**Measure & Integration**
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**Lecture – 19B**
**Monotone Convergence Theorem & Fatou's Lemma**

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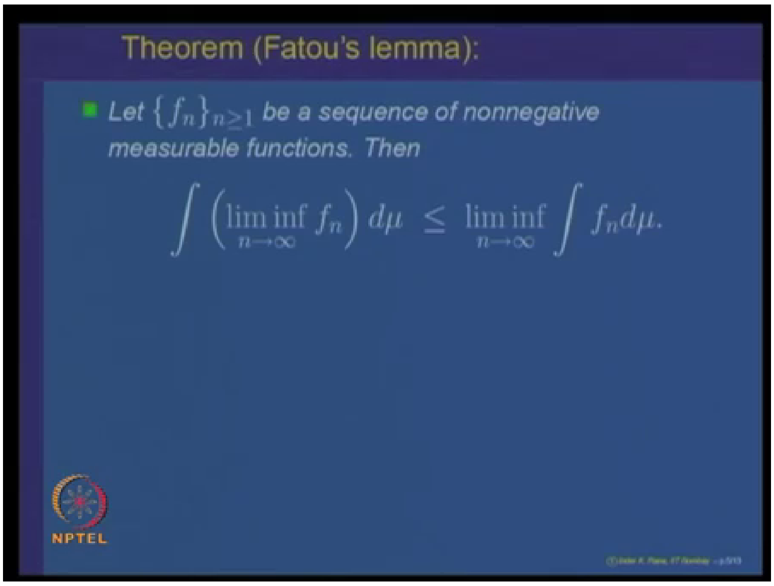


So, we have 2 important results for nonnegative. So, we have got 2 important results for sequences of nonnegative measurable functions, one of them is called the monotone convergence theorem which says if fn is a increasing sequence of functions, nonnegative measurable functions increasing to a function f, then integrals of f ns will converge to integral of f.

So, keep in mind monotone convergence theorem is for a non negative sequence of nonnegative measurable functions which is increasing to f. And in case the sequence of nonnegative measurable functions is not increasing then we have got Fatou's lemma which says that that for any sequence fn of nonnegative measurable functions, the integral of the limit inferior of f ns is going to be less than or equal to is always less than or equal to limit inferior of the integrals.

So, this is these are the 2 important theorems, which help us to relate the limit of the integrals with integral of the limits and we will see applications of this in rest of our course. So, with this we conclude the section on the definition of integral for non negative simple non negative measurable function. So, let us just recall what we have done we have started with defining the integral of nonnegative simple measurable functions, the functions which look like linear combinations of indicator functions sigma ai indicator functions of ai; for them we defined the integral to be nothing, but ai time summation of ai times mu of ai, we showed it is independent of the representation and we proved various properties of the integral for nonnegative simple measurable functions.

And then we looked at the class of nonnegative measurable functions, since the every nonnegative measurable function is limit of sum sequence of nonnegative simple measurable functions increasing to that function f. So, we defined integral of the nonnegative sim measurable function to be nothing, but the limit of the integrals of that sequence of nonnegative simple measurable functions increasing to it. And we showed this integral is independent of the limit of a sequence sn you choose you select which increases to f, and then we proved various properties including monotone convergence theorem and fatous lemma.

And now let us look at how can we define the integral for a function, which is not necessarily nonnegative. So, for that we will do the following; we will start with. So, let us look at a function f.

