Measure & Integration
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Lecture – 21A
Dominated Convergence Theorem and Applications

Welcome to lecture 21; on Measure and Integration. In the previous lecture, we had started looking at the properties of sequences of integrable functions and we started proving an important theorem called Lebesgue dominated convergence theorem. Let us continue looking at that and after that we will start looking at the special case of integration on the real line and that will give us the notion of Legbsgue integral.

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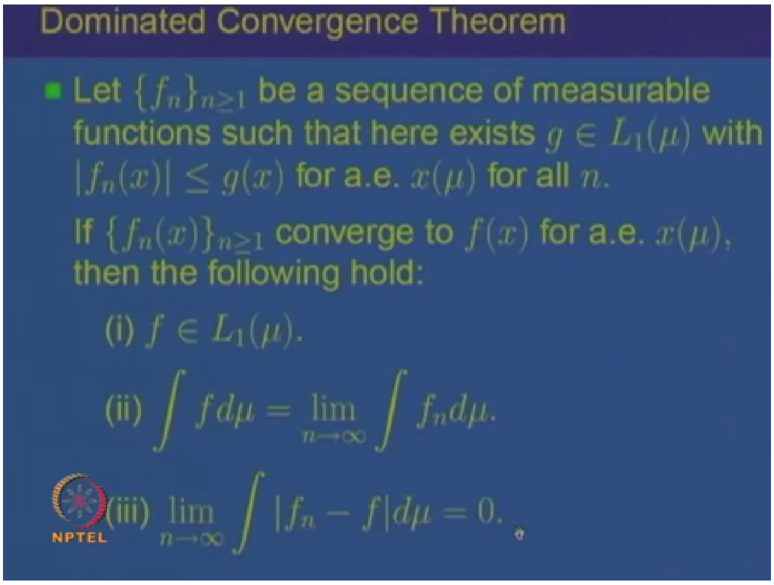

So, let us recall what we had started proving namely dominated converges theorem which says that if f n is a sequence of measurable functions; such that there exists a function g belonging to l 1; such that all the f n s are dominated by this integrable function g; almost everywhere x for all n. And if f n converges to f; then the limit function is integrable and integral of f is nothing, but the limit of integrals of f n's.

So, the theorem basically says that if f n is a sequence of measurable functions; all of them dominated by a single integrable function g, then all the f n's become integrable of course, and if f n's converges to f; then f is integrable and integral of f is nothing, but the limit of integrals of f n's. Now, we had proved this theorem in this particular case, when instead of this almost everywhere that mod f n's are dominated by g everywhere and f n's converge to f everywhere.

So, to extend this case to almost everywhere; we have to do on minor modifications.

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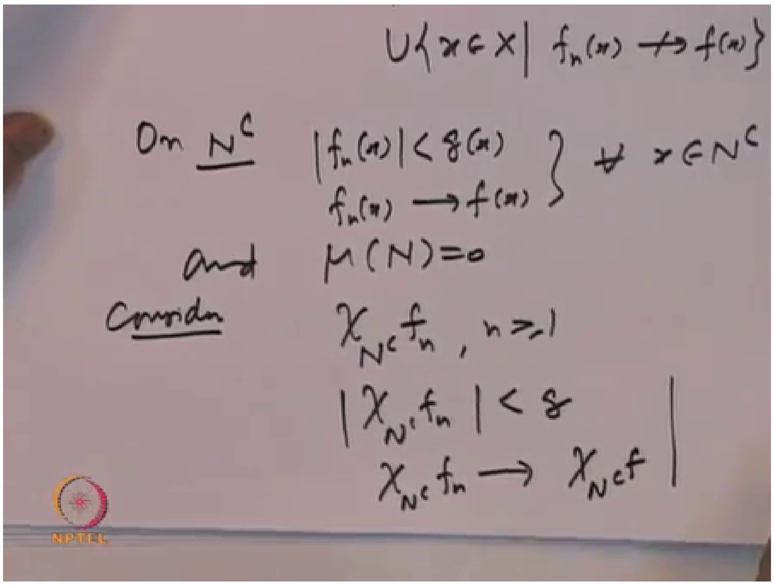

So, let us look at. let us define the set N to be the set of all x belonging to X; where mod f n x is not dominated by g. So, g of x or union the set of all those points x belonging to X; such that f n x, does not converge to the function f of x; so, on N complement; we have f n x is less than g x and f n x converges to f of x. So, for every x belonging to N complement and mu of the set n is equal to 0. Because we are saying that this mod f n x is less than g x almost everywhere. And f n x converges to f of x almost everywhere; so, the set where it does not hold; there is a set n and that set has n has got measure 0.

