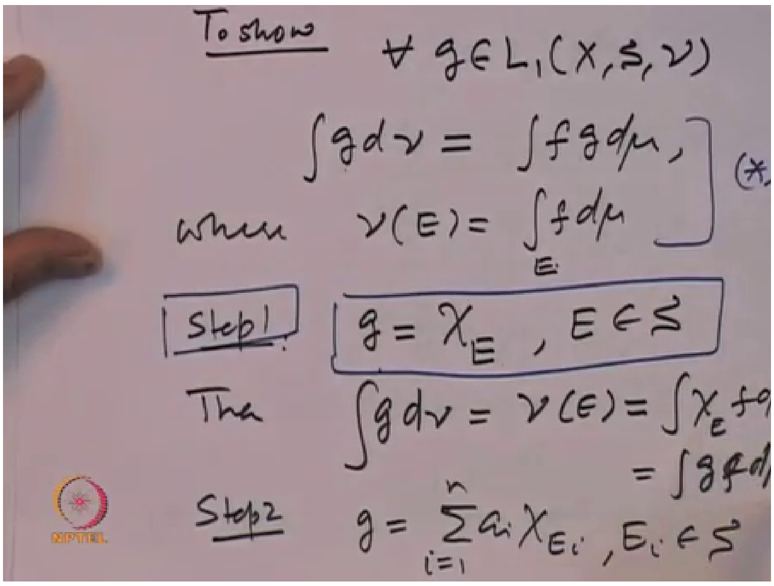
Lecture – 21 B
Dominated Convergence Theorem and Applications

So, let us see how the simple function technique is used to prove this result.

(Refer Slide Time: 00:23)



(Refer Slide Time: 00:25)

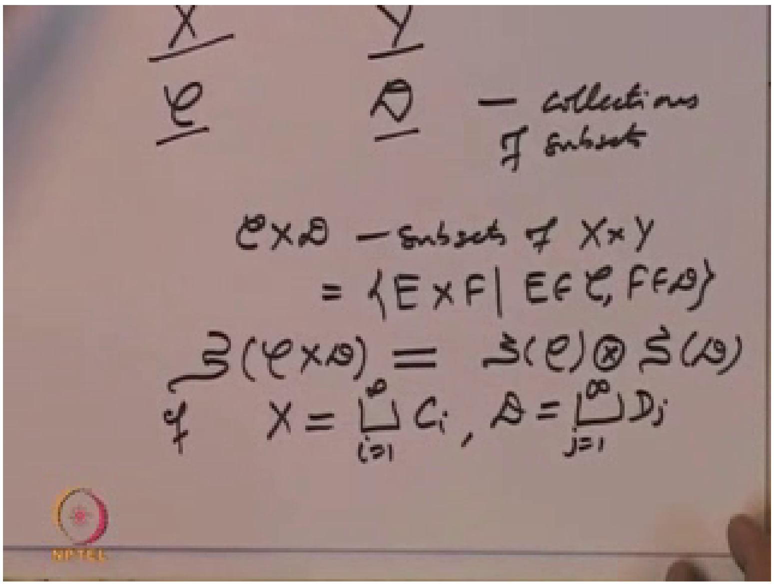


So, let us start. So, we want to show that for every g belonging to L 1 of X, S, nu integral of g d nu can be represented as integral f g d mu. So, where recall, we defined nu of E to be equal to integral of f d mu over E. So, this is the, this is the property we want to prove. So, this is the property star we want to prove for every function g.

So, as we said let us first. So, check this property. So, step 1, let us take g is g is a L 1 function. Let us say g is a function which is indicator function of E. Let us take g as the indicator function of E, E belonging to S. In that case, the integral g d mu, the left hand side is nothing, but nu of E right because g is this is the indicator function so, this integral of nu of E which by definition is equal to integral chi E of f d mu. And So, chi is g. So, this is equal to integral g f d mu.

