Measure & Integration
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Lecture – 20 B
Properties of Integrable Functions & Dominated Convergence Theorem

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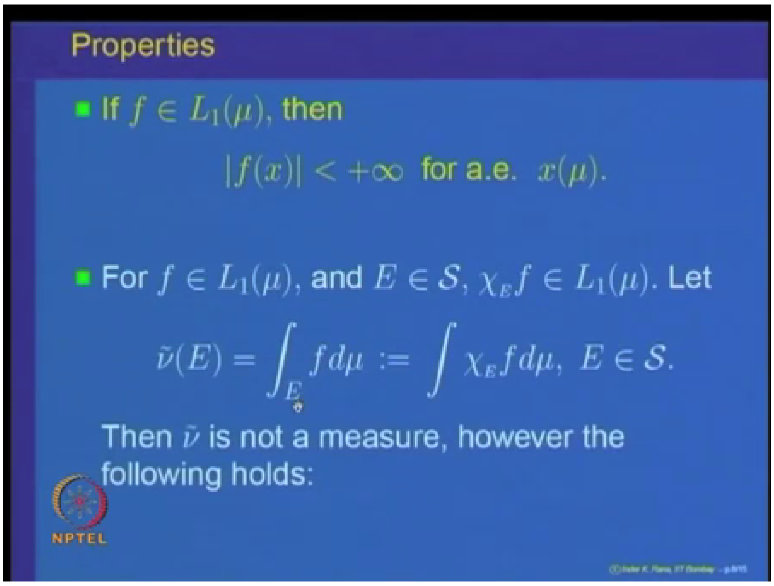

Now, let us look at the additive property of the integral over the sets. So, this is just now we have proved that f. So, another property that if f is integrable, then f must be a finite number for almost all X; so, a similar argument as before. So, let us prove that property also; it says if f is a integrable.

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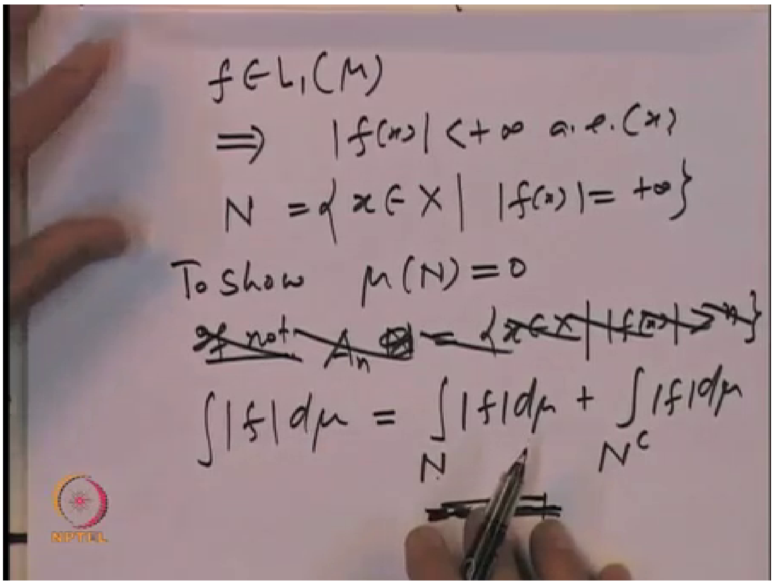

Then this implies that mod f of x is finite almost everywhere x. So, let us write once again the idea is; let us write the set x n to be the set where mod f of x is equal to plus infinity either. So, f of x is equal to infinity or equal to minus infinity. So, together we have put them together in the set n. So, to show that the set mu of N is equal to 0; so, if not so; once again, if not let us write N as x belonging to x such that such that mod f of x is bigger than say some quantity is not equal to 0. So, if this is not equal to 0 and f of x n is the set where f of x is equal to plus infinity. So, let us write f of x bigger than N; oh, let us write, sorry, let us write A n to be the set where f of x is bigger than n, then each set a n is in the sigma algebra, right. So, we want to; sorry; this is not required. So, let us; this is not required because we want to just want to show that f is finite.