So, now let us consider the sequence; indicator function of N complement times; f n. So, this is a sequence of functions which are dominated by; so, this is a sequence of functions for n bigger than or equal to 1 and they satisfy the property. So, namely this the indicator function of N compliment f n mod of that is less than g; for all x everywhere. And the functions converged to the indicator function of N compliment f.

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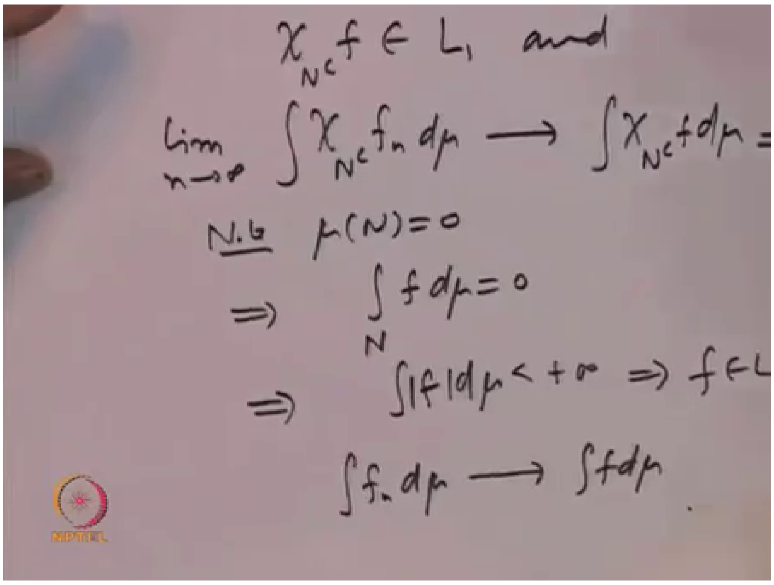

So, by our earlier case; what we get is the following namely that indicator function of N complement times f is L 1 is integrable and integral of f n limit n going to infinity indicator function of N compliment times f n; the integral of that converges to the integral of indicator function of N compliment times f. So, that is by the earlier case when everything is true for all points. So that means, so this is same; but note that mu of N is equal to 0.

So, that implies that is the integral of f over N d mu is equal to 0 and we already know that on N complement; f is integrable. So, that together with this fact implies integral of mod integral of mod f d mu is finite; implying that f belongs to l 1. And this equation which said that integral of a f n over N complement converges to integral of f over N complement and mu of N being 0; together gives us the condition that integral of f n d mu converges to integral f d mu.

Because integral of f n d mu is same as integral of f n over n plus integral over N complement and integral over N compliment converges to integral over N complement of f and on n both are 0, so this gives us the required result. So, that is how from almost everywhere conditions are deduced from the fact that something holds everywhere.

So, this dominated convergence theorem holds for whenever the sequence f n is dominated by g and f n converges to f almost everywhere. Then f limit function is integrable and integral f converges to integral of f n. So, as I said this is one of the important theorems which helps us to interchange; the limit and the integral sign.

Let us look at some more minor modifications of this theorem; one thing more we can even reduce that integral of mod f n minus f d mu also converges to 0.