So, what it says, it say that the required property star holds, when g is the indicator function. And now let us take a nonnegative simple function. So, that is step 2: Let us take g is sigma a i chi of E i, i equal to sum 1 to n; where E i is belong to S. And our claim is that this property holds for this g also. So, we are saying that next step is to verify that the required property.

(Refer Slide Time: 02:46)

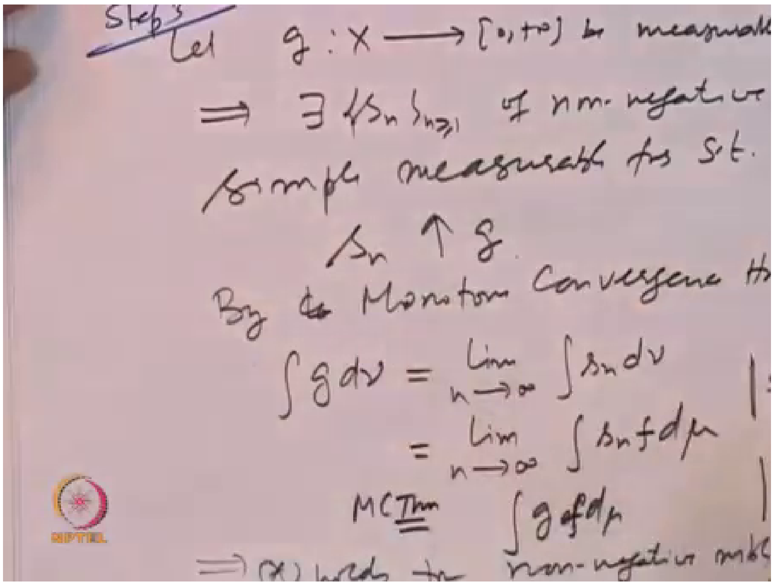


So, integral of g d nu by definition is equal to integral of sigma a i indicator function of E i d nu and what is that; by inherit property of the integral it is sigma i equal to 1 to n of a i, that is scalar times nu of E i, because integral of the indicator function is the measure.

So, that is equal to sigma i equal to 1 to n, a i and nu of a i by definition is integral chi E i of f d mu right that is the definition of nu of E i is this.

So, which I can write it again as sigma i equal to 1 to n of a i we can take it out. So, let us. So, this is and again by the linearity property of that is integral a i chi E i times f d mu, but this is nothing, but my function g. So, this is integral of g f d mu. So, what we are saying, that if g is summation a i chi E i, then using linearity property this is same as integral goes in. So, that is a i integral of the indicator function of E i that is nu of E i. And nu of E i by definition is integral over E i of f d mu. And again using linearity property of the integral I can shift it outside. So, it is integral of summation a i chi E i times f d mu which is g. So, it says. So, the required property holds. So, star holds, for nonnegative simple functions g. So, that is what I said simple function technique. And now let us try to prove it, that this property also holds, when g is a nonnegative measurable function. So, let us look at now g.

(Refer Slide Time: 05:11)