So, now observe that integral of mod f d mu, I can write it as integral over N mod f d mu plus integral over N complement mod f d mu, right because N and N compliment together make up the whole space and just now, we observed that integral of mod f over a set E is a measure. So, integral of mod f over the whole space can be written as integral over the set of mod f over N plus integral of mod f over N complement and on N, the set is on N mu of N is equal to 0. So, the first integral is 0. So, it is equal to 0 plus and sorry yes no let us observe this is.

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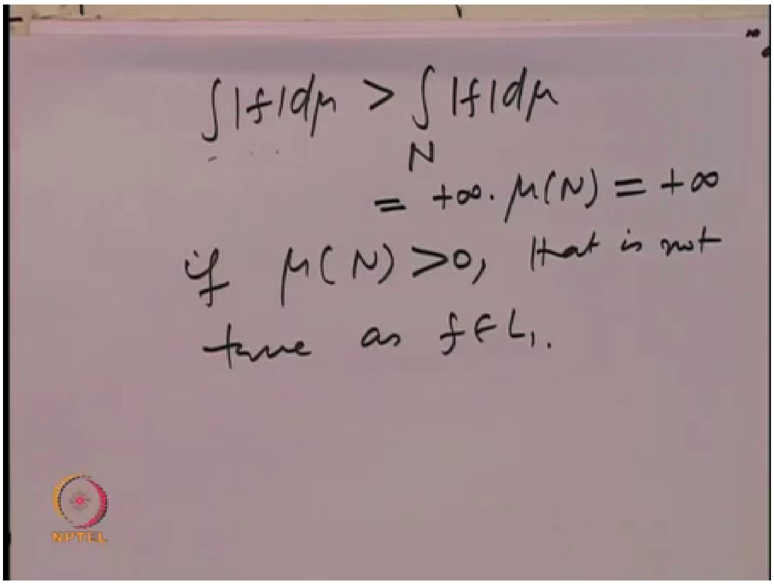

So, if mu of N is bigger than 0, then what will happen. So, let us assume mu of N is bigger than 0, then this integral is equal to this integral plus integral over N plus integral over N compliment so; that means, integral of mod f d mu is always bigger than integral over integral over N mod f d mu right because I am just; so, what we are doing. So, integral of mod f is integral over N plus integral over N complement, let us just drop the second term. So, integral of mod f over the whole space is going to be bigger than integral over N f d mu.

So, once that is true and on N; the function takes the value plus infinity. So, this is going to be equal to plus infinity multiplied with mu of N. So, if mu of N is bigger than 0, then this will be equal to plus infinity if mu of N is bigger than 0 and that is a contradiction that is not possible not true as f belongs to L 1. So, this integral must be a finite quantity and here we are saying in that case it will be equal to infinity. So, that proves that if a function is integrable, then it must be finite almost everywhere. So, now, let us come back to the question if f is integrable and E belongs to the set S, then the indicator function of E times f is integrable that we have just now observed.

So, we let us write nu tilde of E as before the integral of f over E and we observed that this number may not be a nonnegative number; however, is still has the property something similar to that of countable additive property for measures namely.

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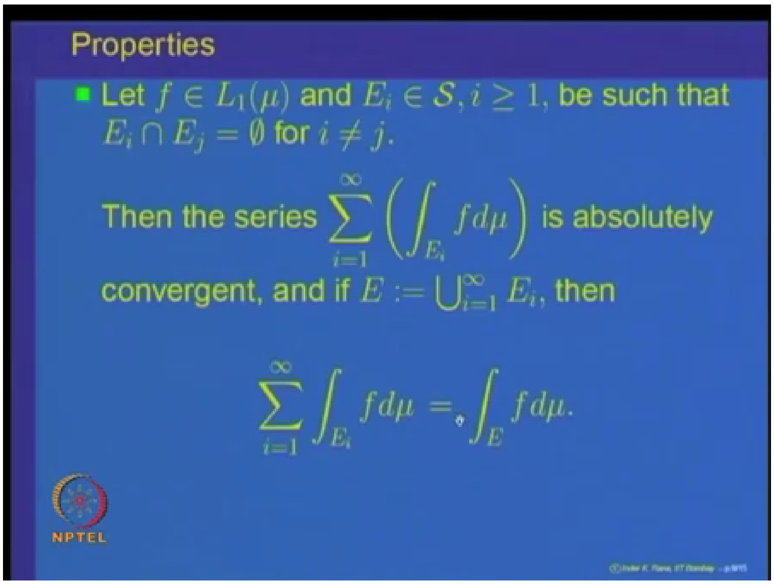

