Measure & Integration
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Lecture - 20 A
Properties of Integrable Functions & Dominated Convergence Theorem

Welcome to lecture 20, on Measure and Integration. In the previous lecture we had started defining, what is called the notion of a function to be an integrable function, and then we started looking at some of the properties of integrable functions.

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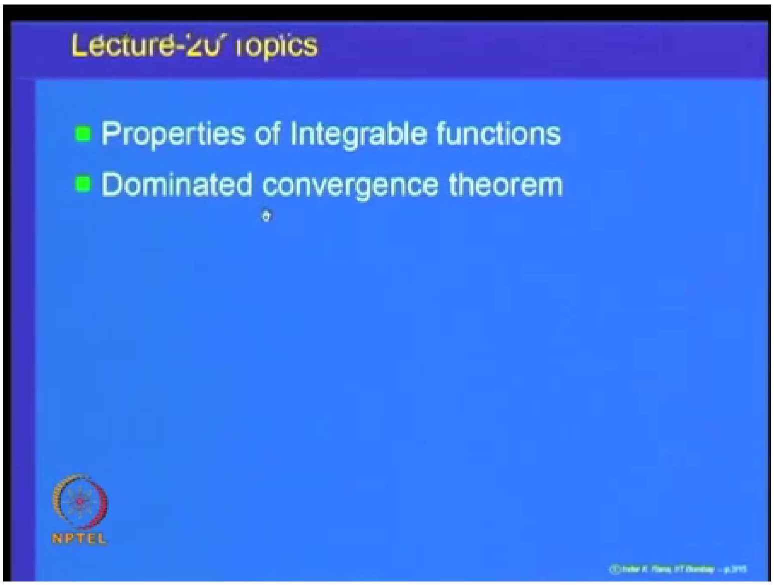

And let us just recall what is integrable function, and then we will start looking at various properties of this, integrable functions, and we will prove one important theorem called dominated convergence theorem.

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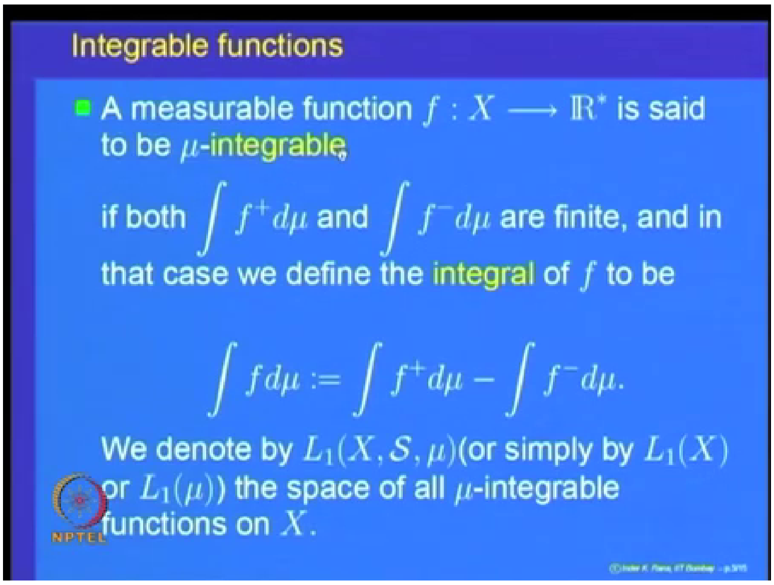

So, if you recall we said a measurable function f on x to extended real valued measurable function, f is set to be integrable with respect to mu, and written as mu integrable if both the integral of the positive part of the function and the negative part of the function are finite.

So, we say f is mu integrable if integral f plus d mu, and integral f minus d mu, both are finite numbers. And in that case we say the integral of f is equal to integral of f plus minus integral of f minus. So, let us just once again emphasize, saying that a function is mu integrable is if and only if both f plus and f minus are having finite integrals, and the integral of f is written as integral of f plus minus integral of f minus.

The class of all integrable functions on the measure space X S mu is normally denoted by l 1 capital L lower 1 X S mu or sometime so, we drop S and mu if they are clear from the context, that what are the sigma algebras, or what is the measure, or sometimes we just emphasize mu, because we want to we know what is x and what is s. So, these are various notations used for denoting integrable functions L 1, X S mu, or L 1 bracket X or L 1 bracket mu. So, this is a space of all mu integrable functions. We will started looking at the properties of this functions.

