Measure & Integration
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Lecture - 02 B
Algebra and Sigma Algebra of a Subsets of a Set

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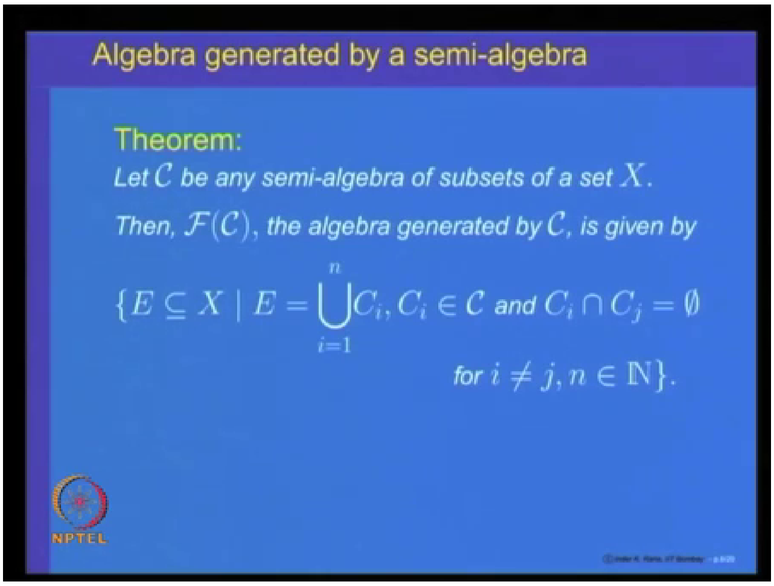

In general, it may not be possible to give a description of the algebra generated by a collection of subsets of a set x. In a special case, when the starting collection C is at least and semi-algebra of sub sets of x, it is possible to describe the algebra generated by it, and that is our next theorem. So, we want to describe the algebra generated by a semi-algebra. The theorem states the following let C be any semi-algebra of subsets of a set X then f of C the algebra generated by C is given by the following collection. So, it is a collection of all subjects E in X size at E can be written as a union of sets finite number of sets C is i say 1 to n with each one of the element C i is in the collection in the semi-algebra C and this C is are pair wise disjoint.

So, what we are claiming is that the algebra generated by a semi-algebra is nothing but the finite disjoint union of elements of C. And we have seen a illustration of this in the previous lecture when we describe C was the collection of all intervals and we will took a f of C the finite disjoint it the union of intervals and showed that that was an algebra. So, that is a typical model example or illustration of describing the algebra generated by a semi-algebra. So, let us prove this that the algebra generated by the semi-algebra is nothing but as described above.

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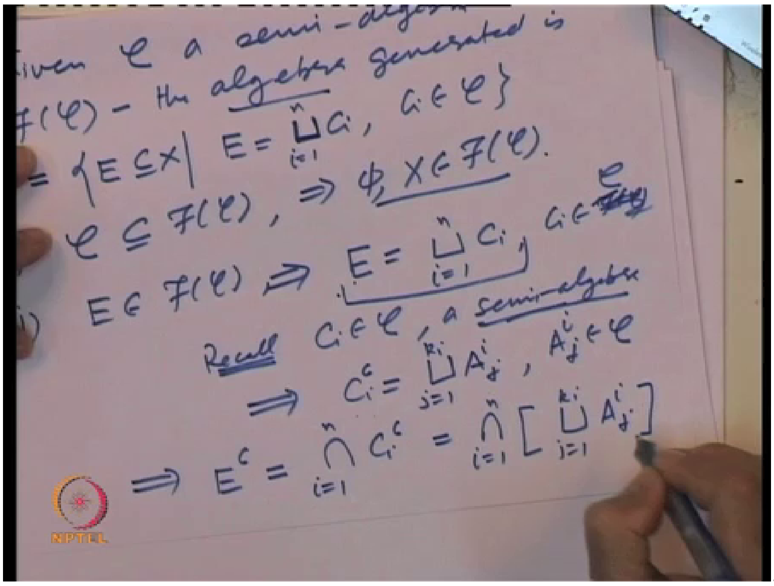