So, let us state that property that if you take sets E and S in the sigma algebra S which are pair wise disjoint. So, and E is the union of this sets, then the claim is that the series which is integral of E i S f d mu summation 1 to infinity, this series is absolutely convergent and if we write E as the union, then integral of f over E is equal to summation of integral of f over E i S. So, essentially we want to say that the integral of f of a integrable function over a set W can be written as in summation integral of E i S where E i S are pair wise disjoint and this is; we are saying its always possible for f to be integrable function and because why we are saying absolutely convergent.

So, here is one should note that when E is equal to union E i; it does not matter whether you write as E 1 union E 2 union E 3 and so on or any other order say E 2 union E 1. So, it does not the union does not depend on the order in which you write the sequence E i so; that means, in this series the summation should not depend upon the order of the terms and all are nonnegative; that means, we should prove that these are absolutely convergent and that is what we want to prove that if E i S are pair wise disjoint, then the series integral over E i of f d mu summation 1 to infinity is a absolutely convergent series and this integral of f over E is summation of integral over E i S.

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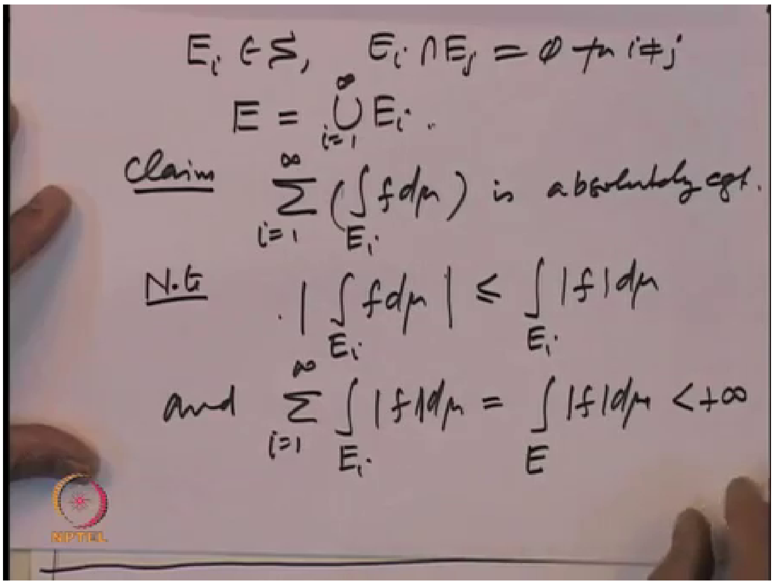

So, let us prove this property. So, let us we have got E i is a sequence of sets in the sigma algebra they are pair wise disjoint is equal to empty set for i not equal to j and E is equal to union of E i S 1 to infinity. So, first we want to show. So, first claim is that the series summation i equal to 1 to infinity of integral over E i of f d mu. This is absolutely convergent absolutely convergent. Let us observe; what is absolute convergent means; that means, absolute values of this terms that this is a series of nonnegative terms that must converge, but. So, for that let us note that absolute value of integral E over E i of f d mu is less than or equal to integral of mod f over E i d mu, right.

So, that is the property of integrable functions that absolute value of the integral is less than or equal to integral of the absolute value and for nonnegative function. So, mod f is a nonnegative function and E is a disjoint union of sets. So, that implies that integral over E i of mod f d mu if i sum it up i equal to 1 to infinity that is same as integral over E of mod f d mu and f being integrable that is a finite number. So, here we are used 2 things; one for non negative measurable functions integral over a set is a measure.

So, integral of mod f d mu over E i summation one to infinity is equal to integral of absolute value of f over E and f being integrable this is finite. So, that proves that the series integral f d mu over E i is absolutely convergent, right because this sum is less than or equal to sum. So, from these 2 it implies that this series is absolutely convergent.

So, once the series is absolutely convergent its sum is equal to sum of the partial sums. So, now, we can easily write. So, this implies that the claim holds.