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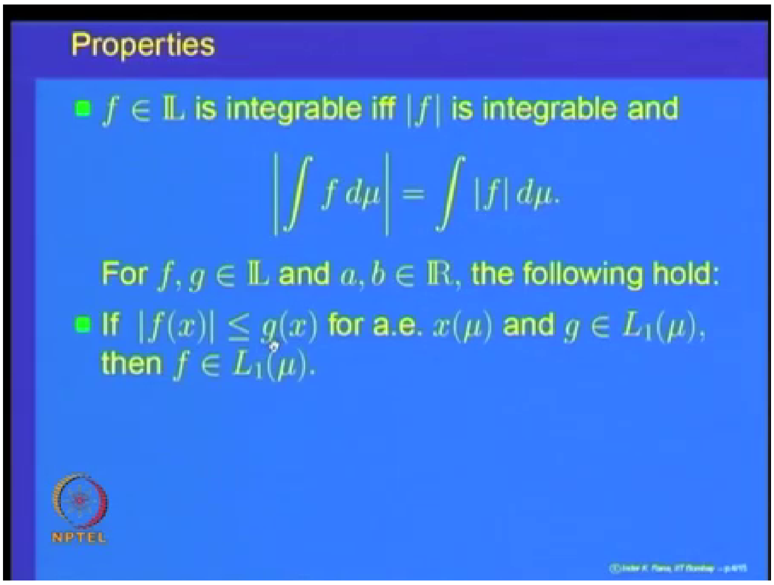

The first important thing we observed was a function, f is which is measurable is integrable, if and only if mod f which is a nonnegative function is integrable; that means, to check whether a function a measurable function is integrable or not, it is to enough to look at the integral of the function mod f, and see whether that is finite or not. And in that case and this is always true, and that for the integrable function integral of a mod f integral mod of the integral of f d mu is less than or equal to integral of a mod f d mu. So, this is a important criterion this is a equitant definition equitant way of defining integrability of a measurable function, namely mod f is measurable this is not equal. So, this is a wrong here it should be less than or equal to so, integral of mod of integral f d mu is less than or equal to so, this is a typing mistake here, they should have been less than or equal to integral of mod f d mu

Let us recall some other properties that we had proved, we said if f and g are measurable functions, and mod f is less than or equal to g of x for almost all x, with respect to mu and g is integrable then f is also integrable; that means, if f function f of x is dominated by an integrable function then that measurable function automatically becomes integrable.

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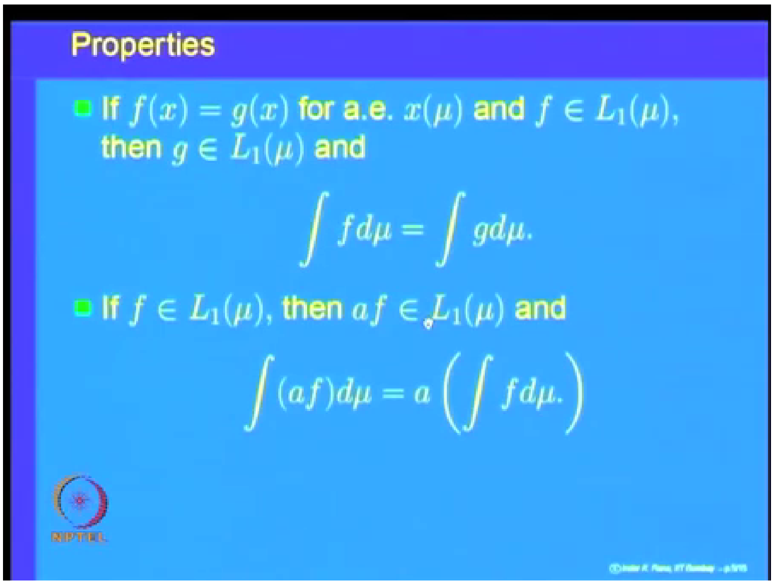

And we also prove the following property namely if two functions f and g are equal almost everywhere, and one of them is integrable say f is integrable then the function g is also integrable; and integral, of f is equal to integral of g. So, that essentially says that the integral of the function does not change if the function is change, if the values of the function the change almost everywhere.

So, f equal to g almost everywhere f and g measurable functions, and one of them say f integrable implies g is integrable, and the integral of the two are equal. We also proved the following property namely f is a integrable function, and alpha is a any real number then alpha time f is also integrable and the integral of alpha f is equal to alpha times the integral of f. So, we continues this study of properties of integrable functions and next we want to check the integrability property namely if f and g are integrable functions.