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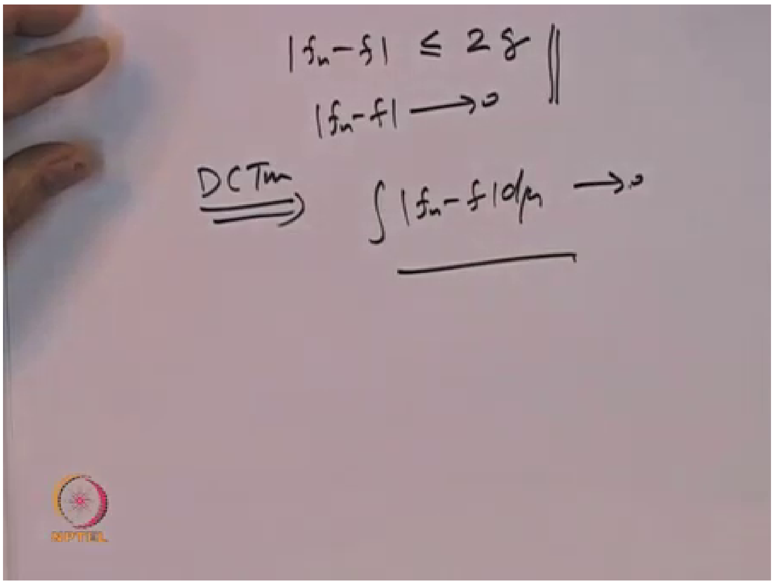

So, to deduce that part; we just have to observe that mod f n minus f is less than or equal to twice of g and mod f n minus f goes to 0. So, again an application of dominated convergence theorem; dominated convergence theorem which we put just now, implies that integral of mod f n minus f d mu goes to 0. So, that is a another modification, another consequence of the dominated convergence theorem.

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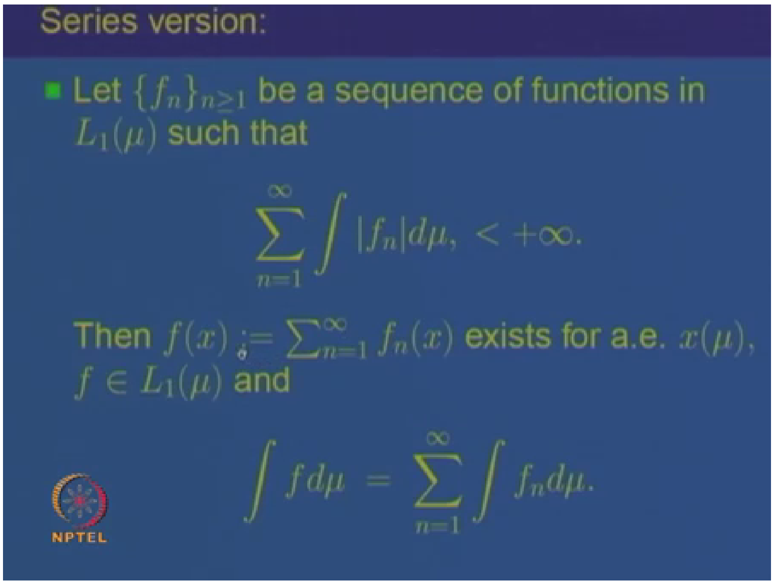

Let us prove what I call as the series version of this theorem; namely that if f n is a sequence of functions which are integrable and integrals of f n summation 1 to infinity. So, sum of all the integrals f n mod f n’s are finite; then the conclusion is that the series f n x converges almost everywhere. And if you denote the limit; the sum being f of x, then that function is integrable and integral f is equal to summation of integral f n’s.

So, essentially this theorem says that if the summations of mod f n’s are finite; then this the series f n x is; I guess is convergent almost everywhere and integral of f is equal to integral of summation of integral f n’s. So, that is again interchange limit essentially. So, let us see how from dominated convergence theorem; we can get this.

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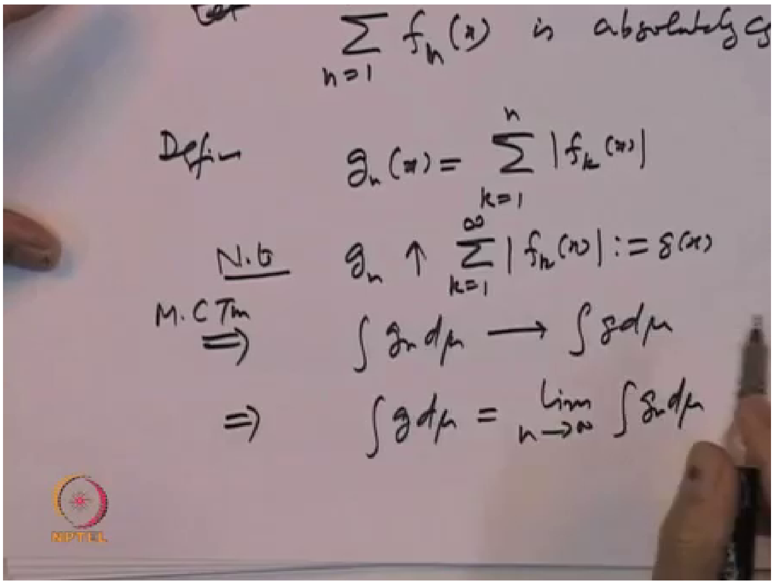