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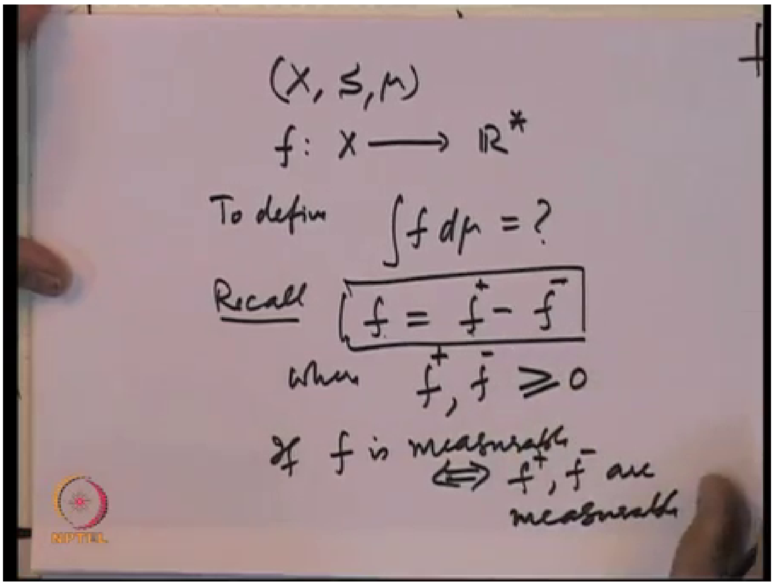

So, keep in mind we have got a major space X as mu, which is complete; f is a function which is defined on X taking extended real valued and we want to define. So, to define integral f d mu, this is what we want to know what it should look like and of course, we would like this integral to have nice properties. So, we would like to have it to be a fun diagonal to be a linear operation.

Now, recall. So, let us recall the function f can be written as f plus minus f minus. We can split into 2 parts the positive part and the negative part of the function where f plus and f minus both are nonnegative functions, and if f is measurable of course, with respect to the sigma algebra s, that is true if and only if both have plus f minus r measurable. So, f can be written. So, this is the clue how we should go about.

So, f can be written as a difference of 2 nonnegative measurable functions if f is measurable, and integral of nonnegative functions is defined. So, integral of f plus is defined integral of f minus is defined, and if our integration is going to be linear it is all, but necessary that our integral.

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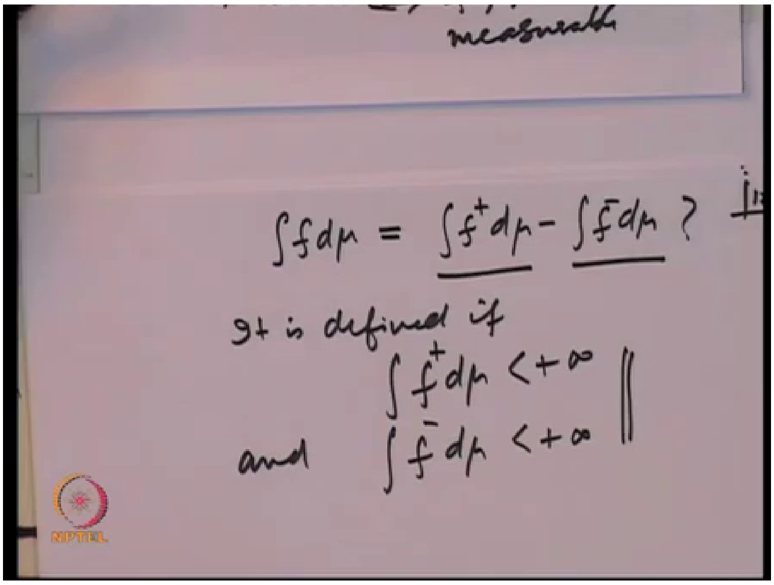


So, we should define integral f d mu whatever be the way we define, it should have the property this is f plus d mu minus integral of f minus d mu. So, this is what we would like to have; this is defined this is defined now, the question is the difference defined.

So, the difference will be defined if both of these quantities are finite numbers. So, the left hand side it is defined if f plus d mu is finite and integral f minus d mu is finite. So, that says so; that means, whenever f is a measurable function. So, that integral f plus d mu is finite integral f minus d mu is finite we can define its integral to be equal to integral equal to integral of f plus d mu minus integral f minus d mu. So, with that we let us define what is the integral; what is called a integrable function.