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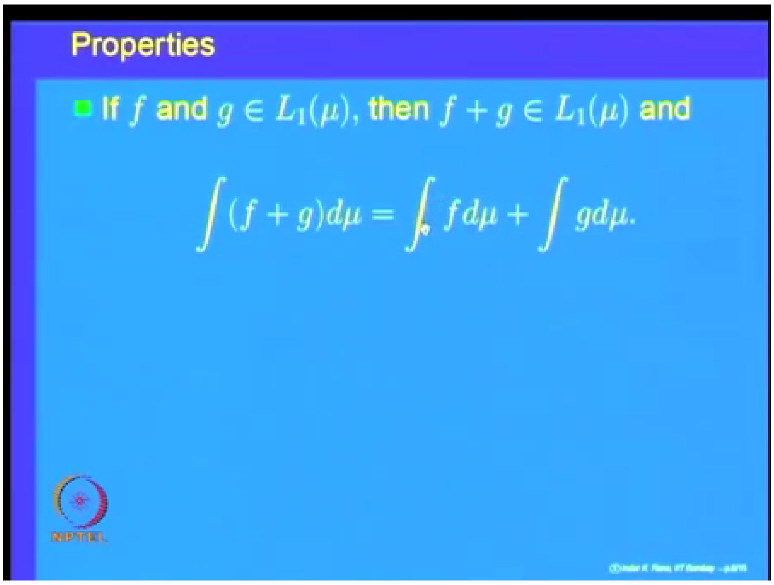

Then we want to show that, f plus g is also integrable and integral of f plus g is equal to integral f plus integral g. So, to prove this property let us look at what we are given.

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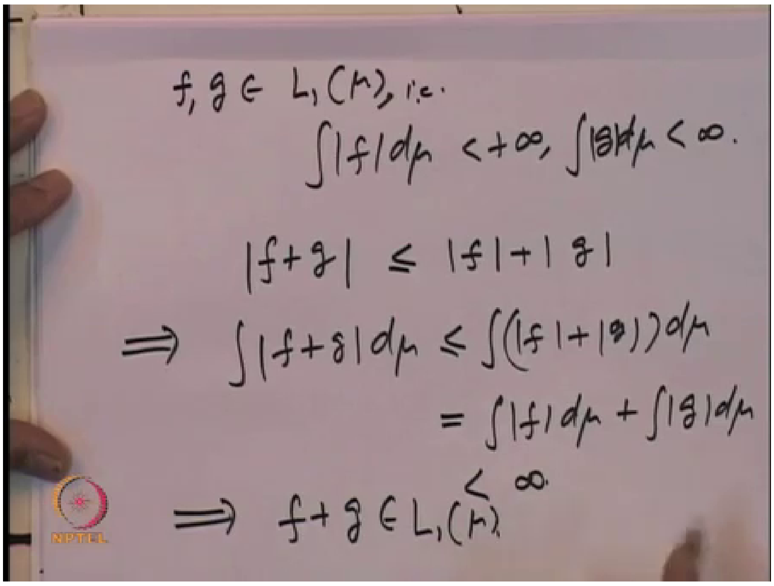

So, we are given that f and g are integrable functions, that is integral mod f d mu is finite, and integral of g d mu is also absolute value of g, with respect to mu is also finite. So, to check whether the function f plus g is integrable or not we have to look at the integral of f plus g absolute value, and show that integral of absolute value of f plus g us also finite, but that follows easily because absolute value of f plus g is always less than or equal to absolute value of fn plus absolute value of g. So, and all are nonnegative measurable functions.

So, using the property of the integral for nonnegative measurable functions, this implies that integral of mod f plus g mu is less than or equal to integral of mod f plus mod g d mu, and that by linearity is same as integral mod f d mu plus integral mod g d mu, and we are given that both of them are finite so, this is finite. So, implies that f plus g is integrable. To compute the integral of f plus g, we have to go back to the definition of the integral.

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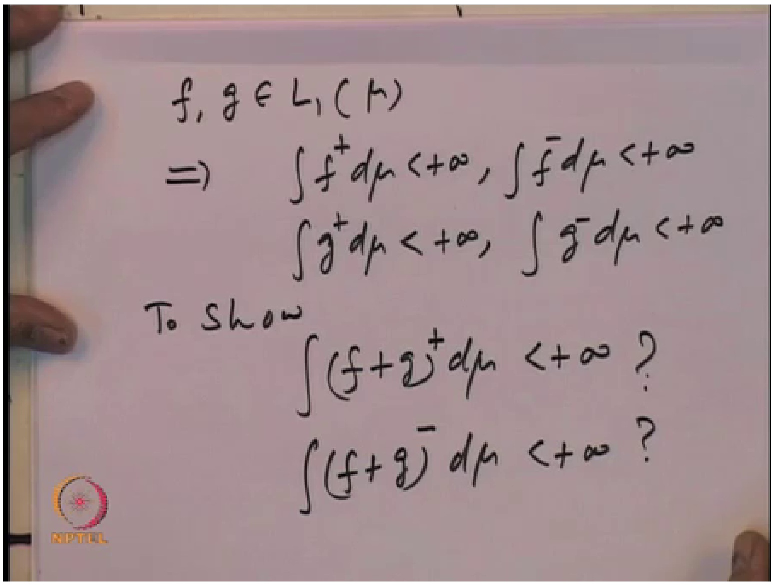

So, f and g integrable, so that implies integral of f plus d mu is finite, integral of f minus d mu is finite, integral of g plus the positive part of g is finite integral of g minus d mu is finite, and we have to show so to show integral f plus g plus d mu finite, and integral f plus g minus d mu is finite. So, these two properties we have to show this, somehow we have to let the positive part of f plus g with the positive part of f and positive part of g and similarly the negative part of f plus g, with the negative part of f and negative part of g.