So, let us define; to show that this series is convergent almost everywhere; we will actually show that, so we show is absolutely convergent. So, for that let us define say g n of x to be equal to summation mod f k of x; k going from 1 to n, the partial sums of the absolute values 1 to n.

So, let us observe that this sequence g n. So, note g n is a sequence of nonnegative measurable functions and g n's are increasing to some function so, that is there going to increase to k equal to 1 to infinity; mod of f k of x; let us call that as g x; they are increasing to the function g of x. So, that implies by monotone convergence theorem; we have integral of g n; d mu must converge to integral of g d mu.

But integral of g n d mu; so, that is same as same integral g d mu is equal to limit n going to infinity of integral g and d mu, but what is integral of g n; g n is sum of absolute values of f k 1 to n. So, by linearity property this is nothing, but limit of n going to infinity of summation 1 to n; k equal to 1 to n of integral mod f k; d mu and this limit is nothing, but 1 to infinity and that is given to be finite. So, let us write that.

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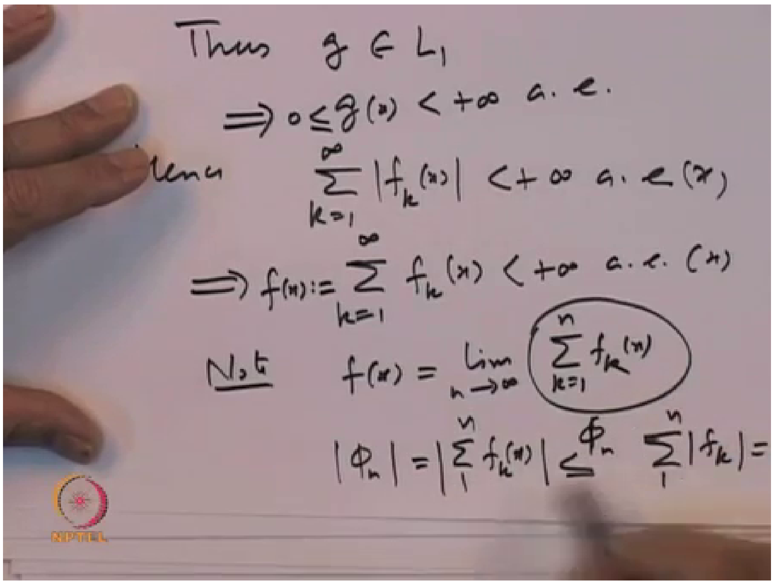

So, this is equal to summation k equal to 1 to infinity of integral mod f k d mu; which is given to be finite. So, hence what do we get is thus g is integrable; g belongs to g is a integrable function. So, saying that g is integrable implies, we call that if a function is integrable and g is a nonnegative function. So, g of x is finite almost everywhere; that is nonnegative function which is a integrable function must be finite almost everywhere.

So, we get that g is finite almost everywhere and what is the function g; g is nothing but the limit of the absolute values of f k x. So; that means, that prove that the series; so, hence sigma k equal to 1 to infinity mod f k of x is finite almost everywhere x. And once a series is absolutely convergent; it is also convergent so, that implies that sigma k equal to 1 to infinity f k x is finite for almost everywhere x.