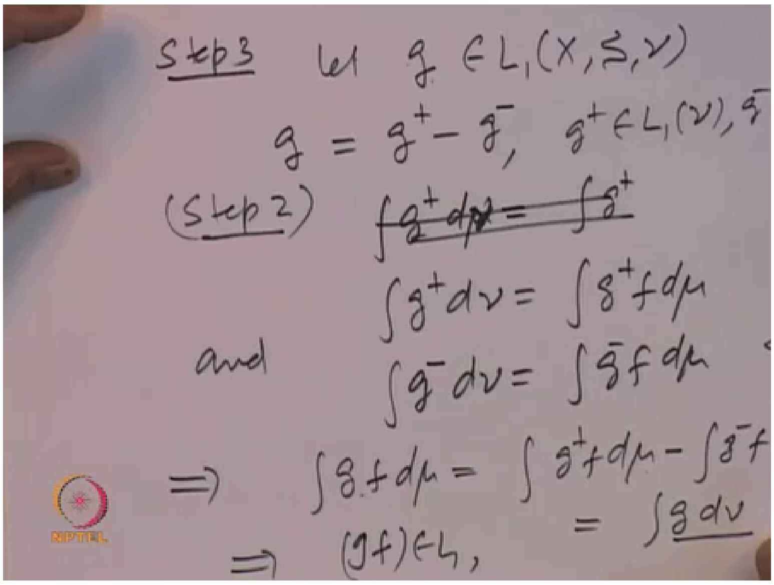


So, let g on X to be measurable. Then we know by the property of measurable functions implies; there exist a sequence S n of nonnegative there is sequence S n of nonnegative simple measurable functions. Functions, such that S n increases to the function g, so then by Lebesgue by Modern Convergence theorem integral of g d nu with respect to nu must be equal to limit, n going to infinity integral s n d nu, but for nonnegative simple functions just now we proved this the star holds; that means, this can be written as limit.

So, by step 2: I can write this as integral of s n times f d mu. So, with the integration of a nonnegative simple measurable function with respect to nu, can be converted into the nonnegative simple measurable function multiplied by f d mu. And now at this stage, we observe that if s n is increasing to g, then s n time's f will be increasing to g time's f.

And all are nonnegative simple measurable all are nonnegative measurable functions. So, once again Monotone Convergence theorem is applicable, and this limit is nothing, but integral of g f d mu that is; so, once again we have used. So, this step this step was by our step 2 that the property holds for non negative simple measurable functions. Integral with respect to nu is integral with respect to mu of the product function. And now once again we are applying Monotone Convergence theorem. So, first integral of g is equal to limit of integral s n s d nu by Monotone Convergence theorem, and now by our earlier step this is equal to integral of s n f d mu again by Monotone Convergence theorem it goes back so; that means, this implies that star holds for nonnegative measurable functions. And now let us come to the last part namely:

(Refer Slide Time: 08:22)



So, let us final step 3: is let g belong to L 1 X, S, nu. G be a integral function. Now then what is g equal to, g plus minus g minus, where g plus is a nonnegative measurable function g minus is a non negative measurable function and by step 2. So, by step 2: we know that integral g plus d mu is equal to integral of g plus sorry d nu. So, let me write it again integral g plus d nu is equal to integral g plus of f d mu and integral of g minus d nu is equal to integral g minus f d mu. So, that is by step 2. And now because g is L one, so, that implies g is equal to g plus g minus. So, g plus is in L 1 of nu and g minus also belongs to L 1 of nu right.

A function g is integrable, if and only if; its positive part and negative parts are integrable so; that means, this quantities right they are all finite, these are all finite quantities so; that means, what and f is nonnegative. So, that implies integral of g f d mu is equal to integral of g plus fd mu minus integral g minus f d mu. By definition of the positive part and the negative part of the function g f f is nonnegative. So, the positive part of the function g f is same as g plus times f and the negative part is nothing, but g minus f. And now both of these are finite quantities so; that means, implies that g f is L 1 and by these two this is same as integral of g d nu. So for a so the step 3: for a g which is integrable, we have deduced that this property is true. So, this is step 3. So, this is what I called the simple function technique. So, let me go back and show you once again, what we have done? So, we wanted to show that, so this is property star, we wanted to show for every function g which is L 1.

First step is, so this is my step 1: that look at the functions g which are indicator functions. So, I want to verify this for the indicator function, g to be the indicator function. So when g is the indicator function, so this left hand side is integral of over E of the constant function 1. So, this is equal to integral d nu is nu of E, which by definition is integral f over E, which I can write as integral f E; so, that is true. So, step 1 is to verify the required thing holds for characteristic function. And step 2: by using the property that the integral is linear, we show it is true for every non negative simple functions.