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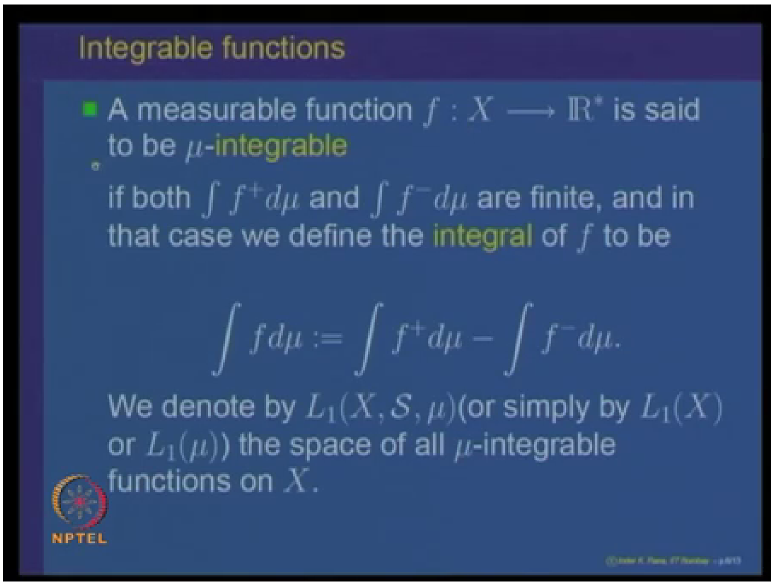


So, a measurable function f defined on X, taking extended real valued real values is said to be mu integrable of course, mu is the measure underlying space which is fixed. So, it is said to be mu integrable, if both integral of f plus d mu f plus is a nonnegative measurable function, f minus is a nonnegative measurable function. So, by our earlier discussion both these numbers f plus d mu and f minus d mu are defined.

So, if they are both finite in that case we say that the function f is integrable, and its integral is defined as integral of f plus minus integral f minus. So, integral of f is defined as integral of the positive part of the function minus the integral of the negative part of the function. So, whenever a function f is defined on x, we say f is integrable if both the positive part and the negative part have finite integrals and in that case we defined the integral of f. So, written as integral fd mu to be integral of f plus minus integral of f minus.

So, we will denote by the symbol capital L lower 1 X, S mu to be the class of all mu integrable functions. So, in case it is clear what is x and what is mu and what is S, we can simple. So, sometimes simply write it as L 1 of X or simply L 1 of mu. So, if we understand what is underlying measure space.

So, the space of integrable functions either it will be a explicitly written as L 1 x s mu or sometime simply as L 1 of X or L 1 of mu. So, this is the class of all integrable functions; that means, all functions f such that integral f plus is finite integral f minus is finite and in that case integral of f is defined as integral f plus minus integral of f minus.

So, for all integrable functions f belonging to L 1 of X, we have integral f d mu we will now study the properties of this integral.

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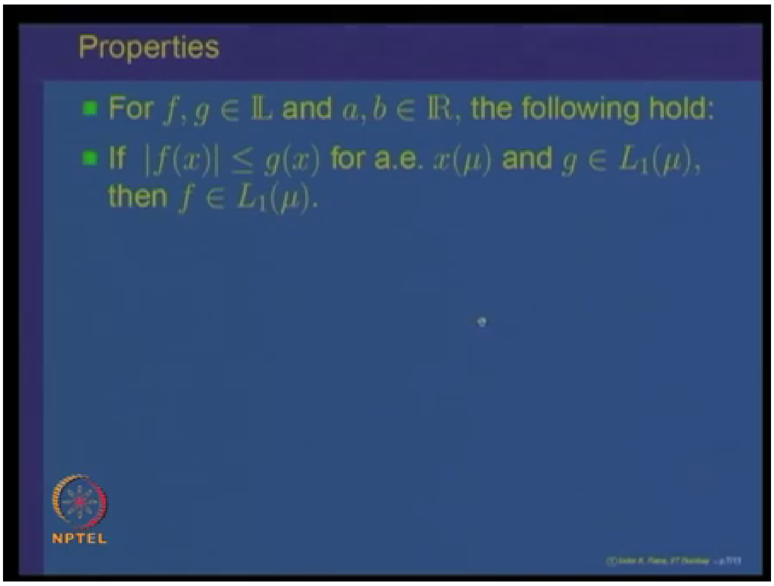