So, let us denote this limit by f of x; so, this is f of x and so, as observed earlier. So, note; so f of x, we can write also f of x as the limit n going to infinity of summation k equal to 1 to n; f k x. And if this functions are called as something say phi n; then note that mod phi n, if this is called as phi n; then what is mod phi n? mod phi n is absolute value of 1 to n f k x; absolute value of that and that is less than or equal to summation 1 to n mode of f k 1 to n; and that is nothing, but our g n which is less than or equal to g.

So, these partial sums which we called as phi n's are all dominated by g and phi n's converge to f; so by dominated convergence theorem. So, what we have got is; so, we have got.

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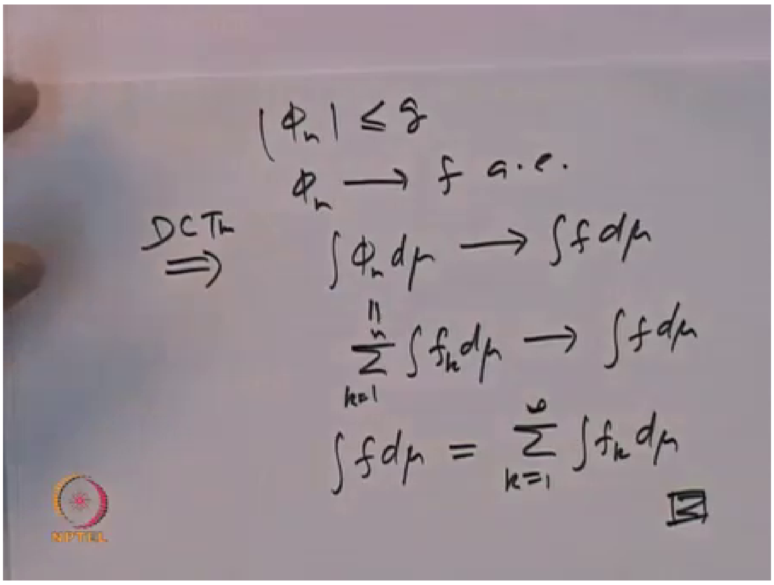

So, all the phi n’s are less than or equal to g and phi n’s converge to f almost everywhere. So, that implies by dominated convergence theorem; that integral of phi n’s d mu must converge to integral of f d mu. But this is nothing, but this phi n’ this is what we called as phi n that is summation of 1 to n. So, this is nothing, but summation of 1 to n of integral k equal to 1 to n of integral f k; d mu must converge to integral f d mu.

And that is same as saying that integral f d mu is equal to summation k equal to 1 to infinity; integral f k d mu. So, that proves the theorem namely if f k is a sequence of functions, which are integrable and the sum of the integrals is finite; then the series f n x; n 1 to infinity itself is convergent almost everywhere and the limit function is integrable and integral of the limit function is equal to summation of integrals of f n’s. So, this will referred to as the series version of dominated convergence theorem.

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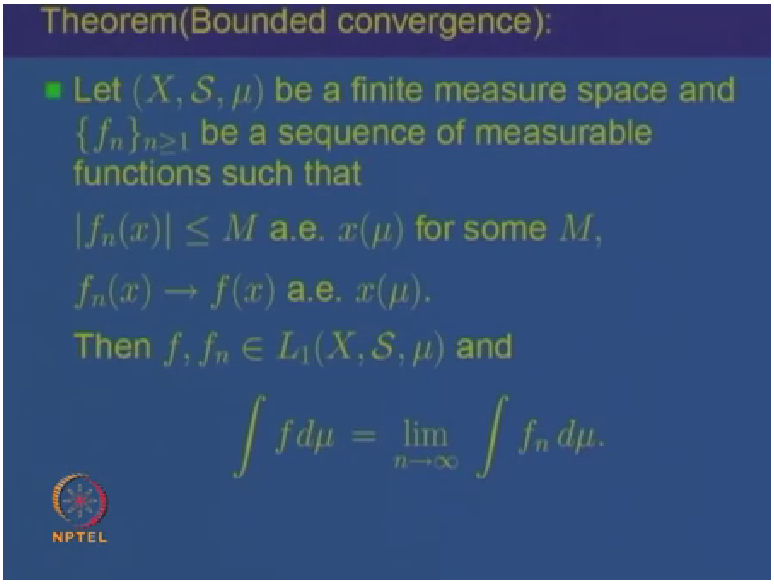

There is another interpretation of the dominated convergence theorem; when the underlying measure space is a finite measure space; then one has that if X S mu is a finite measure space and f n is a sequence of measurable functions; such that all of them are dominated by a single constant M; almost everywhere and f n x converges to f of x; then integral f n s converges to integral f.