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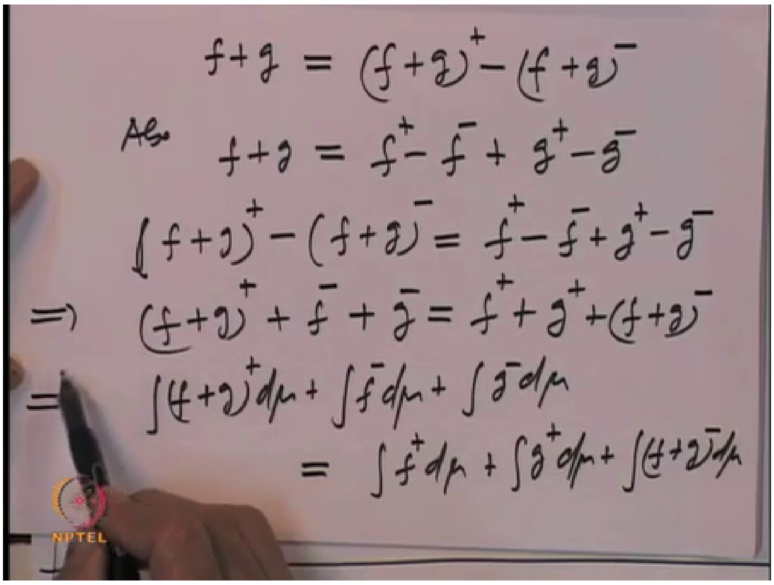

And that is done as follows, so what we do look at f plus g, by definition we can write it as f plus g positive part minus f plus g the negative part. So, that is by the definition of the positive part and the negative part of the function also, f plus g we can also write it as decompose f into positive part and the negative part, so that is f plus minus f minus and similarly write g as g plus minus g minus.

Now, from these two it follows that integral of f sorry not the integral from this it follows that, f plus g positive part minus f plus g the negative part is equal to f plus minus f minus plus g plus minus g minus right. So, from these 2 equations it follows this is, so and now what we do is all the negative terms we shift on the other side of the equation. So, this implies that f plus g plus, plus f minus plus g minus is equal to f plus, plus g plus, from here and this term on the other side will give me plus f plus g minus.

So, yes rearrange the terms, and now observe that the left hand side is a nonnegative function and the right hand side is a nonnegative function so by the properties of integrals for nonnegative functions this, implies that integral of f plus g plus d mu plus integral of f minus d mu plus integral of g minus d mu, so that is the integral of the left hand side is equal to integral of f plus d mu and plus integral g plus d mu plus integral f plus g minus d mu.

So, from this equation by using the properties of integral for non negative functions the linearity property the integral of the left hand side is equal to integral of the right hand side, and integral of the left hand side consists of integral of f plus g plus plus integral of f minus plus integral of g minus and, that is equal to integral of f plus plus integral of g plus plus integral of f plus g minus. And now we observe that in this equation all the terms are finite quantities are real numbers, that is because f plus g we have already shown is integrable, so this first integral of f plus g plus that is finite integral f minus is finite and similarly or the all the terms are nonnegative real numbers.

We can again manipulate them, and treat shift comes on the left hand side and right hand side the what we will do is this term f plus g minus on the right hand side, will bring it on the left hand side, and the terms f minus d mu integral and integral g minus d mu we shifted on the right hand side.

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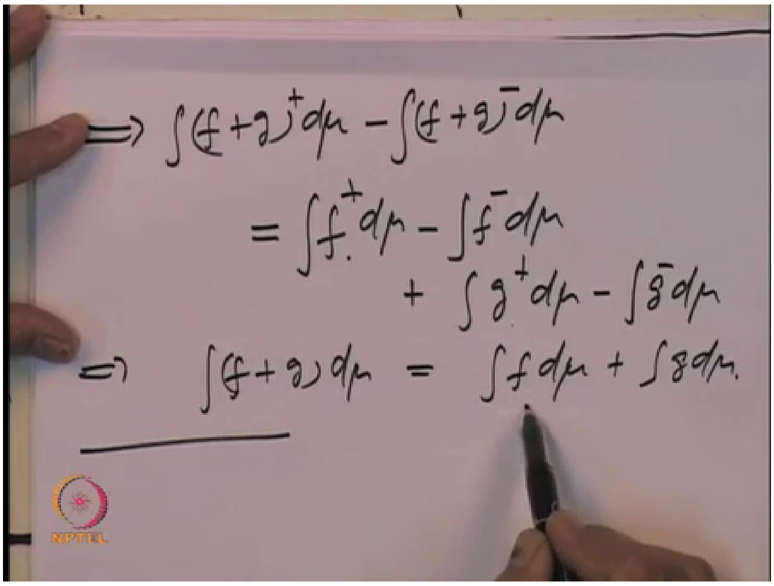