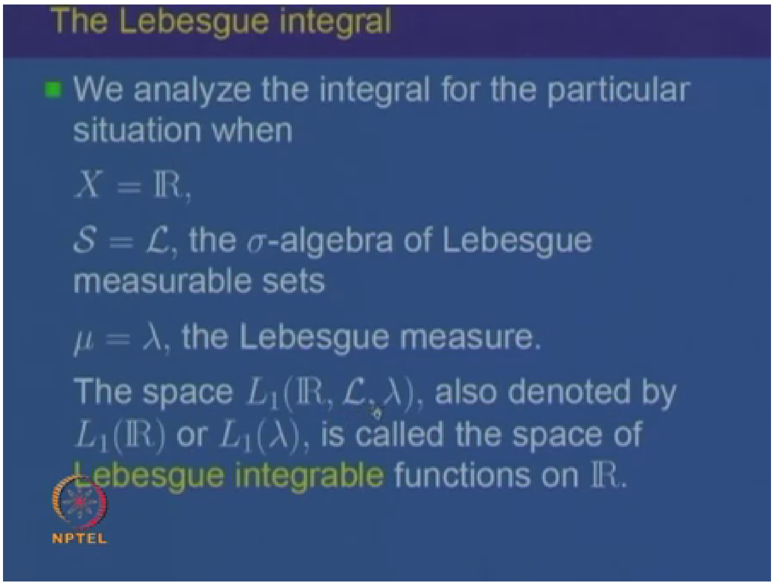
So, take g a nonnegative simple measurable function and apply. So, g is equal to integral of nonnegative a i indicator function a i and interchange and so, it required property holds. So, the step 2 as that the required property holds for nonnegative simple for nonnegative simple measurable functions and then using an application of Monotone Convergence theorem so that is step 3. That if g is a nonnegative measurable function then we know it is a limit of nonnegative simple measurable functions increasing limit. So, an application of Monotone Convergence theorem together with the earlier step gives us that integral of g d nu is equal to integral of g f d mu.

So, that is the next step to show that it holds for nonnegative measurable functions. And once that is done, the final step that it holds for all integrable functions is via; splitting the function g into the positive part minus the negative part. And g integrable means both are integrable and for each one of them, the required claim star holds. So, by putting them together we get that the required claim holds property star holds for all functions g. So, which are L 1. So, this is what I normally call as the simple function technique. So, while proving results, about integrable functions one uses quite often the simple function technique.

And while proving some properties about subsets of sets recall; we had the sigma algebra Monotone class theorem technique. So, for proving properties about sets, one uses the Monotone class sigma algebra, Monotone class technique and for proving results about integrals one normally uses what is called the simple function technique.

So, with this we have defined and proved general properties about integral of functions on sigma finite measure spaces. Now we will try we will specialize this property this construction when X is real line. So we want to specialize this thing, for the real line. So, let us see what we get.

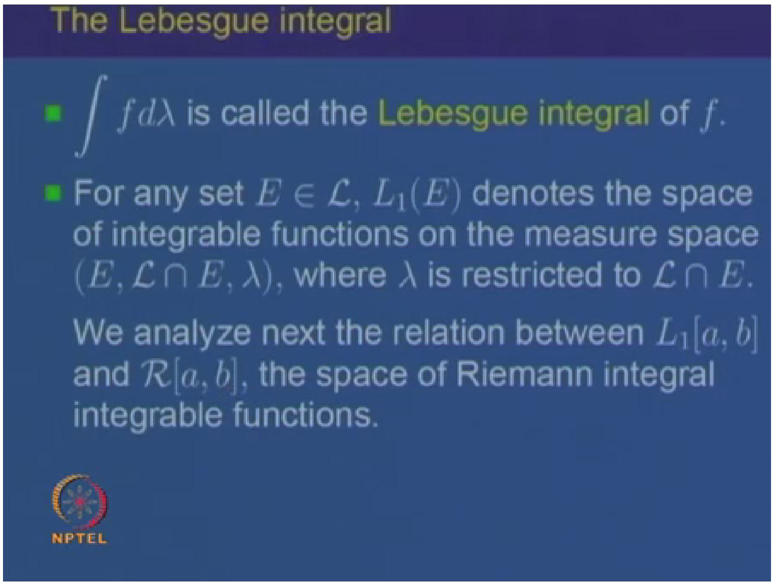
(Refer Slide Time: 14:39)