So, this is a particular case of dominated convergence theorem when the underlying measure space is the finite measure space. And the only thing to observe here is that because let us see how does this follow from; is a dominated convergence theorem.

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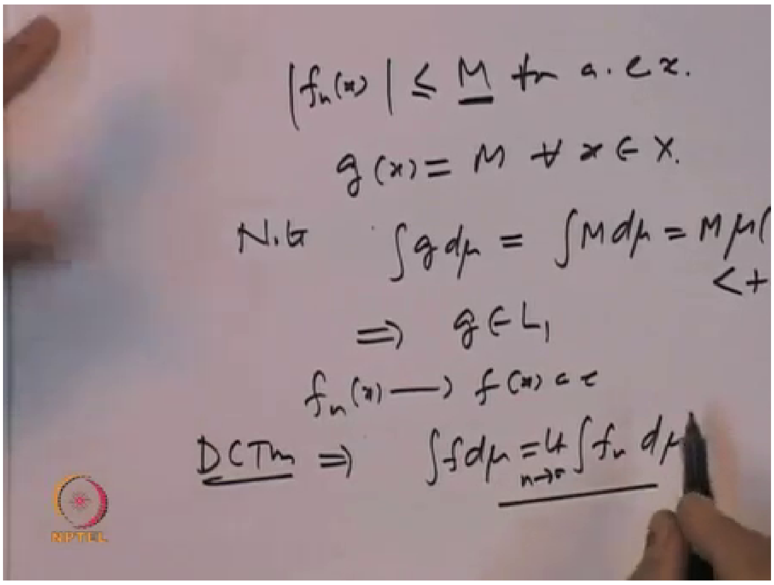

So, we are given that mod f n x is less than or equal to M; for almost everywhere x. So, now, look at this; so, look at the constant function M. So, look at the function g of X which is equal to M for every x belonging to X. So, the constant function is measurable; so, note that g is a nonnegative measurable function because it is constant function note that. So, its integral g d mu is equal to is equal to integral the constant function M d mu and that is equal to m times the measure of the whole space x; which is finite.

So, what we are saying is on finite measure spaces a constant function is always integrable. So, implies g is L 1; so, f n x bounded by M. So, that is constant function and that is a integrable function; once we have that and f n x converges to f of x almost everywhere. So, now, by dominated convergence theorem is applicable and that implies integral fd mu is equal to integral f n d mu limit n going to infinity.

So, the main thing is on finite measure spaces a constant function becomes integrable because of this reason. So, this is what is called bounded convergence theorem and it is quite useful; when underlying measure space is a finite measure space. So, let us look at what we have proved till now; we have looked at the space of integrable functions and proved linearity property and an important theorem called dominated convergence theorem.

So, if you recall for nonnegative measurable functions; we had two theorems one was monotone convergence theorem namely that was theorem; when f n is a sequence of nonnegative measurable functions increasing to a function f, then integral of f is equal to limit of integrals. So; that means, interchange of limit and integration is possible by monotone convergence theorem whenever the sequence f n is monotonically increasing and sequence of nonnegative measurable functions.

The second theorem which involved sequences of measurable functions was again for nonnegative measurable functions and that was called fatous lemma. So, there we do not emphasize; we do not require the sequence f n be nonnegative and measurable; we only want the sequence f n to be a sequence of nonnegative measurable functions; they need not be increasing. So, for such a sequence we had that integral of the limit inferior of the sequence f n is less than or equal to limit inferior of the integrals of f n; so, that was Fatou's lemma.

And now we have the third theorem; dominated convergence theorem which again helps you to interchange the notion of integral and the limiting operation under the conditions that all the f n's dominated by a single integrable function.