So, the first property is let us fix functions f and g which are integrable and f and g to be which are f and g to be real number a n g to be real numbers. So, if f and g are measurable functions at mod f of x is less than g of x for almost all x, and g belongs to L 1 then the claim is f is in L1. So, this is a very simple property we want to check namely that if.

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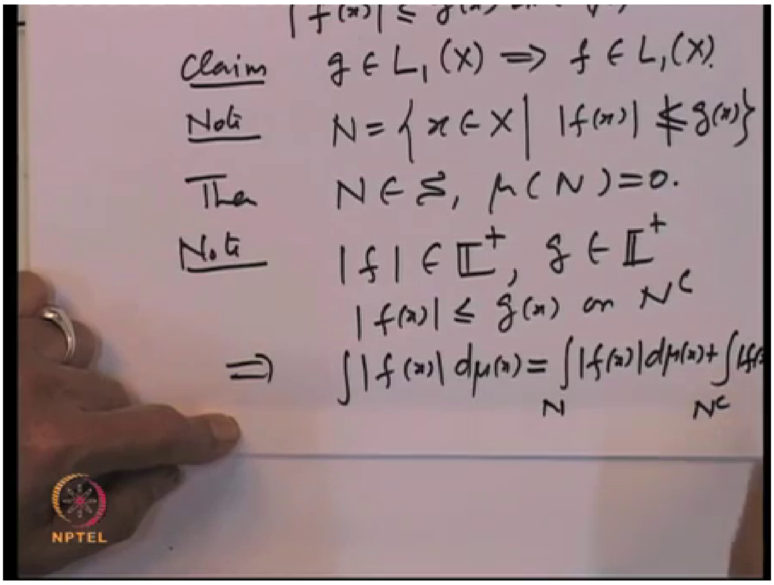


So, let us f and g are measurable functions on x and we are given that mod f of x is less than or equal to g of x almost everywhere mu.

So, from here we first claim is that if g is L 1 of X, then that implies that f is in L 1 of X. So, to prove that let us observe. So, note where given f x is less than or equal to g x almost everywhere X. So, let us define n to be the set of all points x belonging to x where mod f of x is not less than or equal to g of x where this property is not true. Then we know that n belongs to the sigma algebra and mu of n is equal to 0 n of n is equal to 0.

Now, note because mod of f is a nonnegative. So, mod f belongs to L plus g is nonnegative measurable. So, g belongs to L plus and on is almost everywhere. So, mod f of x is less than or equal to g of x on n complement. So, that is what is given to us. So, thus implies integral mod f x, d mu x which we can write as integral over N mod f x d mu x plus integral over N complement mod f x d mu and now let us observe. So, let us now. So, observe that the set n as got measure 0. So, this party is 0 and N complement on N compliment f is mod f is less than or equal to g.

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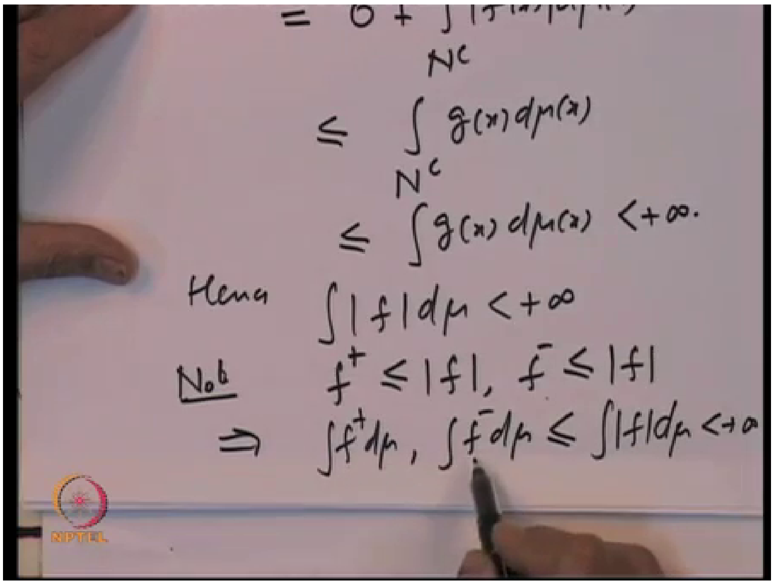