So, that gives us the property, so shifting, implies, that integral of f plus g plus d mu minus this term will give you integral f plus g minus d mu so this term we have shifted, and shift these two term on other side is equal to integral f plus d mu that is this term and this minus f integral of f minus bringing on this side will give you integral f minus d mu plus integral of g plus d mu which is already there and integral of g minus from the left hand side will give you integral of g minus d mu.

So this rearrangement of the terms here, once again give you that integral of f plus g plus d mu and the integral of the negative part of f plus g is equal to integral of f plus minus integral f minus plus integral g plus, and now by the definition the left hand side is nothing but integral of f plus d mu and the right hand side is integral f d mu plus integral g d mu. So, that proves the linearity property of the integral, that if f and g are integrable functions not only f plus g is integrable; integral of f plus g is equal to integral of f plus integral of g d mu.

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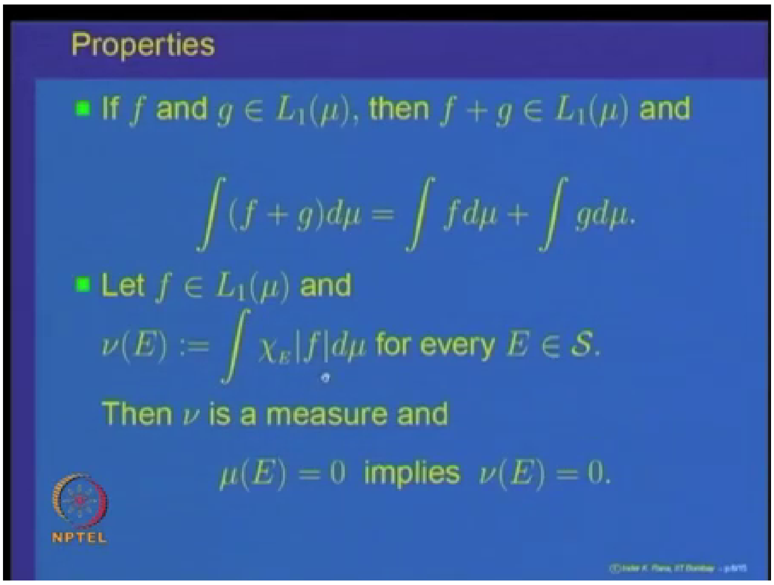

So, that is the linearity property of the integral, we have proved the basic properties of the integrals namely the integral of a function which is integrable of course, it is a finite quantity and it is linear namely, if you take a function f multiply it by scalar alpha, then alpha times f is integrable; and the integral of alpha f is equal to alpha times integral of f, and similarly if f and g are integrable, then f plus g is integrable; and the integral of f plus g is equal to integral of f plus integral of g.

Let us look at some more properties of this integral which are going to be useful later on. Let us look at the next property, so for a integrable function f; so, f in l 1 of mu let us look at we have already shown, that if you mod f is a nonnegative measurable function, and if you multiply it by the indicator function of asset E, then we already shown that this is again a nonnegative measurable function, and of course this function is less than or equal to integral of mod f. So, nu of E is going to be always a finite quantity, so the claim is nu is a measure infinite measure, and it has the property that mu of E equal to 0 implies nu of E equal to 0. So, whenever asset E has got mu measure 0 the measure of nu also is going to be equal to 0.

And this basically follows from the properties of the integral for non negative functions, because if f is integrable, then mod f is a nonnegative measurable function and its integral is a finite quantity. So, nu of E is a finite measure for every e the function chi E times mod of f is less than or equal to mod f, so this integral is going to be less than or equal to integral of mod f which is finite, and obviously for nonnegative functions we have already proved this property, that for a mu of E is 0 then the integral over E is equal to 0, so this property follows from our earlier discussions.

