Measure & Integrate

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Lecture – 19A
Monotone Convergence Theorem & Fatou's Lemma

Welcome to lecture number 19, on Measure and Integration. In the previous lecture we had started looking at, the properties of integral for non negative measurable functions. We had looked at, the linearity property of the integral for non negative measurable functions, and then we said we will start looking at the limiting properties of a functions, which are nonnegative measurable, and integrals of them.

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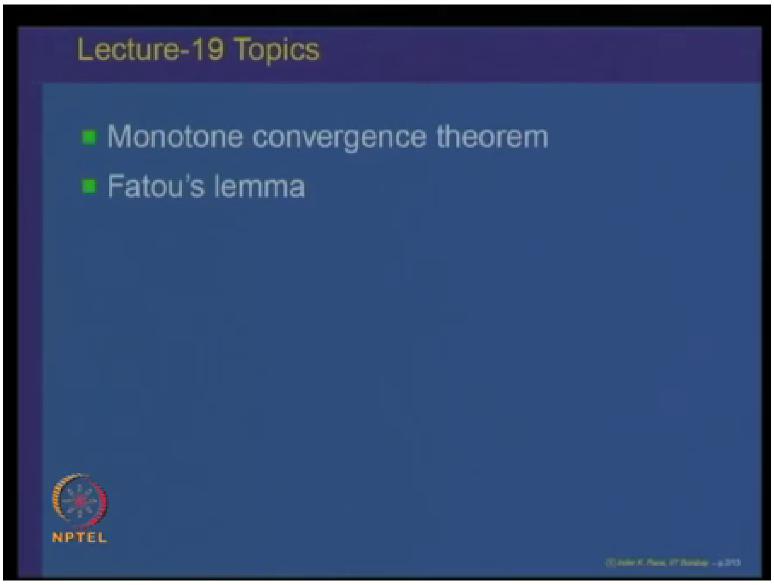

So, today we will prove some important theorems, we will start with proving what is called monotone convergence theorem, and then we will prove fatuous lemma, and then go to define integral for general functions.

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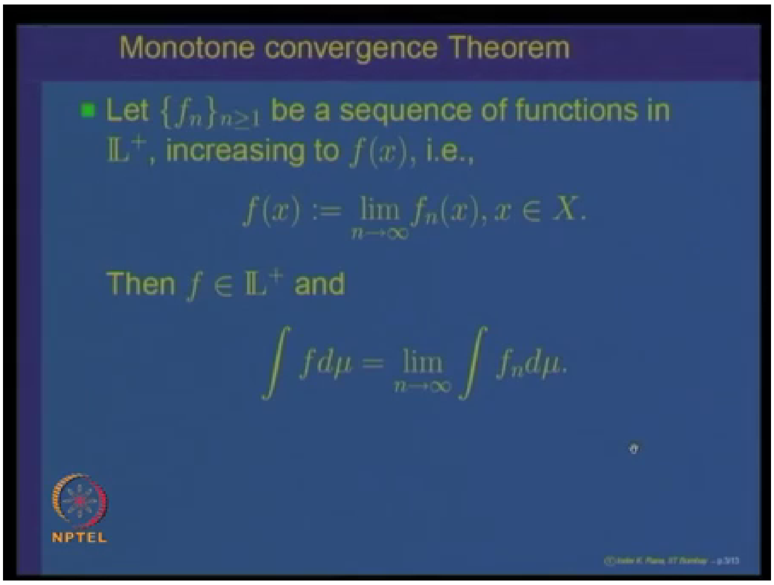

So, let us look at, what is called monotone convergence theorem, monotone convergence theorem says that let f n be a sequence of functions in class l plus; that means, f n is the sequence of nonnegative measurable functions, increasing to a function f of x at a brief point; that means, f of x for every x in x is limit n going to infinity of f n of x. So, we are given a sequence f n of nonnegative measurable functions, which is increasing and the limit is f of x, then the claim is the function f belongs to l plus this, we have already observed and the additional properties that the integral of the limit f d mu is same as limit of the integrals of f n d mu; that means, whenever a sequence f n of nonnegative of measurable functions, increases to f then integral of the limit is equal to limit of the integrals. So, this is 1 of the first important theorem, about convergence of sequences of nonnegative measurable functions and their integrals.