So, let us so given C a semi-algebra F of C the algebra generated by C is equal to all sets such that I can write E as a finite disjoint union of elements belonging C i's belonging to C and note this square bracket square union means that the sets involved are pair wise disjoint. So, let us observe first a few things first of all C is a subset of F of C that should be obvious because every set in C is union of itself single, so E is has representation is union of itself. So, only one set is involved, so E is a subset of and that implies that the empty set and the whole space belong to F of C. So, as a consequence it implies the first property required for F of C to be an algebra and generated will see it later.

So, second thing we are trying to show that F of C is an algebra. So, first property we have checked. Let us take a set E belonging to F of C that means, that implies that E can be written as a finite disjoint n union C i i equal to 1 to n, where C i's belong to f of C , but recall saying that C i is belong to C, sorry. Recall saying that C i is belong to C, and C is a semi-algebra, this is important implies by the property of the semi-algebra that each C i can be written as C i compliment can be written as a disjoint union of elements of C again.

So, let us write it as A i j i equal to 1 to some k i, where a i j is belong to C again and they are disjoint. So, this together with the representation for this implies. So, look at E is a union of C is each C i complement is equal to this so that means, first of all E compliment can be written as intersection of i equal to 1 to n of C i compliment and each C i compliment is a finite disjoint union.

So, this is i equal to 1 to n of finite disjoint union j equal to 1 to k i of A i j right. So, here we have used the property namely that E compliment by De-Morgan laws is intersection of C is because E is union of C i's compliments and each compliment being an element of the semi-algebra can be written as a disjoint union of elements A j i, so that is the representation. Now, we use the fact that intersection distributes over union. So, what kind of this set is this can be written as a union finite number of sets which will involve A i j into so sets of the following type intersection A k l.

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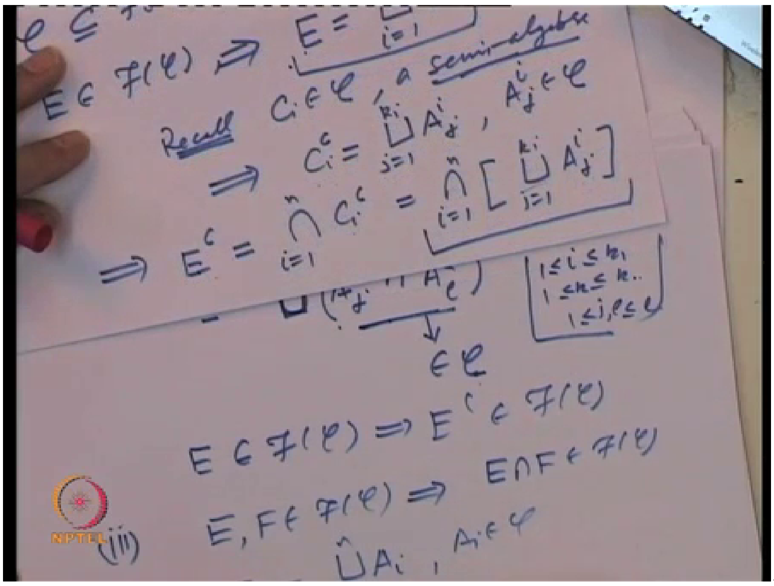

Because when I distribute it over, I will be getting one set A i j and some other set A k l and this intersection of that and unions of those. And these unions, so these sets belong each one of them belong to C. Why does it belong to C, because the collection C is a semi-algebra both of them are elements of the semi-algebra. So, this intersection is element of the semi-algebra, and these sets are pair wise disjoint because if i, k, j and l, if any one of the pairs is different than those sets will be disjoint. So, I can say this is a disjoint union of sets of the following type, where i belong between 1 and k; i, k between 1 and some k some index, and j and l between 1 and l.