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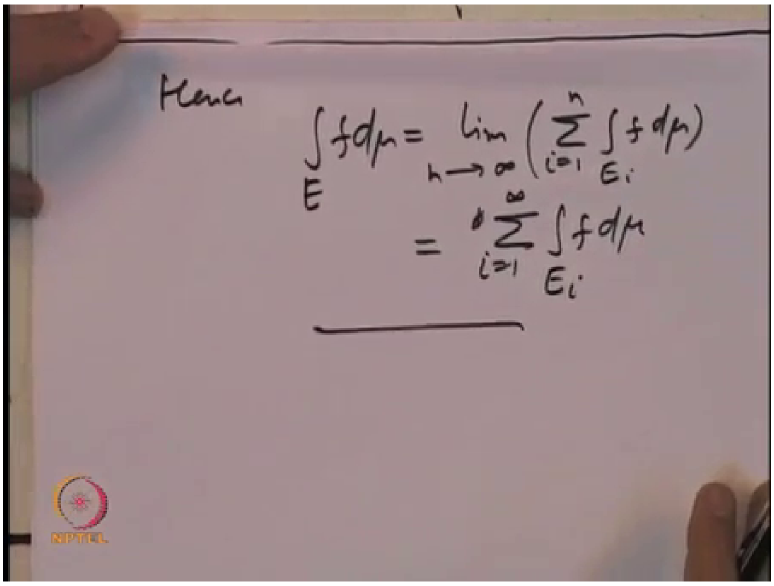
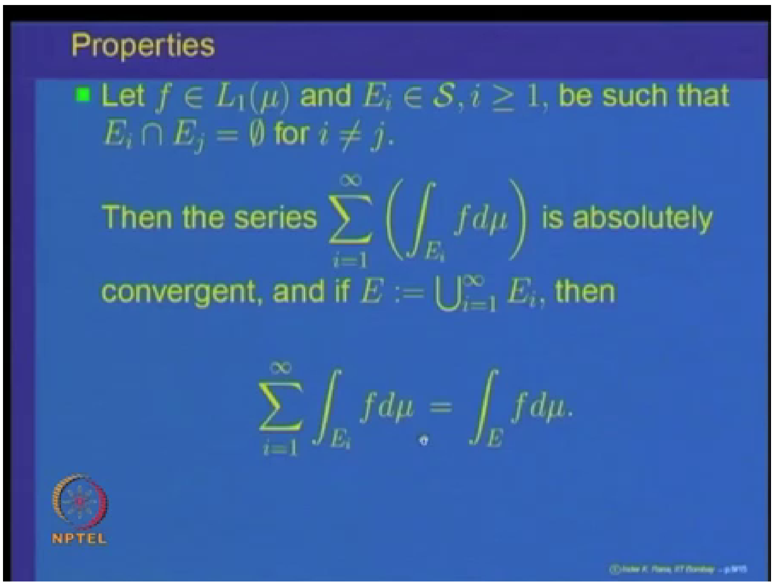


So, the series is absolutely convergent hence integral over E f d mu is equal to limit n going to infinity of the partial sums. So, i equal to 1 to n integral of f over E i d mu and that is nothing, but partial sums and that is same as right saying that sigma i equal to 1 to infinity integral over E i of f d mu. So, that proves that the integral of a integrable function over a set need not be a measure.



But it has we can say this is a countably additivity property of this integral that integral over E is equal to summation of integrals over E i S whenever E is a union of pair wise disjoint sets E i. So, this is the property that we are just now proved. So, these were some of the properties that we have proved about the integral of integrable functions and now we want to prove an important property, we want to analyze this sequences of integrable functions to we want to analyze if the property that if f n is a sequence of integrable functions and it converges to a function f can we say that f is integrable and can we say integral of f n s will converge to integral of f.

We have seen that this need not be true even for non negative functions, but they under some suitable condition we can say that integral of f n s will converge to integral of f and that is an important theorem called Lebesgue dominated convergence theorem. So, let us prove interchange of integral with the limits and look at the theorem called Lebesgue dominated convergence theorem.