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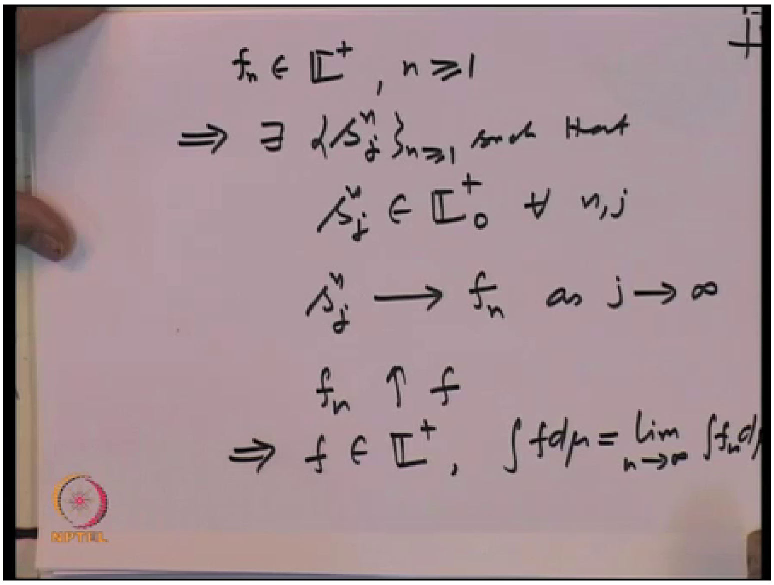

So, let us prove this property, so we are given f n is a sequence each f n belongs to l plus, is a nonnegative a measurable function for every n bigger than or equal to 1 so; that means, that implies there, exist a sequence will denoted by a s n j of functions, n bigger than or equal to 1 such that s n j are nonnegative measurable simple functions for every n and for every j and s n j increases to f of, so let us fix notion which 1 we are going to vary. So, let us say that the upper 1 will be fixed. So, this is going to f n as j goes to infinity. So, for every n fix s n j is a sequence or nonnegative simple measurable functions increasing to f n's, and f n's they increased to f. So, we want to show we have already shown, but will show it again that this implies f belongs to l plus is a nonnegative measurable function, and integral f d mu is equal to limit n going to infinity integral f n d mu.

So, to prove this we are going to use this sequence s n's and construct a new sequence of nonnegative simple measurable functions out of it. So, what will do as the following, so let us write that for n is equal to 1.

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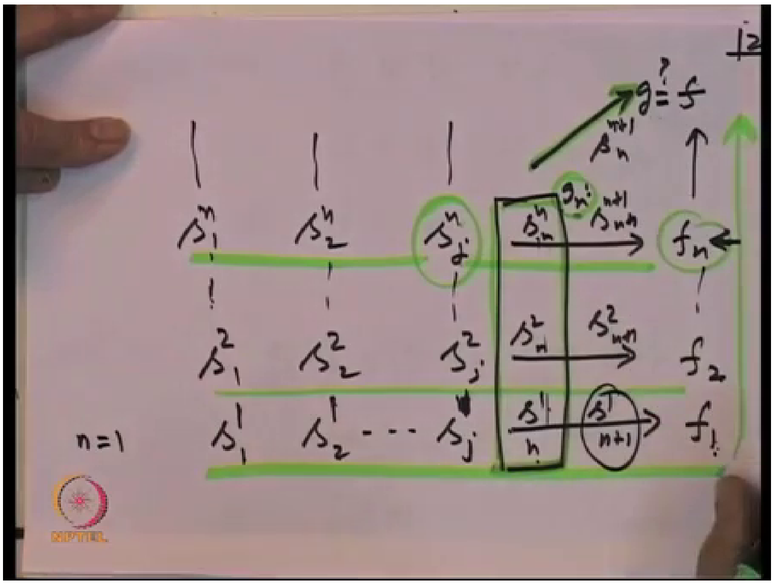

That s 1 1; s 1 2: s 1 j; s 1 say j this converges to f 1. So, the upper index is going to give you, so s 2 1, s 2 2, s 2 j, this increase is to f 2, and in geranial we will have s n 1, s n 2, s n j will increase to f n and so on, and this increases to f.