So, basically what we are saying is E compliment can be represented again as a finite disjoint union of elements of C again and that is following because of the fact that E compliment which is the union of sets is a intersection and that is intersection of distributive property and this. So, it is because the indexes involved are so many is difficult to write all these things, but it should be clear that E complement is a finite disjoint into union of elements of C, so that means, E belonging to F of C implies E compliment belong to F of C.

And let us look at the third property namely if E and F belong to F of C does this imply that E intersection F belong to F of C well that should be obvious because E belonging to F of C means E is a disjoint union of some A i's 1 to n. A i's belonging to C f is a union of some js 1 to m of some sets B j's, j equal to 1 to m where each B j belong to c, so that implies.

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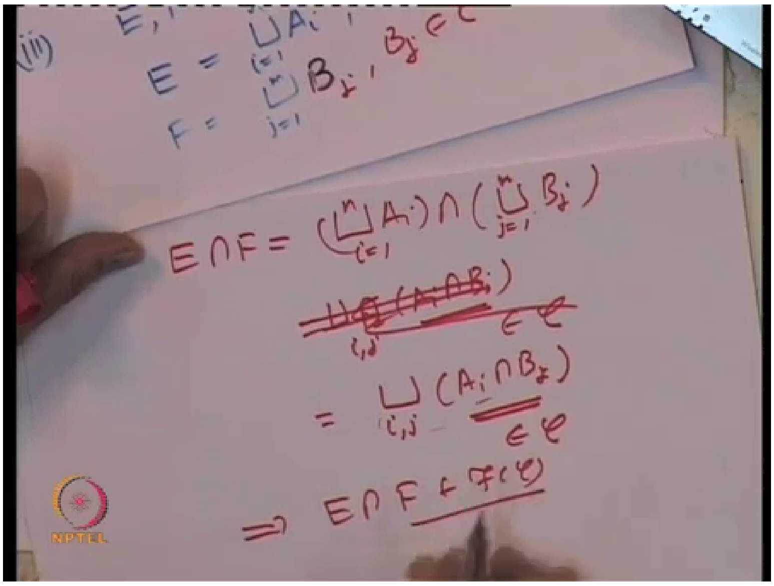

So, let us conclude from here. So, E intersection f will be equal to this union a is intersection of union B j's i equal to 1 to n j equal to 1 to m. And once again using the distributive property, we got this intersection of A i's intersection B j i n j. So, the intersection and each one of them belong to C. So, this is union over i and j. So, let us this by the distributive property will be union over i and j of A i intersection B j. And these elements are elements of C because C is a semi-algebra A i belong to C, B j belong to C and these are again disjoint because they are A i and B j right or disjoint pairs or different pairs they will be disjoint by the property that this unions are disjoint. So, implies that E intersection F also belongs to f of c.

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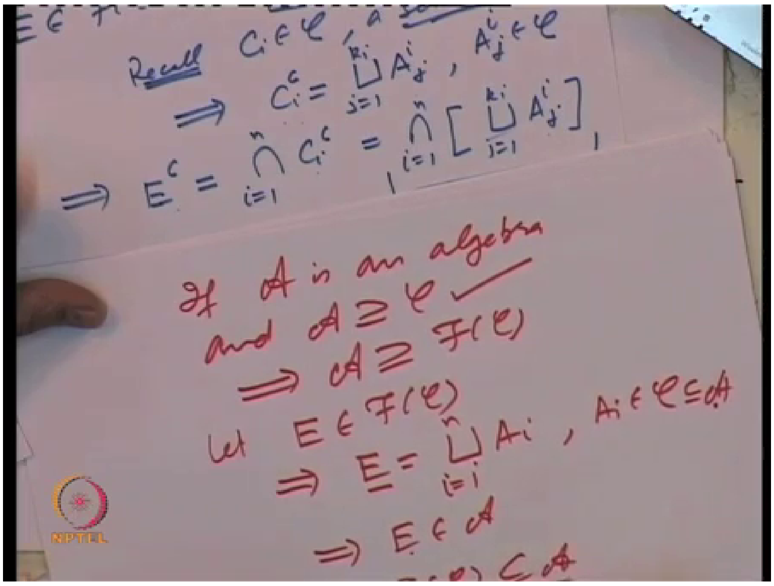