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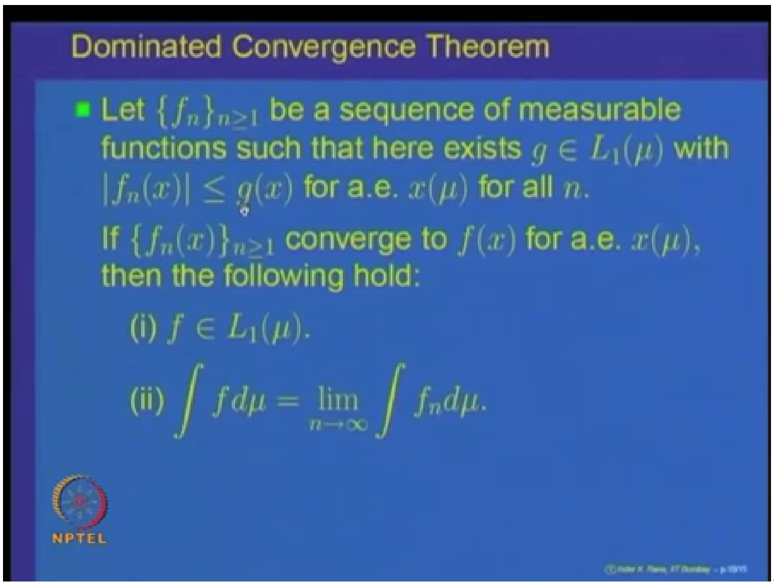

So, it says let f n be a sequence of measurable functions such that there exists a function g which is integrable with the property that integral of mod f n s are less than or equal to g for almost all x nu for all n, then the claim is if f n s converges to f almost everywhere then the limit function is integrable one and. Secondly, integral of the limit function is equal to limit of the integrable functions

So, let us observe once again what is given and what is true we are saying f n is a sequence of measurable functions and all the f n s are dominated by a single function g which is integrable and this dominance could be almost everywhere. So, all the f n s are dominated by a single function g and. So, the conclusion is if f n s converges to f then f is integrable and integral of f is equal to limit of integral of f n s. So, this is what is called dominated convergence theorem and it is an important theorem. So, let us prove this theorem.

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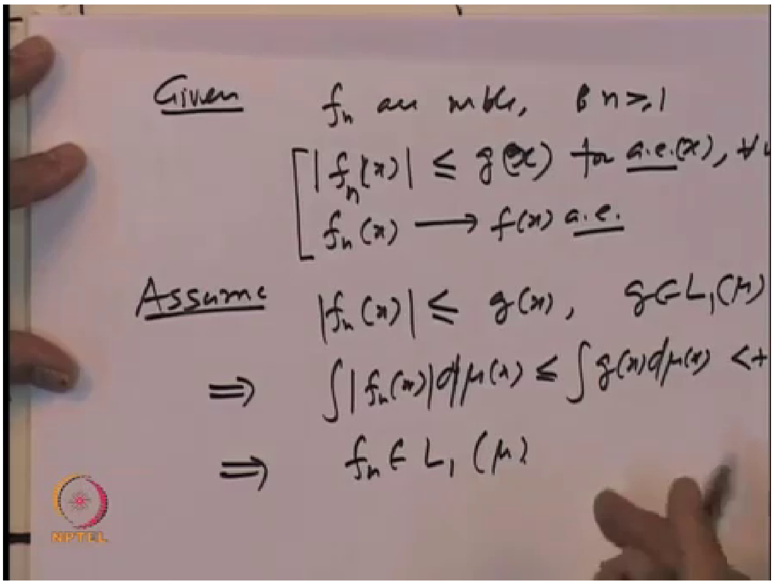

So, we are given that all the f n s are measurable for n bigger than or equal to one mod f n x is less than or equal to g of x for almost all x and for every n and we are given that f n x converges to f of x almost everywhere. So, to prove the required claim for the time being, let us assume that this almost everywhere is everywhere that is the proof is not going to change much. So, we will see that improver. So, let us assume for the time being f n x is less than mod f n x is less than or equal to g of x where g is L one is a integrable function. So, that implies that integral of mod f n x d mu x is less than or equal to g x d mu x and g is integrable. So, that is finite. So, that implies that each f n is a integrable function also.