So, this is equal to 0 plus integral over N complement of mod f x d mu x, and on N complement f is less than or equal to g. So, this is less than or equal to integral over n complement of g of x, d mu x and that is less than or equal to integral over the whole space g of x d mu x which is finite.

So, what we have shown is that in case mod fx is less than or equal to gx, then we have shown that the integral of mod f is finite. So, this says. So, hence integral mod f d mu is finite and now let us note that f plus is always less than or equal to mod f and f minus is also less than or equal to mod f. For any function the positive part is less than or equal to mod f the negative part also is less than equal to mod f. So, that implies that integral f plus d mu and integral f minus d mu both of them are less than or equal to integral mod f and which is finite.

So, we have shown that the integral of f plus and integral of f minus both are finite and whenever mod of f is less than or equal to. So, whenever. So, what we have shown is whenever mod. So, this property is true this implies that integral of f plus and integral of f minus both are finite. So, that implies. So, implies f belongs to L 1 and further.

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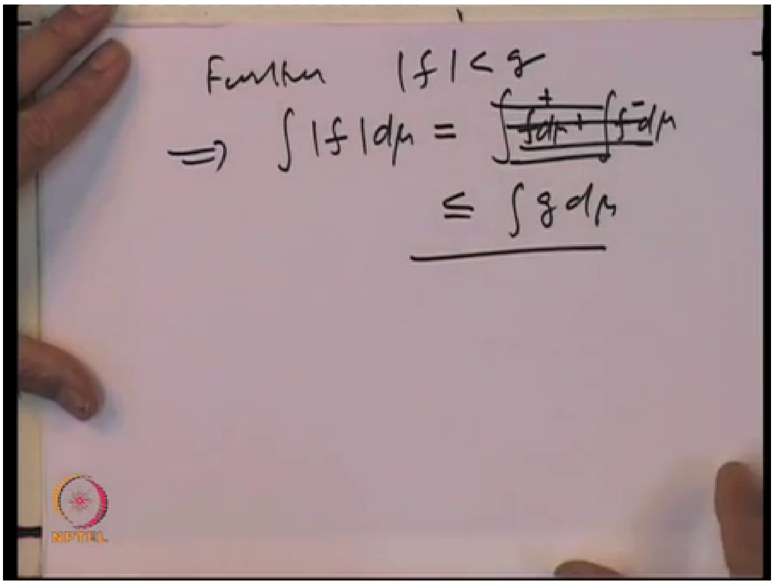


Let us calculate what is integral of mod f d mu. Mod f if you recall is nothing, but f plus plus f minus so; that means, this is equal to integral d mu plus integral of f minus d mu. So, integral of mod f is nothing, but integral of f plus, plus integral of f minus d mu.

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And both of them are finites. So, that we have already observed. So, I wanted to check that integral of mod f is less than or equal to integral of integral g d mu, which we have already actually checked. So, we have already checked that integral of f plus. So, which is less than integral of mod f is less than. So, mod f is less than integral g implies. So, you do not have do this. So, is less than or equal to integral g d mu. So, that follows from directly from that mod f now we have shown is integrable is integral is finite. So, this is less than or equal to integral of g.