So, what we are shown is the following namely if you take if I look at C is any semi and is algebra if I look at this collection of then subsets which are finite disjoint unions then this include C and these an algebra. To show that it is algebra generated, we have only to show that if A is an algebra, and A include C that showed imply A includes f of C. Well, to prove that is obvious because let us take let E belong to f of C with that will imply that E is a disjoint union of elements A i, where A i's belong to C and C is a subset of a. So, C is inside A, so that implies that E is a union of some elements in A and is a algebra; that means, E belongs to A. So, hence f of C is a subset of A.

So, what we are show is that this collection f of C of finite disjoint unions of elements of the semi-algebra is indeed the algebra generated by the semi-algebra. So, we are able to describe completely the algebra generated by a semi-algebra. And note one thing; we have described the algebra generated by a semi-algebra explicitly and as remarked earlier that the algebra generated by a semi-algebra is described. But in general the algebra generated by any collection of subsets we cannot describe it explicitly, one may not be able to say what are the elements of that algebra generated by collection of subsets, so that is not possible always. It is only in the case when it is a semi-algebra C we start with we are able to describe the algebra generated by it.

So, we have looked a semi-algebra, we have looked an algebra of collection of subsets of X and then we have looked the algebra generated by a semi-algebra of subsets of X. Next, we go to the next level of collection of subsets which are slightly stronger and that is called the semi sigma algebra of sub sets of X now, but before that probably let us look at another property, namely how if you generate more algebras or sigma algebras out of a given collection.

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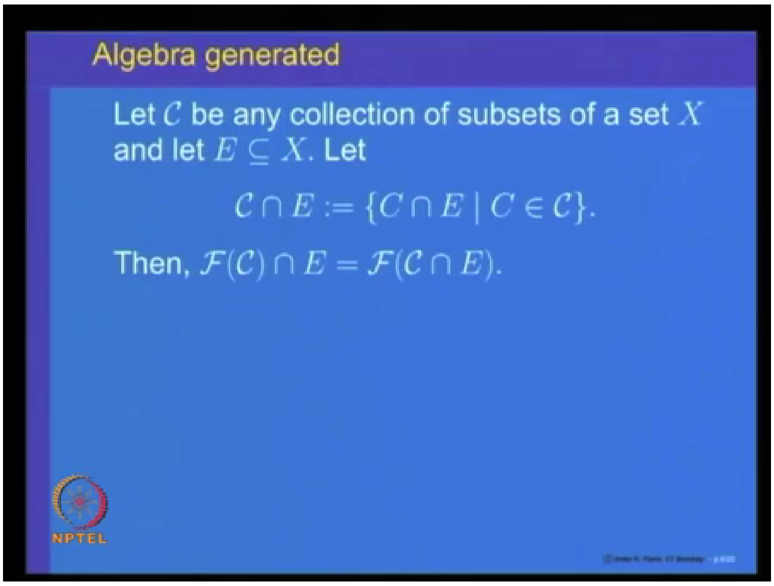

So, let us take C any collection of subsets of a set X and let E be a fix set in X. Let us write C intersection E to be the collection of sets of the type C intersection E C belonging to C note that C intersection E, C is a collection of sets and E is a set. So, C intersection E is just a notation for all sets of the type C intersection E. And note this are all subsets of the given set X. So, the claim is that given any collection C and given any set E let us looks at C intersection E and generates the algebra by this collection of subsets of. So, F of C intersection E is same as the algebra generated by C intersected with E. So, this is a very useful thing of restricting in some sense restricting the sigma algebras goes restricting the algebras.