So, let us observe that as we go from left to right, as we go from left to right, this is increasing. So, everywhere left to right it is increasing and down to up that also is increasing, so every sequence if you look at them. So, this is the array of nonnegative simple measurable function, each row is increasing to the function on the right side and this is increasing upwards. So, let us out of this I am going to define. So, let us look at the function.

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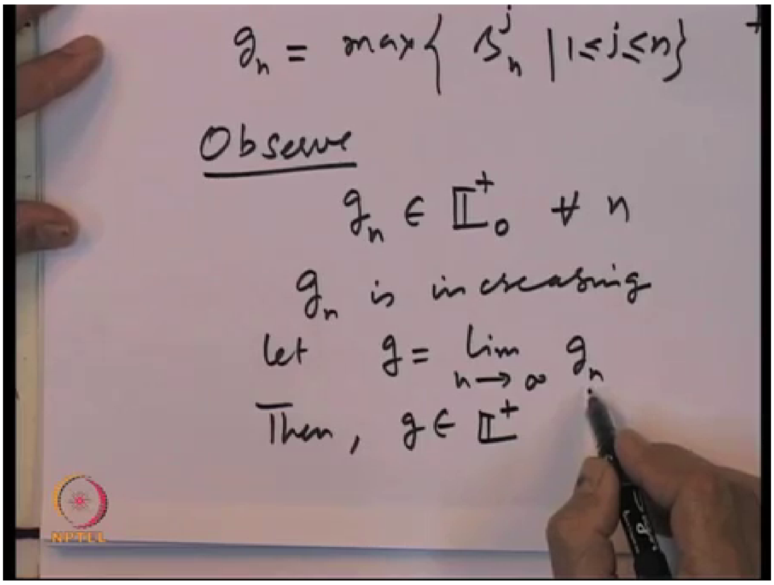

So, let me define from this a function define g n to be the function, which is maximum of s n j j between 1 and n. So, look at so in a sense what way I am doing is in this picture look at the 1 say let us say here is s 1 n and, here is s 2 n and, here is s n n. So, I look at this column. So, s 1 we are looking at the column s n 1 s n. So, let us look at this column say and call that maximum of this to be g n. What is g; so, g n is the so, let me write again, so g n is the maximum so, define g n equal to maximum of s j n j 1 2 n.

So, let us observe that each g n is a maximum of nonnegative simple measurable functions. So, each g n is a nonnegative simple measurable function for every n, and g n is increasing because at the next stage n plus 1. So, all this is going to be bigger the next stage, if you look at g n plus 1. So, that is going to be s 1 n plus 1, s 2 n plus 1, and so on s n plus 1 n plus 1 n and s n plus 1 n. So, all this 1 is going to be bigger than everything on the left hand side, and these are we are looking at the maximum. So, in the maximum of this is going to be bigger than or equal to maximum of this, because at each the right inside a function is bigger than the left hand side function.

So, this is going to give us, that is g n is increasing sequence of functions. Let us write let g b equal to limit n going to infinity of g n, so g is. So, all these g n's are increasing and they are going to increase to some function g. So, what we are going to show is g is equal to f ok. So, that is what we are going to check. So, let g b equal to, so then clearly by definition g is a nonnegative simple measurable function, because it is a limit of increasing sequence of nonnegative simple measurable functions. So, g belongs to l plus also.

Let us observe also, each g n is less than or equal to f n, for every n right. So, that is because say g n is the maximum of this. So, and the maximum of this each 1 of them is less than f 1 is less than f 2 is less than f n, so the maximum of these g n's is just going to be less than or equal to this a f n for every n and f n is increasing to f. So, that will imply that, so g n is less than or equal to f n for every f and f n is less than or equal to f.