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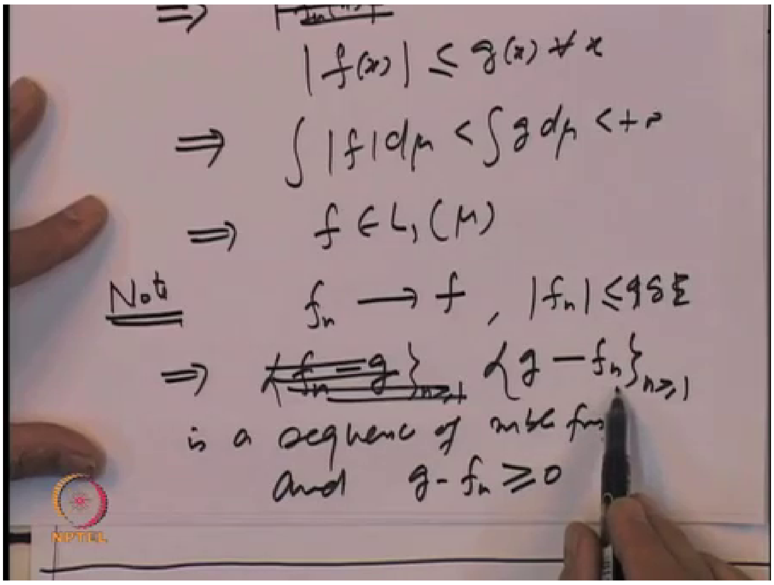

Mod f n converges to mod f right because f n converges to f and each mod f n is less than or equal to g. So, that implies. So, this implies this implies that mod f n x is mod f n x is less than or equal to g and that converges. So, that implies that mod f x is also less than or equal to g of x for every x. So, that once again implies integral mod f d mu is less that integral g d mu which is finite. So, this again implies that f is in L 1 of mu.

So, under the given conditions we are shown if f n s are dominated by a integrable function and f n s converge to f, then f is a integrable function. So, to look at the limits let us observe. So, note f n converges to f right and mod f n is less than equal to mod g. So, this implies that look at the sequence f n. So, look at the sequence f n minus g. So, look at the sequence f n minus g. So, look at the sequence this is a sequence of measurable functions of course, they are mod a f n is less than or equal to g. So, this will be negative.

So, let us look at we want nonnegative. So, let us look at g minus f n look at this sequence instant. So, this is a sequence of nonnegative measurable functions because mod g is bigger than mod f n is bigger than or equal to g that is given. So, g minus f n; so, is a sequence of measurable functions and let us observe g minus f n is bigger than or equal to 0 because g is bigger than or equal to f n. So, this is a sequence of nonnegative measurable functions; and so, let us write that g minus f n is a sequence of nonnegative measurable functions and it converges to g minus f because f n converges to f.

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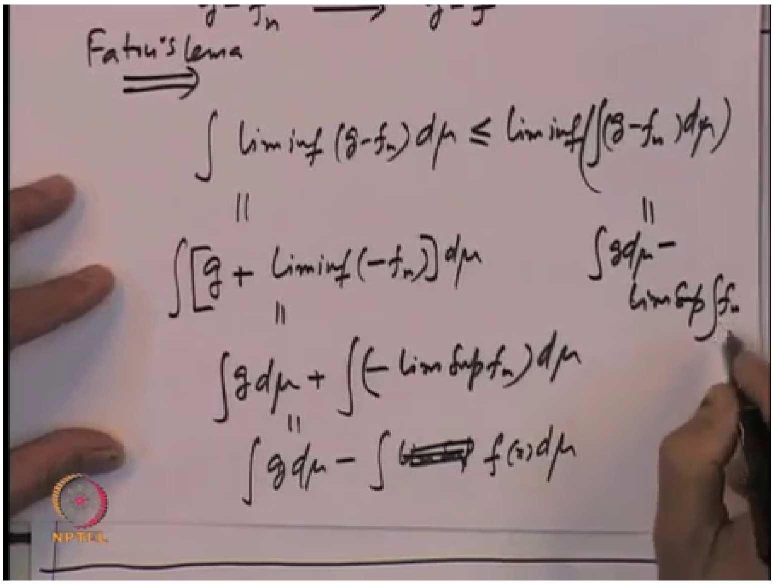

So, now we can apply Fatou's lemma. So, implies by Fatou's lemma that integral limit inferior of g minus f n d mu will be less than or equal to limit inferior of integral of g minus f n d mu. So, that is application of Fatou's lemma. So, recall we had Fatou's lemma which was applicable for functions which are not necessarily sequence of functions is not necessarily increasing. So, look at this now let us compute both sides. So, what is the left hand side? So, this is equal to integral limit infimum of g minus f n is g minus limit inferior plus limit inferior of minus f n right. So, that is the left hand side d mu and that is equal to integral of g.