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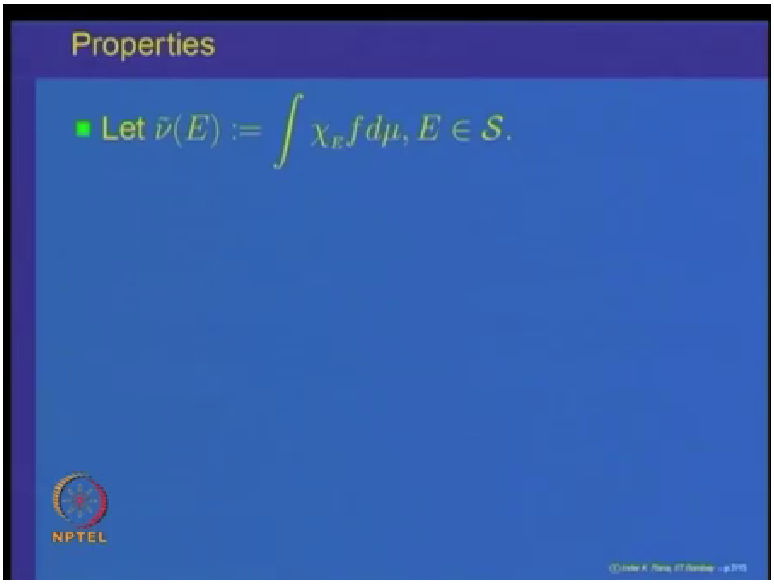

Let us look at integral of for the integral function not the integral of mod f, but let us look at the integral of f time's indicator function of E. And if you recall we had already shown, that if f is measurable and E is a set in the sigma algebra then chi E times f is a is again a measurable function, and just now we have observed that this number is going to be a finite number because, this is again a integrable function.

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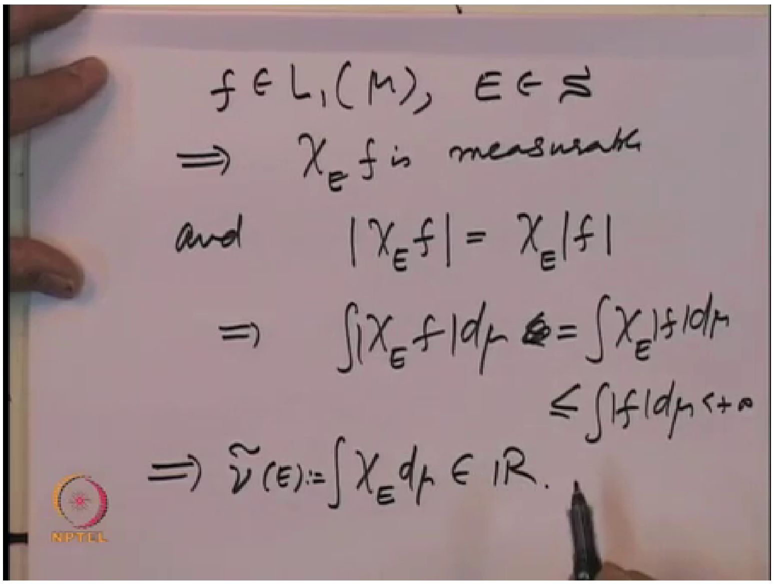

So, let us just observe this property once again that, if f belongs to l 1 of mu and E is a set in the sigma algebra, then this implies that chi E time's f is a measurable function, so that we have already seen because, f is a measurable function indicator function of E is a measurable functions of product or measurable function its measurable, and we observed just now, if you look at the absolute value of chi of E times f that is same as indicator function of E because that is negative into absolute value of f. So this implies that the integral of chi E times f absolute value d mu is less than or actually is equal to integral chi E of mod f d mu, and which is less than or equal to integral mod f d mu which is finite. So, what does it imply?

So, this implies that integral chi E d mu is a, so this we are denoting by nu tilde of E so, this is a real number is a finite the real number.

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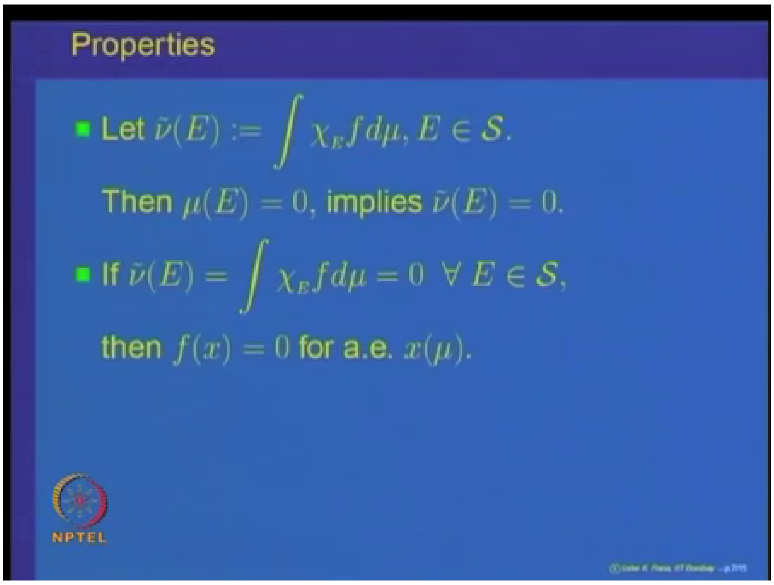

So, that is the observation and we want to claim, that mu of e is equal to 0 implies that nu tilde of e is also equal to 0. So, the claim is that this claim is that this property, so let us prove this property.