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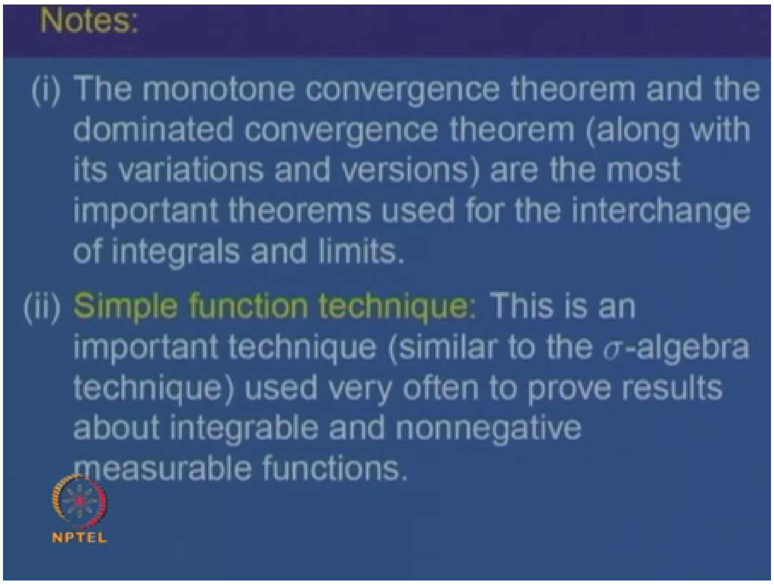

So, these are the three important theorem which help us to interchange limit and the integral signs. Let us at this stage emphasize one more point about this technique of integration and so, basically for integral; we started with simple functions and then we go to a nonnegative functions and then we defined it for integrable functions.

So, this process of step by step defining the integral can be is useful in proving many results and I call it as the simple function technique. So, this is a technique which is used very often to prove some results about integrable functions and non negative measurable functions. So, what is the technique; let me outline that and then I will give a illustration of this.

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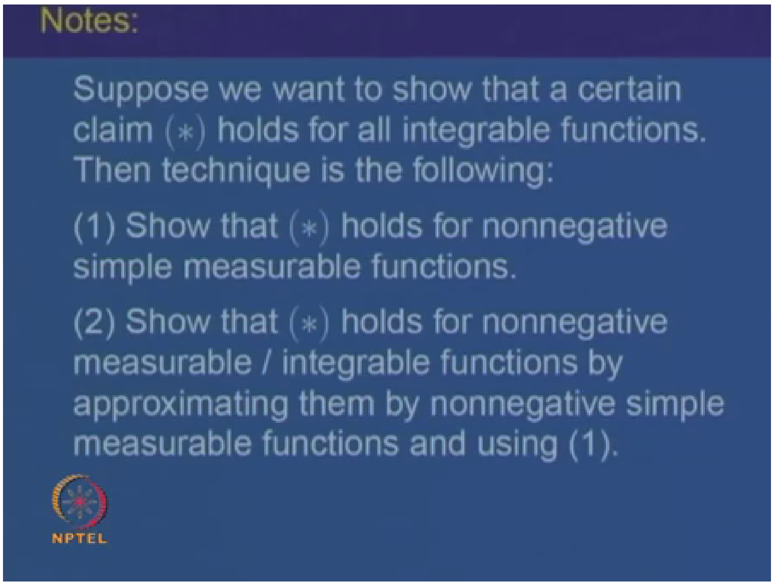

Suppose, you want to show that a certain property, say let us all that property as star holds for all integrable functions. So, to prove that the property holds for all integrable function; the technique is as follows basically that show that this property star holds for all non negative simple measurable functions.

So, if you want to show a property holds for all integrable functions; first show that it holds for the class of nonnegative simple measurable functions. And next show that star holds for non negative measurable; integrable function by using the fact that non negative measurable functions are limits of increasing the limit of simple measurable functions. So, and use their one use is normally the monotone convergence theorem. So, using monotone convergence theorem; one extends the property star from simple measurable functions to nonnegative measurable functions or nonnegative integrable functions.