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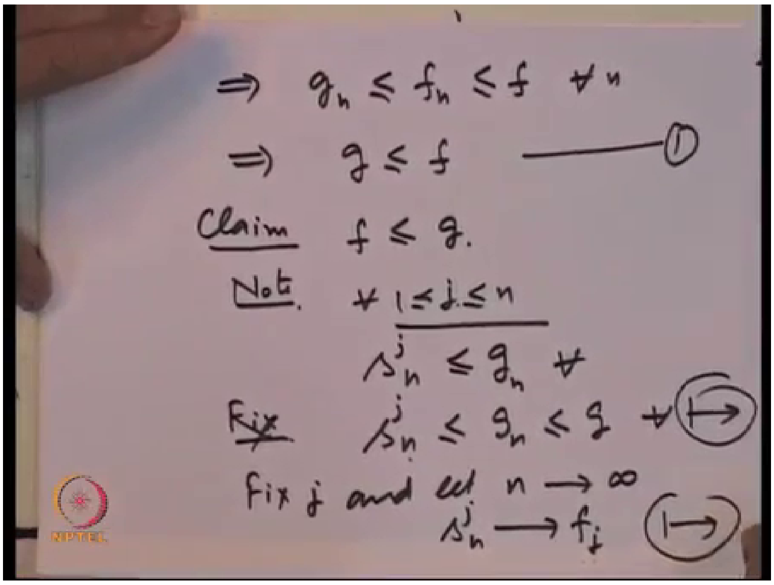

So, implies that g n is less than or equal to f n, and if less than or equal to f for every n. So, hence and g n is increase in to g. So, that implies g is less than or equal to f. So, that is 1 observation that the function g is less than or equal to f, and we claim that the other way round is also true. So, claim is that f is also less than or equal to g. So, let us note that for every j between 1 and n if I look at s j n; g n is the maximum of this. So, this is less than or equal to g n for every n. So, this is less than or equal to g n for every n and j, j between less than this. If we fix n g n is less than or equal to so, in and this is less than or equal to g so, s j n is less than or equal to g n, is less than or equal to g, for every j between 1 and n and for every n. So, let us now fix j and let n go to infinity.

So, as n goes to infinity what happens. So, this converges f n, so note that as n goes to infinity s j n goes to f j. So, from this and this, these 2 observations s n j is less than or equal to g, for every so, if we fix j and let n go to infinity then n is crossover j, and s n j and as n goes to infinity converges to f of g. So, this implies f of g is less than or equal to g for every j, so this we; so we get.

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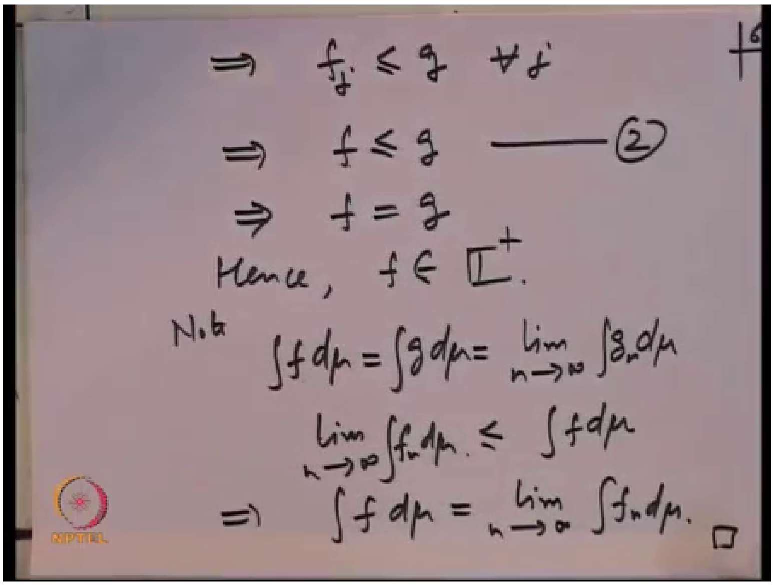


So, implies that f j is less than or equal to g for every j, and fns f j are increasing. So, this implies that f is also less than or equal to g. So, we have already shown g is less than or equal to f and now we are saying f is less than or equal to g. So, this implies that f is equal to g. So hence, 1 observations from here is hence that g belongs to l plus. So, f belongs to l plus. So, we have once again proved that, if f n are increasing to a function f and f n's are nonnegative measurable then f is also nonnegative measurable, and now note, that integral of f d mu is same as integral of g of d mu, because f is equal to g and this is equal to limit n going to infinity of integral g and d mu, because g n's are nonnegative a simple measurable increasing to g.

So, by definition this is so, but each g n is less than or equal to f. If you recall so, each g n is less than or equal to f. So, integral of g n will be less than or equal to integral of f. So, limit of integral of fns will be less than or equal to integral f.