So, while you looking at the special case, when X is real line. The sigma algebra is L that of Lebesgue measurable sets and the measure mu will be the lambda the Lebesgue measure. So, we will be working with the measure space X S mu which is same as real line Lebesgue measurable sets and Lebesgue measure. So, the spaces of all integrable functions on this measure space are L and lambda is called the space of all Lebesgue integrable functions and is also denoted by L 1 of R or L 1 of lambda. So, this is the space of all Lebesgue integrable functions.

So, we want to study, this space of Lebesgue integrable functions in some more detail.

(Refer Slide Time: 15:34)



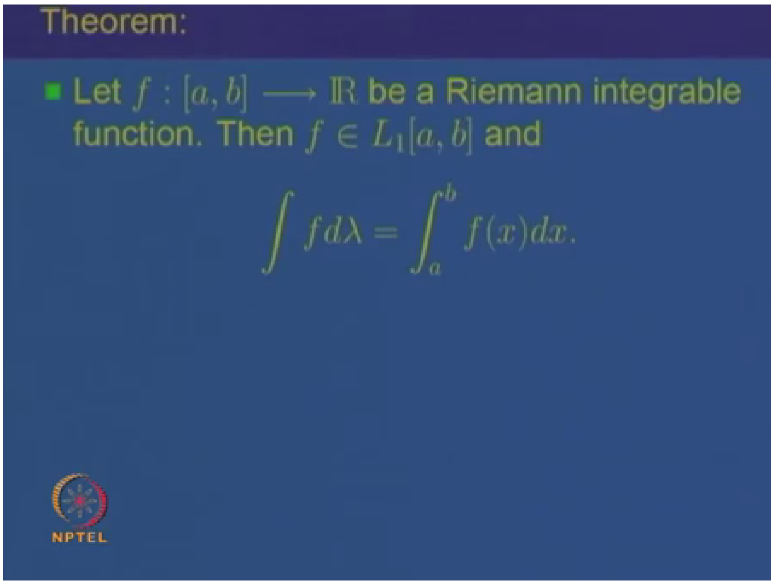
So, let us first agree to call integral f d lambda to be the Lebesgue integral of the function f. So whenever, f is integrable or nonnegative integrable f d lambda will be called the Lebesgue integral of f. Sometimes we have to look at functions which are defined on subsets of E. So, for any subset E, which is Lebesgue measurable, L 1 of E will denote the space of all integrable functions on the measure space E, so underlying set is E, L intersection E is the collection of all Lebesgue measurable sets inside E and lambda is the Lebesgue measure restricted to subsets of L intersection the sigma algebra L intersection E.

A of particular interest, for the time being is going to be, the set when E is a closed bounded interval a, b. So, we will start looking at the space L 1 of a, b that is the space of all Lebesgue integrable functions defined on the interval close bonded interval a, b and we also have the space r a, b namely the space of all Riemann integrable functions on a b. So, we want to compare these two spaces on one hand we have got the space of Lebesgue integrable functions on a, b. On the other hand, we have got the space of a

Riemann integrable functions on a, b, and we want to see the relation or establish a relationship between the two and that was one of the starting points for our discussion of this subject namely; the space of Riemann integrable functions had some difficulties, some problems, some drawbacks, and for which we wanted to extend the notion to a larger class and this is the larger class L 1 of a, b.

So, what we are going to show is R a, b the space of all Riemann integrable functions is a subset of L 1 of a, b and the notion of Riemann integral is same as the notion of Lebesgue integral for Riemann integrable functions. So, that is called the relation between the Riemann integral and the Lebesgue integral.