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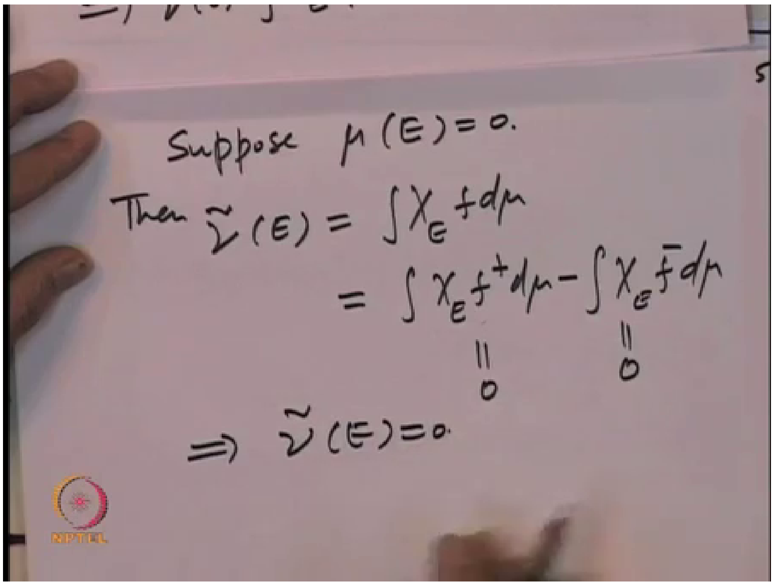

So, suppose mu of E is equal to 0, then what is nu tilde of E nu tilde of E by definition is integral chi E of f d mu, which is same as the integral of chi E f plus d mu minus integral chi e f minus d mu. And now we observe mu of E equal to 0 chi E of f plus is a nonnegative function and for properties of nonnegative functions, imply if the set has got measure 0, then the integral of this is equal to 0. So, the first integral is equal to 0, second integral is equal to 0 by properties of integrals of nonnegative measurable functions, so implies nu tilde of E is equal to 0.

What we are saying is mu of E equal to 0 implies nu tilde of E is also equal to 0, so that is a property we are proving here, but keep in mind that nu tilde of E is defined as a real number for every E belonging to s, but it is not a non negative number because f may not be nonnegative function. So, we cannot say nu tilde of E is a measure we will look at this property a bit later it may not be a measure, but it has some property similar to a measure.

Here is another important property, let us look at again the same value nu tilde of E, which is equal to integral of f over E d mu, and suppose this is equal to 0 for every set E in the sigma algebra, then the claim is in this function f must be equal to 0 for almost all x belonging to mu.

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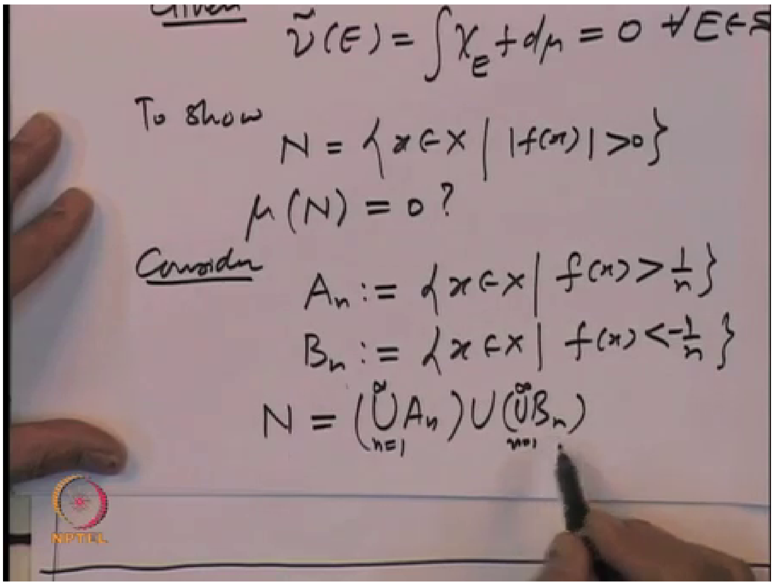

So, let us prove this property namely, so given nu tilde of E which is nothing but integral chi E time's f d mu is equal to 0 for every E belonging to S. So, that is what is given to us, and we want to show that if you take the set N, which is x belonging to X such that mod f of x bigger than 0, if we write this set N, then note that this set N is a set in the sigma algebra and we want to show that mu of n right, we want to show of f is 0 to almost everywhere, and this is the set where f is not 0, so we want to show this is equal to 0, so this is the problem we want to show.