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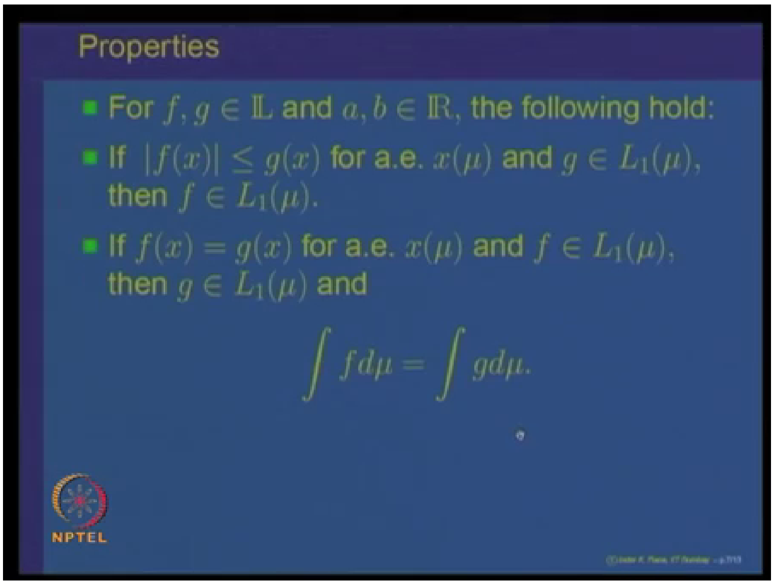


So, this proves the first property namely if f and g are measurable functions, and mod f x is less than or equal to g x for almost all x and if g is integrable then f is integrable.

So, what we are saying is if a function f of x which is measurable is dominated by a function g which is integrable, then the function f also becomes integrable. Let us look at next the property that if f and g are equal almost everywhere and f is integrable, then g is integrable and the integrals of the 2 are equal and that property is something similar to that we have just now shown a similar analysis will work.

So, let us look at we have got 2 functions f and g and f of x is equal to g of x almost everywhere mu. So, let us write the set N x belonging to x, where f x is not equal to g x then by the given condition mu of N not equal to that is equal to 0, and f x equal to gx for every x belonging to N complement, right. So, now, let us look at. So, we have given that the function f is f belonging to L 1.

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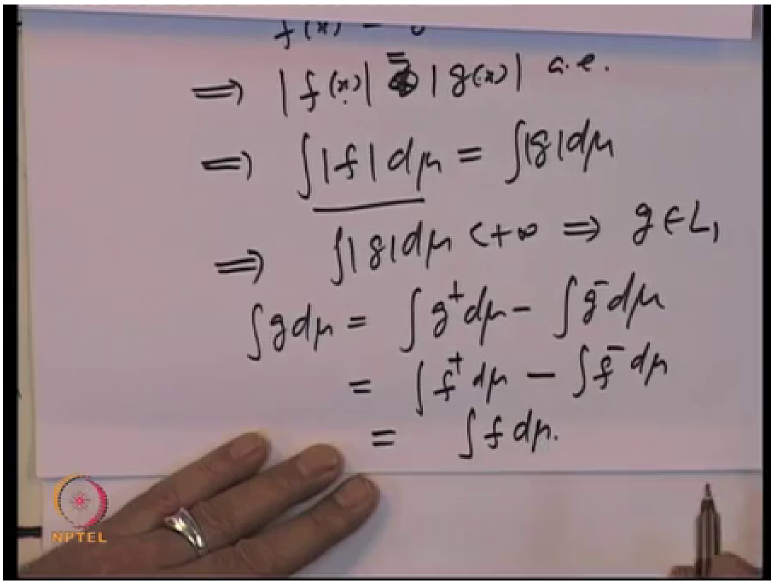

So, implies we want to show that g belongs to L 1 and that is because if f x is equal to g x almost everywhere, then that implies that mod f x is also less than or equal to mod gx almost everywhere.