So, what we are saying is C any collection of subsets of a set X, E contained in X fixed define C intersection E to be C intersection E where C belongs to C. So, what are these sets. So, note that it is C intersection E. So, this is a subset of, so this is a collection of subsets of the set E. And now I can look at F of C intersection E. So, what will be F of C intersection E it is the algebra of subsets of E generated by C intersection E generated by the collection C intersection E; and the claim is that F of C intersection E is same as you first generate the algebra by C. So, this will be the algebra generated by the collection of subsets in C the subsets of X. So, this is the algebra of subsets of X, take its restrictions to E and that is same as this.

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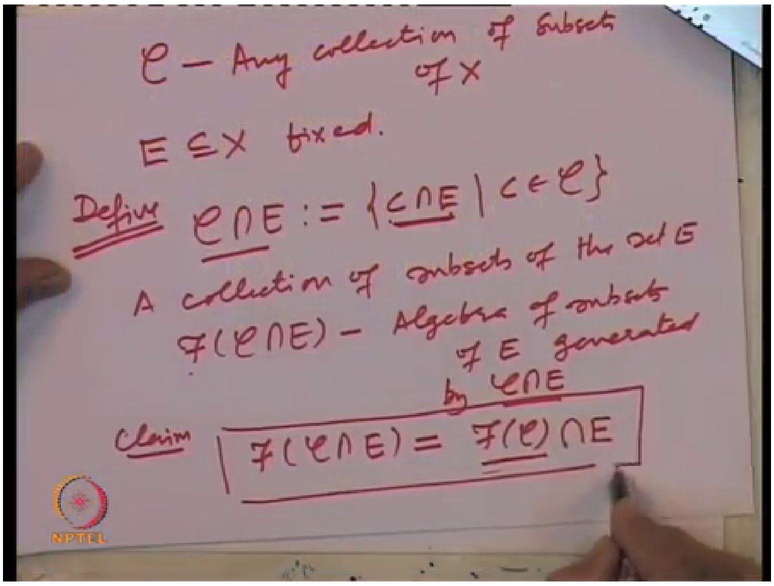

So, this is the theorem. Now, this is result that we say is true. So, let us prove this.

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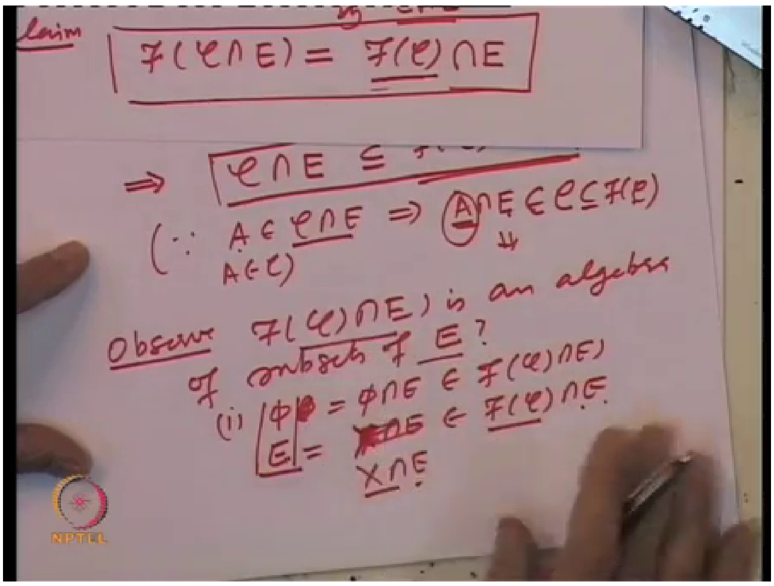

So, to prove this let us first observe note. So, first of all we observe that C is contained in f of C right given any collection C of subsets of a set X, F of C is algebra generated by c. So, by the very definition C is a subset of f of C, so that implies that C intersection E is a subset of F of C intersection E is that clear. So, if I take a set in C intersection E that is going to look like that is going to look like here C intersection E, and C belongs to E. So, that is also an element in F of C. So, is f of C intersection E.