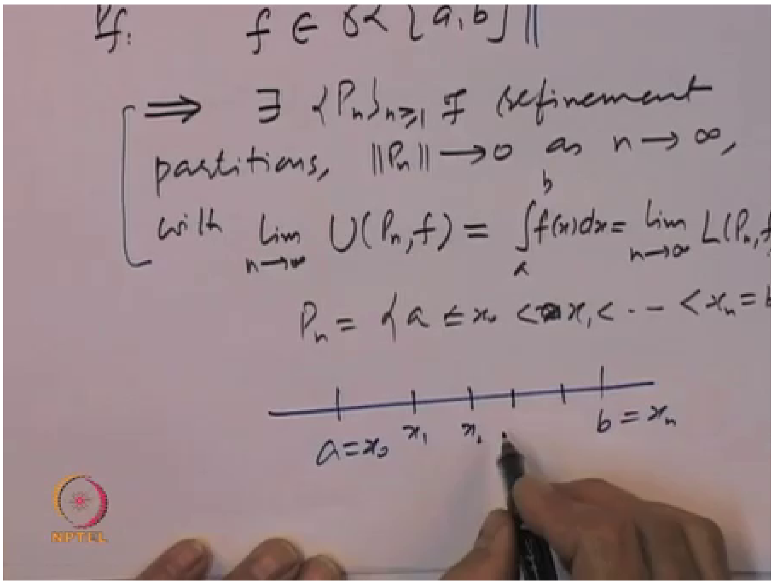
(Refer Slide Time: 17:52)



So, to be more specific we want to prove the following theorem, namely; if f is defined on a close bounded interval a, b is Riemann integrable function, then f is also Lebesgue integrable and the Lebesgue integrable is same as the Riemann integral of the function f. So, this is what we wanted want to prove.

So, let us start looking at, how do we prove this? So, the proof of the theorem

(Refer Slide Time: 18:23)

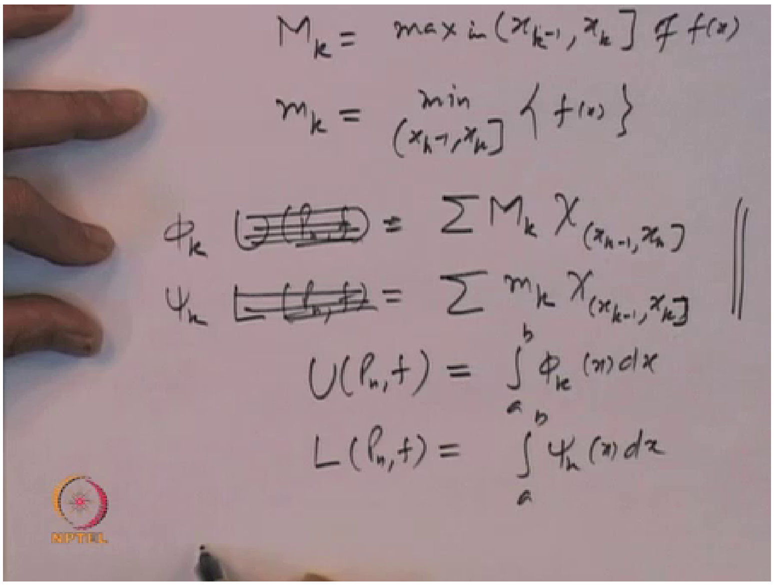


So, we are given that the function f belongs to R a, b, it is a Riemann integrable function; that means, so, let us recall how is the Riemann integral of a function define. It is defined via; limits of upper sums and lower sums of partitions. So, implies there exist a sequence P n of refinement partitions with norm of P n going to zero as n goes to infinity, partitions n going to infinity with the upper sums of P ns with respect to f, limit of that is same as integral the Riemann integral of f is same as the limit of the lower sums L P n of f. So, that is the meaning of saying that a function f is Riemann integrable.

So, we can find, Riemann integrable implies there exists a sequence of partitions P n, which are refinement partitions. Refinement means P n plus 1 is obtained from P n by adding one more point. And norm of these partitions the maximum length of the sub intervals goes to 0. And integrability means that the upper sums and the lower sums, both converge to the same value and that is the Riemann integral of the function f. So, this is this is the property of saying that f is Riemann integrable.