Now, let us look at consider the set say for example, let us look at let us write the set say A n to be the set, where x belongs to X say that f of x is bigger than one over n, and similarly let us write B n to be the set of x belonging to X where f of x is less than minus one by n. so, now the claim is that the set N is nothing but union over A n union over B n n equal to 1 to infinity union of union n equal to 1 to infinity; that means, all this sets A n's and B n;s if take their unions, that is precisely set n where n is what is that set N; N, is the set where f of x is not equal to 0.

So, if f of x is not equal to 0, then either f of x is positive or f of x is negative, so if it is positive then it is going to be bigger than 1 over n for some n, if x is positive and bigger than one over n then it is going to belong to 1 over n or if f of x is not 0 and it is negative. That means, it is negative, so it is going to be less than 1 over minus 1 over n for some n so it belong to B n. So, every set N every point x in n either belongs to A n or belongs to B n. And obviously, if x belongs to A n or B n then f of x is not equal to 0, so it belongs to n, so N is equal to this.

So, N is written as a countable union of sets and all of these are sets in the sigma algebra S, and we want to show this union has got measure zero, so in case mu of N is not 0 that will mean for some n either A n has got positive measure or B n has got positive measure, because otherwise mu of N will be less than or equal to sigma mu of A n's plus sigma mu of B n's all of them equal to 0. So, let us write what we are saying is the following to show that mu of n is equal to 0.

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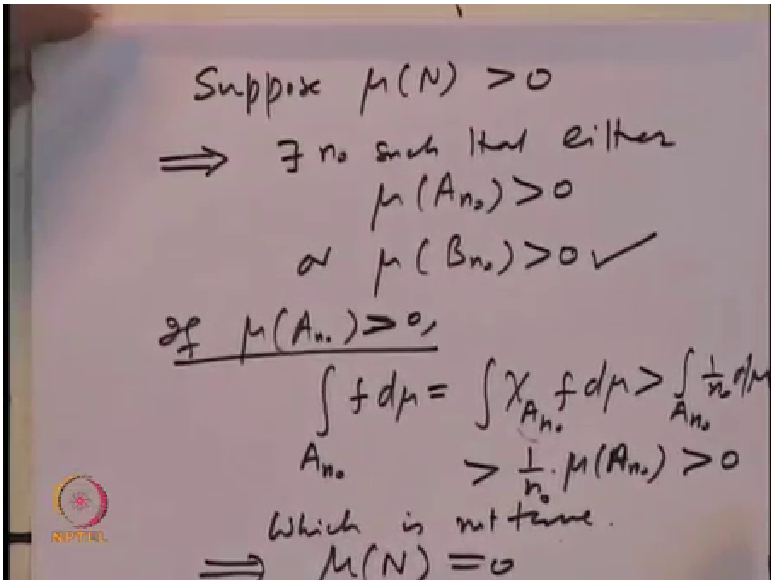

Suppose, mu of n is bigger 0 then that implies, then this condition implies there exists some n naught such that either mu of A n naught is bigger than 0 or mu of B n naught is bigger than 0 because if not, then mu of N will be equal to 0. Let us look at this conditions so, suppose the first one if mu of A n naught is equal to is bigger than 0 then look at the integral; then, integral of f ok then, integral of f over the set A n naught let us look at integral of f over the set A n naught. So, that is equal to integral, so which is same as integral chi A n naught times f d mu.

Now, on the set f A n naught f is bigger than 1 over n, this is bigger than; obviously, integral one over n times n naught times mu of A n naught. So, let us observe that on the set A n naught outside A n naught this function is equal to 0 indicator function of A n naught times that is 0, and on A n naught f is bigger than 1 over n naught. So, this function is bigger than one over n naught right, and outside A n naught is zero so this is going to be bigger than, so it is bigger than integral over A n naught of one over n naught d mu, so that is what we are saying, once that is true and this is nothing but this integral and that is bigger than 0.

So, in case mu of n naught is bigger than 0 integral of f over A n naught is going to be bigger than 0 which is a contradiction, because which is not true because we are given integral of f over every set E is equal to 0 which is not true. So, if this holds then is a contradiction similarly if this holds one can prove it is a contradiction, than the integral of f over B n naught will be less than strictly less than 0 not equal to 0. So, in either case both of these are not possible so, our assumption that mu of n naught is bigger than 0 must be wrong and hence mu of n so, implies that the measure of the set N is equal to 0, and N was the set where f of x is bigger than 0.

So, this set has got measure zero so, this is what we wanted to prove, so we have proved the property that if integral of a function, over f is a integrable function embedded integral over E is equal to 0 for every E belonging to S, then f must be equal to 0 almost everywhere ok.

So, this is a very nice property and useful property.