So, let me write because if a belongs to C intersection E implies A intersection E belongs to C, which is in F of C. So, A is an element in C, so C intersection E. So, A belong that means, A belongs to C. So, this is in C, so that is in F of C. So, implies this is an element of F of C intersected with A E and that is precisely the meaning of this. So, we have got C intersection E because of this, this is true. And now let us observe that F of C intersection E is an algebra of subsets of E, this is an algebra of subsets of E, why obviously, empty set belongs to this because empty set can be written as empty set intersection E, so that belongs to F of C intersection E.

And second now note this is a subsets E. So, what is the whole space that is E. So, E is equal to E intersection E, so that is X, so that is X intersection E and that belongs to F of C intersection E. So, I can write E as X intersection E X is an element of F of C and E is here, so that belong to F of C. So, the empty set and the whole space that is they belong to this collection.

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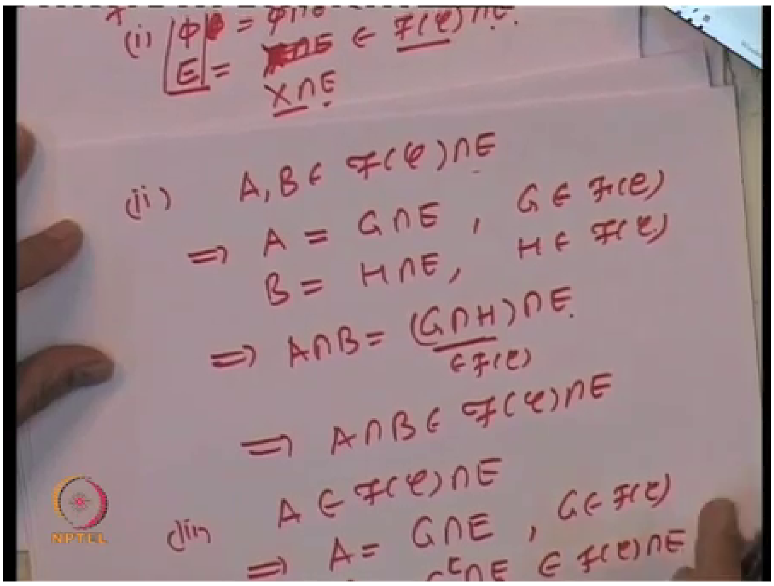

What is the second observation? So, let us look at the second observation from here. So, let us take two sets A and B belonging to f of C intersection E, that means, this A is equal to. So, if a belongs to; that means, C intersection E that means, A can be written as some let us write G intersection E where G belongs to looking at F of C and B is also in this. So, B is written as some H intersection E where H belongs to F of C. So, that implies A intersection B is written as G intersection H intersection E. And now G and H both belong to F of C. So, this belongs to F of C and that is in A, so implies A intersection B belongs to F of C intersection of E.

So, and finally, let us look at third if A belongs to F of C intersection E that means, A can be written as some G intersection E, where G belongs to F of C . So, now, let us look at A compliment, but keep in mind the A compliment is in E. So, this is nothing but G complement intersection E. See this complement is in E, we are taking the compliment inside set E because we are looking at the collection of subsets of E. So, this belongs to F of C intersection E. So, what we have shown is that C intersection E is subset of this and this collection is an algebra, so that implies the property. So, this implies. So, let us observe.