So, now from here, let us look at what is U P n, f upper sum. So, let us write down the partition P n as something. So, let us say P n looks like a. So, interval is a to b, so a the point x naught less than x 1 less than x n which is equal to b. So, let us say that is the partition P n. So, in the picture it will look like, here is a, here is b. So, this is the x 0, this is x n, and here is x 1, x 2 and so on. So, to construct the upper sums, what one does? To construct the upper sums, one looks at the maximum value of the function in this interval and the minimum values in this interval.

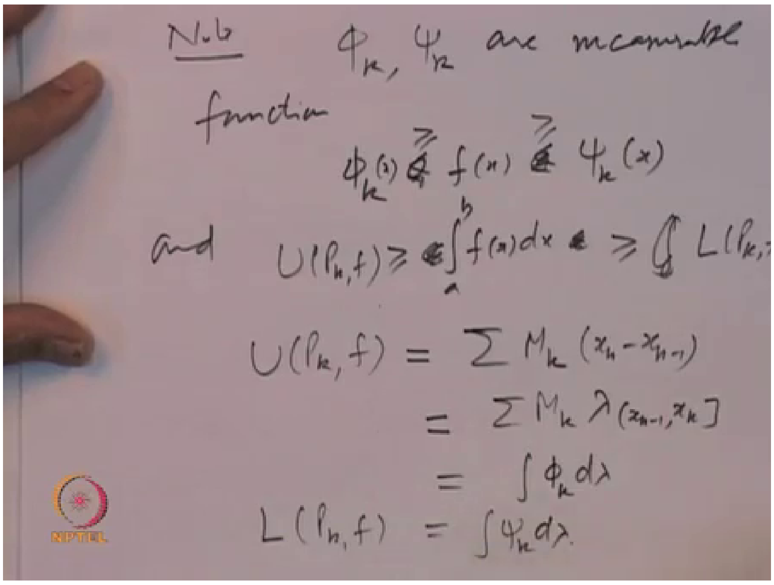
(Refer Slide Time: 21:30)



So, let us write let us let us write M k to be the maximum value of the function, in the interval x say x i minus x k. So, let us x k minus 1 to x k. I am just trying to be trying to make the intervals disjoint. Maximum in this interval of maximum in this interval maximum of maximum of f of x maximum in of f of x. And similarly M k, let us write it is a minimum in the interval x k minus 1 to x k of f of x. Only at the end points you have to make it close, but that is not going to matter much. So, that then we define, what is U P n, f? that is, essentially looks like summation of the maximum value into the indicator function of that sub interval. And the lower sum with respect to P n, f looks like summation small m k, the minimum value of the function in that sub interval x k minus 1 and x k.

So, let us let us do one thing, sorry this is upper sums they are not these are not. So, let us let me I am sorry this is not the upper sum. Let us call this when in the interval x k minus 1 to x k the value is capital M k. Let us call that as, the function phi k and when you are taking the minimum value in that interval and summing up, let us call that as psi k. So, these are functions, because they are linear combinations of indicator functions. And the upper sums and lower sums are nothing, but the upper sum P n, f is nothing, but Riemann integral a to b of phi k x d x and the lower some P n, f is equal to the integral of Riemann integral of this function psi k of x d x. That this functions phi k and psi k, which are linear combinations of indicator functions are in fact, non negative measurable functions. On the measure space a, b the interval a, b. So, note: so that is the observation that we should note and then so let us note down.