Right the sets where they are not equal. So, wherever they are equal so that is mod x is equal to gx. So, because on then compliment that will happen. So, this is less than or equal to. So, that implies integral of f x is equal to g x. So, sorry. So, we should say that fx is equal to gx almost everywhere. So, that implies f x is equal to g x mod fx is equal to mod x. So, and just now we have showed that whenever f and g are equal almost everywhere, integral mod f is equal to integral mod g d mu. So, either of them finite implies other is finite we are given this is finite. So, implies mod g d mu is finite.

So, implies once again that g is L 1 and g is L 1. So, integral of gd mu is equal to integral g plus d mu minus integral g minus d mu, but f is equal to g almost everywhere, we ask the reader to verify this so; that means, f of f plus must be equal to g plus, and f minus must be equal to g minus almost everywhere. So, once again this integral is equal to minus integral of f plus d mu, minus integral of f minus d mu which is nothing, but equal to integral of f d mu. So, our integral g is equal to integral f, whenever f and g are equal almost everywhere. So, these are simple properties of our integrable functions that we have looked at. So, if f is equal to g almost everywhere, and one of them is integrable then the other is integrable and the 2 integrals are equal.

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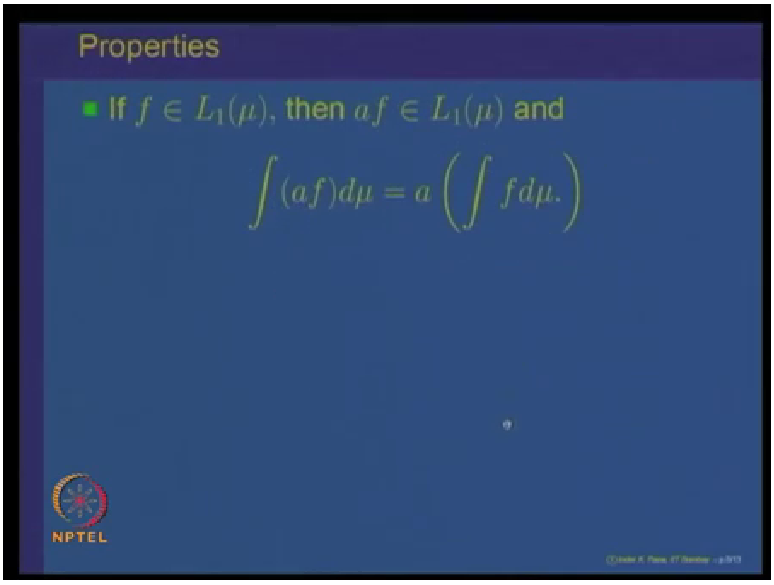


Next let us check the property of linearity. So, if f is L 1 then we want to check that alpha f is also in L 1 and alpha f of d mu is equal to alpha times integral of f d mu. So, to check that property let us observe one thing that saying that just now we looked at this kind of analysis namely if f belongs to L 1 of X it is same as if and only if mod f belongs to L 1 of X right.

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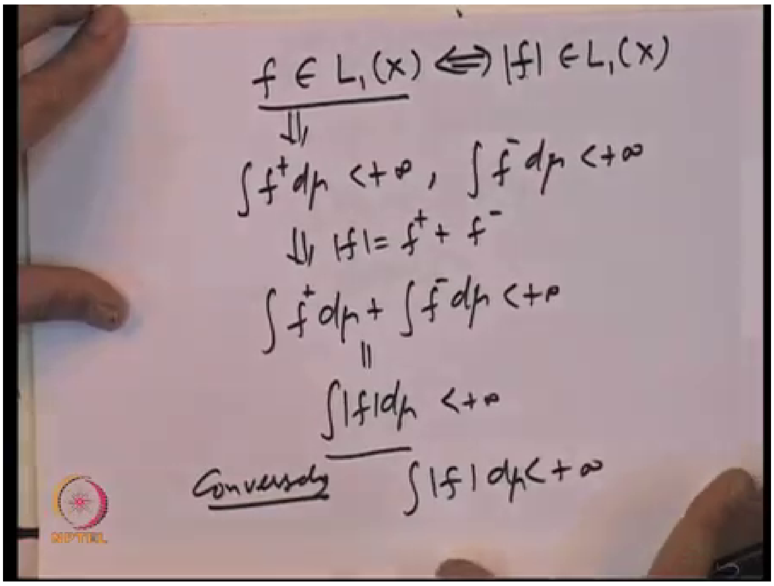