So, look at that is equal to integral of g d mu and what can you say about this f n is a all are integrable functions. So, everything is finite. So, this limit inferior of minus f n is equal to minus limit superior of f n s d mu. So, this is a property of limit superior and limit inferior that limit inferior of minus f n is equal to minus of limit superior. So, this is equal to. So, this is equal to integral of g d mu minus integral limit superior of f n s. So, limit f n is convergent. So, limit superior is same as f of x f of x d mu d mu x. So, that is a left hand side and let us see; what is the right hand side once again. So, limit inferior of integral. So, this is equal to limit inferior of integral integral of g minus f n is integral g and that does not depend upon limit. So, is integral g d mu and then limit inferior of minus. So, that will be minus limit superior of integral f n d mu.

So, from these 2; so, this is less than or equal to this. So, what does that imply? So, that implies that integral g d mu Minus integral f d mu, right.

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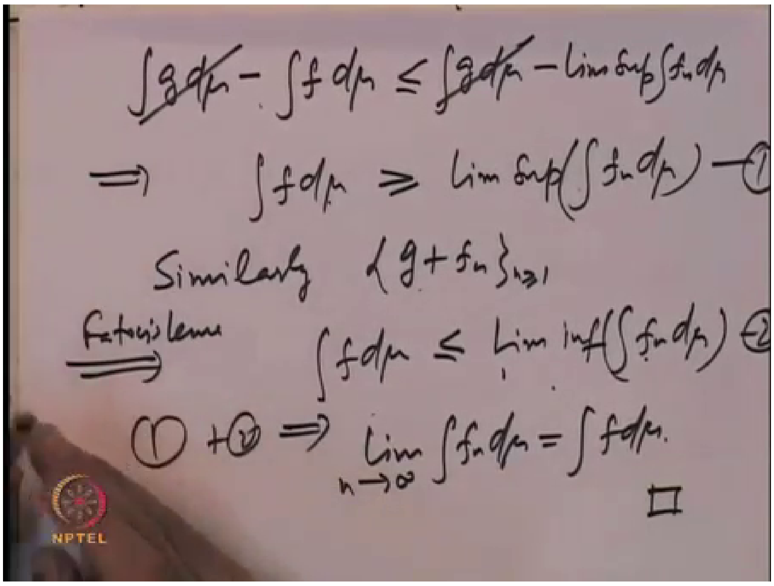

So, I am just writing integral; this is less than or equal to integral g d mu minus limit superior of integral f n d mu, right. So, everything is finite. So, I can cancel out this and negative sign gives you other way inequality. So, it implies integral f d mu is bigger than or equal to limit superior of integral f n d mu. So, looking at the sequence g minus f n we got this is that g minus f n is nonnegative converges to g minus f gives us this.

Similarly, if I look at the sequence g plus f n that is again a sequence of nonnegative measurable functions and application of Fatou's lemma. So, Fatou's lemma will give me that integral; integral of f d mu is less than or equal to limit inferior of integral f n d mu. So, a similar application of Fatou's lemma to this sequence will give me this. So, 1 and 2; so, 1 plus 2 together imply that integral f d mu is bigger than limit superior that is always bigger than limit inferior and that is bigger than integral f d mu implies that the sequence. So, limit integral f n d mu exists this limit exist and is equal to integral f d mu. So, that proves dominated convergence theorem.

So, the proof of dominated convergence theorem is essentially very simple it is just a straight forward application of Fatou's lemma because mod f n is less than or equal to g implies g minus f n and g plus f n both are sequences of nonnegative measurable functions. So, apply Fatou's lemma and we have the conclusion that integral of f is equal to limit of integrals of f n s. So, we have proved this under the conditions that f n converges to f n x converges to f of x and f n x is dominated by g of x for every x the modification for this for almost everywhere things is simple and we will do it next time.

Thank you very much.