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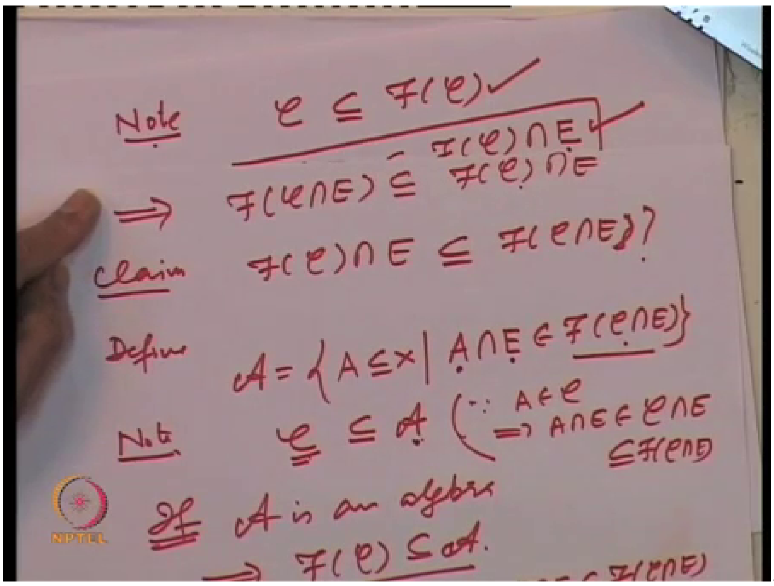

So, this implies that the algebra generated by C intersection E is a subset of F of C intersection E, because this collection is inside this and this is an algebra. So, the algebra generated being the smallest must come inside. So, we get this property. Now, we have to prove the other way in equality. So, claim that F of C intersection E, E is a subset of F of C intersection E. So, this is what we have to prove. So, to prove this let us define a collection A to be all sets A contained in X such that A intersection E belongs to F of C intersection E.

Let us look at this collection A. Note C is contained in A right that is obvious because if I take an element in C then that set A intersection E belongs to C intersection E, which is inside this. Because A belonging to C implies A intersection E belongs to C intersection E, which is inside F of C intersection E. So, C is inside A. So, if A is an algebra, what will this imply C is inside A, A is an algebra there that will imply F of C is inside A. But what is the meaning of F of C is inside A, that means for every element f belonging to F of C have you look at F intersection E that is belonging to f of C intersection E. So, that means, that f of C intersection E is F of C intersection E is a subset of F of C intersection E.

So, to complete the proof we only have to show that a is an algebra of subsets of A is an algebra of subsets of X. So, now, let us look at this, and show this is an algebra of subsets of the set X. So, let us start observing someone to show this is an algebra of subsets of A the set X.

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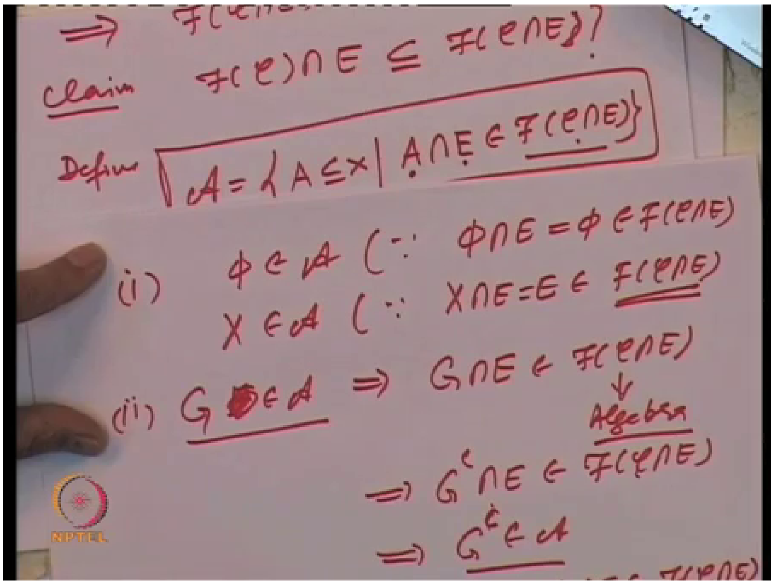