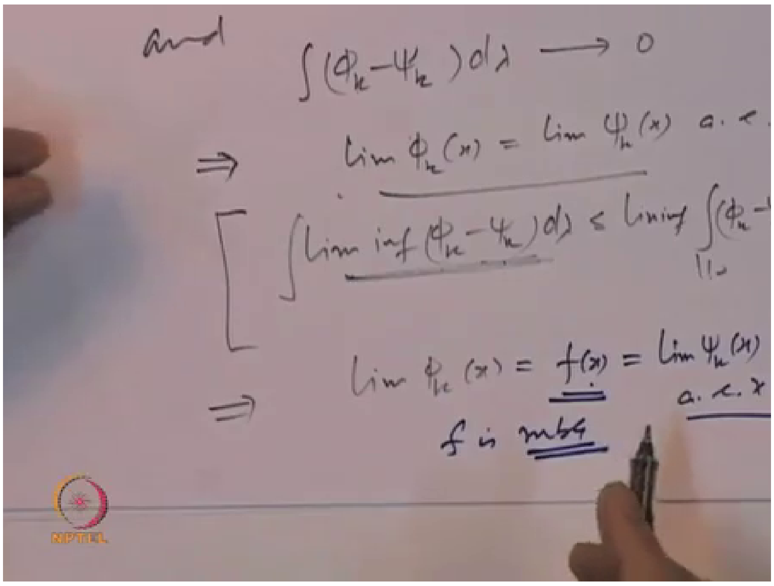
(Refer Slide Time: 24:35)



Note: phi k and psi k are measurable functions are measurable nonnegative sorry are simple measurable functions. So that phi k is less than or equal to at every point x is less than or equal to f of x is less than or equal to psi k of x. And as far as the integral is concerned, the integral a to b of f x d x is between the upper sum and the lower sum, so that is the phi k was a maximum sorry this. So, this should be bigger than or equal to like this, because phi k is taken as the supremum, so this is. So, this is upper sum P k of f and that is bigger than or equal to the upper sum sorry; bigger than or equal to the lower sum with respect to P k of f. And in the limit both of them are converging. So, here is the second observation is that the upper sum with respect to the partition of f is same as, so what was it. So, that was equal to sigma M k into the length of the interval x k minus x k minus 1. So, that is the upper sum that is also a Riemann integral.

In fact, this is also equal to so, the length. So, this is the length of so you can write this is the length. So, M k times length of x k minus 1 and x k, which is same as the Lebesgue integral of the function phi k d lambda. So, this is, this is the important observation that we should keep in mind. That the building blocks for Riemann integral which are these step functions are also Lebesgue integrable and the Riemann integral of the step functions phi k and psi k are same as the Lebesgue integrals of phi k and psi k. So, similarly the lower sum P k, f is equal to integral of psi k d lambda. And now essentially the idea is to put them together. So, because phi k and psi k they are between these two. So, let us look at integral of. So, look at the sequence.

(Refer Slide Time: 27:25)



So, consider the sequence psi k minus phi k minus psi k. Recall, phi k is bigger than f x is less than psi k. So, phi k minus psi k is nonnegative for every k and saying that the upper sums and lower sums converge to the same value is saying that the integral of phi k minus psi k phi k minus psi k d lambda that goes to 0. So, that goes to 0. So, because the phi k d lambda is the upper sum, this is the lower sum and that goes to 0.

So, that implies, so that implies that limit so; that means, this implies that the limiting function f is trapped in between so that means, limit phi k x is equal to limit psi k x almost everywhere, Why is that? So, that we can reduce from the fact that, applying fatous lemma, so to reduce this look at the limit inferior of phi k minus psi k, integral d lambda be less than or equal to limit inferior of integral phi k minus psi k and that is 0. So, this is 0 so; that means, and this is and so, this says that integral of a nonnegative function is 0. So, the function must be 0, almost everywhere and that is same as saying this must be 0, almost everywhere and f is trapped in between. So, that implies that limit phi k x limit of phi k x is equal to f of x is equal to limit psi k x for almost everywhere x. So that proves is equal to.

So, we are falling short of time so; that means that the function f is measurable. So, we will continue the proof of this tomorrow in the next lecture. So, our aim is to prove that the space of Riemann integrable functions is inside the space of Lebesgue integrable functions and the Riemann integral is same as the Lebesgue integral. So, we will continue the proof in the next lecture.

Thank you.