So, why is that? Once again let us do this because this we are going to use it again and again, see saying that f belongs to L 1 this implies integral of f plus d mu is finite and integral of f minus d mu is finite now. So, that implies because what is mod f? Mod f is equal to f plus, plus f minus. So, that implies integral f plus d mu plus integral f minus d mu is finite and this is equal to integral of mod f d mu. So, f belonging to L 1 implies integral of mod f is finite conversely. So, let us look at the converse part that if mod f integral is finite.

So, let us. So, conversely. So, let us this is given to us that integral of mod f d mu is finite. So, once again let us observe that f plus is less than or equal to mod f and f minus is less than or equal to mod f. So, that implies all are nonnegative measurable functions. So, implies integral of f plus d mu is less than integral mod f d mu which is finite, and integral f minus d mu is less than integral mod f d mu which is finite.

So, that implies that f belongs to L 1. So, saying that a function is integrable is equivalent to saying that mod f which is a nonnegative measurable function has got finite integral. So, this property will be used again and again and let us see how that the property is used in our proposition now.

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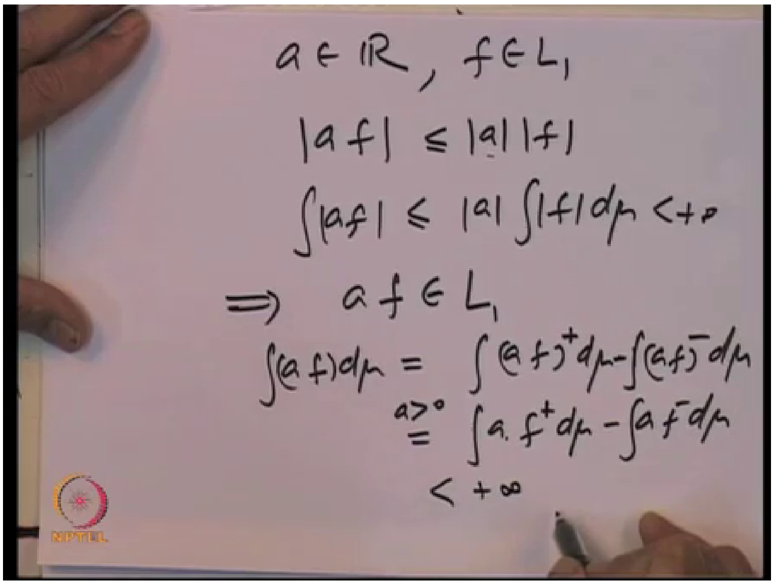

So, a belongs to real line and f is L 1. So, look at mod of a f. So, mod of a f is less than or equal to mod a, mod f all are nonnegative functions.

So, integral of mod a f is less than or equal to integral of this which is mod a times integral mod f d mu which is finite. So, that implies that af is integrable function and. So, now, we can write not only it is integrable, integral of af d mu we can write as integral of af plus d mu minus integral of af minus d mu nd now possibilities either a is equal to 0 in this case af will be 0 and everything is 0 so no problem.

If a is positive then this part is same as a times f plus d mu minus a times integral f minus d mu if a is positive and so this a comes out, because of the property for nonnegative measurable for integral of nonnegative measurable functions. So, this will be finite in case a is less than 0 this becomes a of a minus negative part and again that thing is. So, similarly for a minus. So, that proves the property that if f is integrable and a is a real number, then af is integrable and a comes out.

So, we will continue looking at the properties of integrable functions in the next lecture we will show that in this integral is a linear operation on the space of integrable functions and various other properties of this space of integrable functions and integrable of it. So, we will continue this study in the next lecture.

Thank you.