So, to show that let us observe one empty set belongs to A, because empty set intersection E is empty set which belongs to F of C intersection E. The whole space X belongs to A, because the whole space intersection with E which is E that belongs to F of C intersection E keep in mind this is the algebra of subsets of E generated by this collection. Secondly, let us take a set E belonging to a that implies let us take a sets or a let us take a set G belonging to A, that means, G intersection E belong to F of C intersection E, and this is an algebra of subsets of E. So, this will imply that is G compliment intersection E also belongs to F of C intersection E because it is an algebra of subsets of E. So, its complement should also be inside that and that implies that G compliment belongs to A. So, G belonging to A implies G compliment belongs to A.

And finally, let us conclude that F, G and H belong to a that imply that G intersection E and H intersection E belong to f of C intersection E and that implies G intersection H intersection E belongs to F of C intersection E. Once again, by the fact that this is an algebra F of C intersection E, once again the fact this is an algebra. So, the collection A of subsets of X is an algebra include C, so that must include F of C and that proves the theorem that the algebra generated by a collection C that is F of C.

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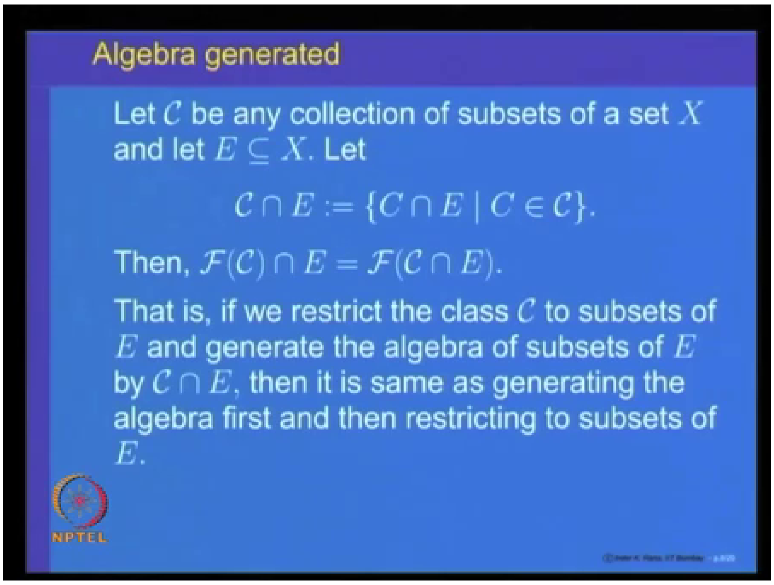

When restricted to A the set E is same as first restrict the collection by which you are generating and then restrict and generate. So, what we are saying is given a collection C of subsets of X, if you restrict the class C to the subsets of E, and then generate the algebra of subsets of E that is same as first generating the algebra and then restricting it that class of sets to that of E. This is going to be very useful later on when we want to restrict collection of sets to subsets of it.

So, let us just conclude do what we have done today, we started with recalling what is an algebra, what is a semi-algebra, what is a algebra. And then we started by observing that in general if a collection C of subsets of a set X is not an algebra we can always generate an algebra out of it, that means, we can show the existence of a smallest algebra of subsets of the set X which includes this collection C. And basically that is the proof is by showing that if I take the intersection of all the algebras which include C, then that is the smallest one and that also had the important property namely observation namely intersections of algebras is again an algebra. So, that was the crucial property crucial observation that helpless to prove at the intersection of all the algebras that include C is also a algebra and that includes because the intersection. So, it has to be smallest and hence that is the algebra generated by.

And then we give examples of how to find algebra generated by a collection for example, if you take singleton sets of any collection of for any set X then the algebra generated if it is the collection of all sets which are either finite or the compliments are finite. And in general, if C is a semi-algebra then the algebra generated by it is nothing but the collection of all finite disjoint unions of elements of that semi-algebra.

So, thank you. So, we will continue next time.