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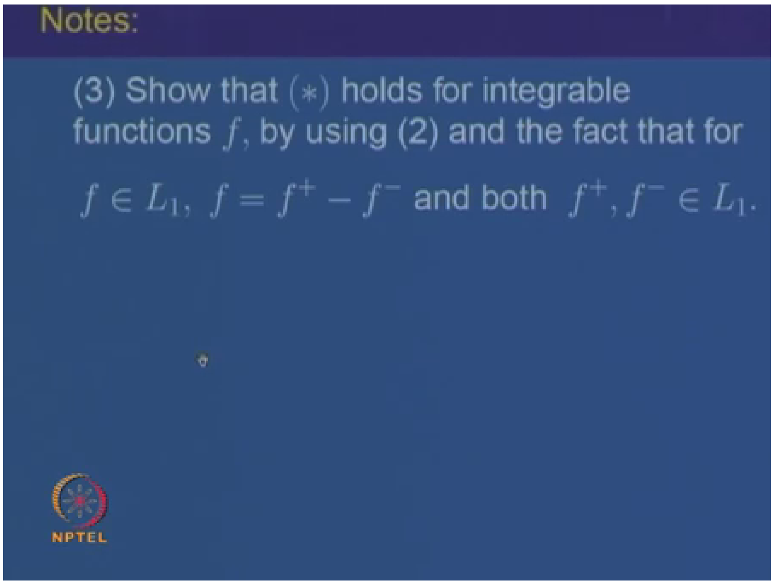

And then keeping in mind that for a function f it can be split into positive part and negative part. So, f can be written as f plus minus f minus and if a property holds for nonnegative functions. So, about integral; so, f for plus that will hold for f minus that will hold and then conclude from there that it holds for f also. So, this is what I call as the simple function technique to prove results about integrable functions.

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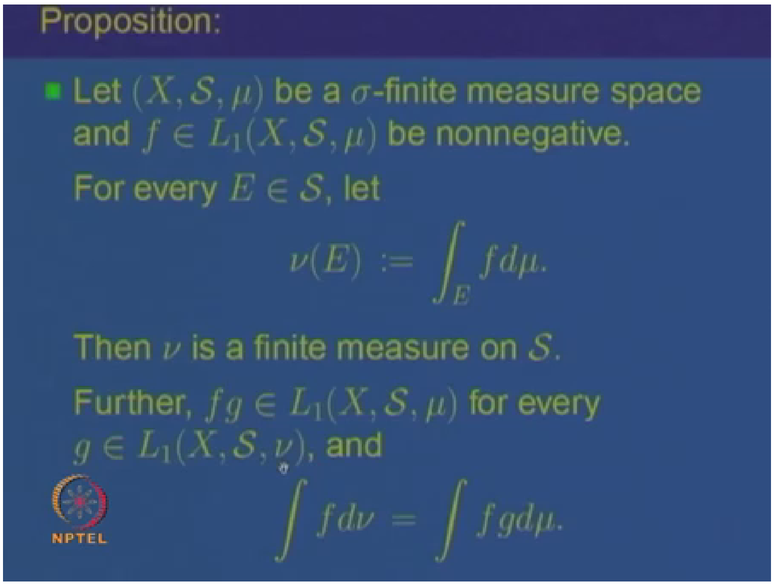

To give a illustration of this; let us look at the following result. Let us take a measurable space X S mu; which is sigma finite measure space and let us look at a function f which is integrable on this measure space and is nonnegative. So, we have got a sigma finite measure space and f is a nonnegative integrable function on this measure space. Let us define nu of E, for every set in the sigma algebra; let us define nu of e to be integral of fd mu over the set E integral of f over the set E is denoted by nu of E; for every set e in the sigma algebra S. Then we had already shown that this nu, the set function nu is in fact, a finite measure on S. So, this we have already proved.

But what we want to prove now is that further if g is any integrable function within on the measure space X S nu; this nu is the new measure. So, if g is integrable on X with respect to nu, then the product function f into g is integrable with respect to mu and this relation holds integral fd mu. So, integral of f with respect to nu is equal to integral of f into g with respect to mu.

So, what we have done is by fixing a function f; which is nonnegative, we have defined a new measure on the measurable space by nu of e to be equal to integral of efd mu. And we are saying; if you want to integrate a function with respect to a function g, with respect to this new measure; then it is same as integrating the function; the product function f g with respect to the old measure mu.