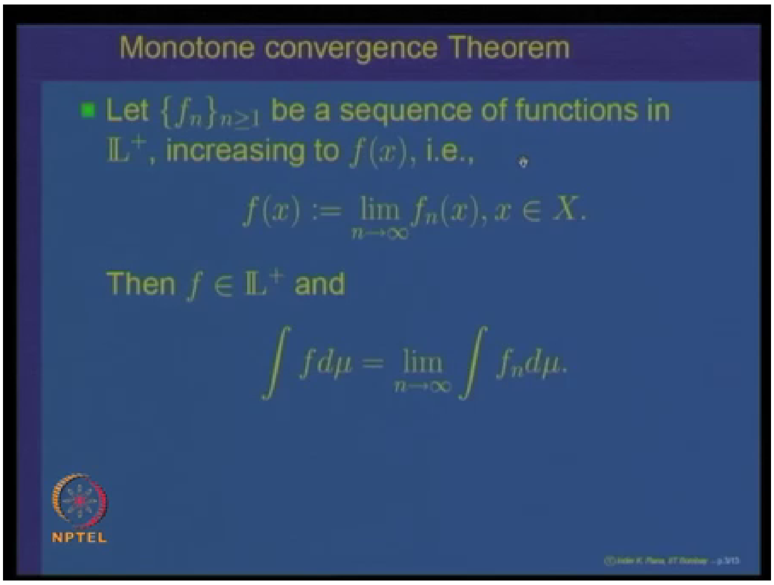
So, this is less than or equal to integral f d mu, or we can even introduce in between. So, g n is less than or equal to f n. So, it is less than or equal to limit n going to infinity integral f d mu, which is less than or equal to integral f d mu. So, what does this imply integral f d mu is less than, or equal to limit f n integral of f n d mu, and that is less than or equal to f d mu. So, that implies that integral of f d mu is equal to limit n going to infinity integral f n d mu. So, that proves the theorem completely integral of f d mu is equal to limit n going to infinity integral of f n d mu.

So, this is a construction which is quite useful in, so this is the kind of analysis 1 must carry out. So, let us go through the proof again so we understand, what we are doing each f j of f n is a measurable function. So, I can look at a sequence s 1 1, s 1 2, s 1 j, s 1 n which is going to increase to f 1, similarly toppers fix is fixed at 2. So, s 2 1, s 2 to 2, s 2 j, s 2 n that increases to f 2 and so on.

So, each row is increasing to the function on the right side and the functions f 1, f 2, f n s are increasing to the function f. So, what we do we look at the maximum of this column. So, what is this column this column is the maximum or the functions s 1 n, s 2 n and s n n. So, call this as g n. This function is called g n, so the observation is e g n is a maximum of nonnegative simple measurable function. So, it is nonnegative simple measurable each g n is less than or equal to f n, because you are going only going up to this corner only.

So, each g n is less than or equal to because s 1 n is less than f n, s 2 n is s n, f 2, f 1 is less than f 2 and so on. So, this says g n will be less than or equal to f n and each f n is less than or equal to f. So, each g n is less than or equal to f n is less than f. So, if you write the limit of g, g n to be, so write the limit of this to be equal to g then g is less than or equal to f by this simple construction.

Also for any fixed j let us look at s n j. So, let us look at s n j, where j is fixed and n is going to vary, so as n varies what happens to this functions. So, for every fixed j this sequence of functions is going to be s n j is less than or equal to g n and g n is less than or equal to f. So, we let g less than or equal to f. So, s n j is less than or equal to g n for j between for between 1 and n. So, that will give us that f is also less than or equal to j. (Refer Slide Time: 17:26)



So, that will prove the theorem that limit of increasing sequence of nonnegative measurable functions, if f n s is equal to sequence of nonnegative measurable functions increasing to f then integral of f d mu is equal to limit of n going to infinity integral f n d mu. So, this is called monotone convergence theorem, monotone because we are looking at monotonically increasing sequences, f n and convergence because we are looking at the convergence of the integrals of integral f n d mu. So, this proves monotone convergence theorem.

A let us remark we have proved the theorem monotone convergence for f n is a increasing sequence. So, naturally the question arises will the similar result hold if I have a decreasing sequence f n of nonnegative measurable functions, that results unfortunately is not true.

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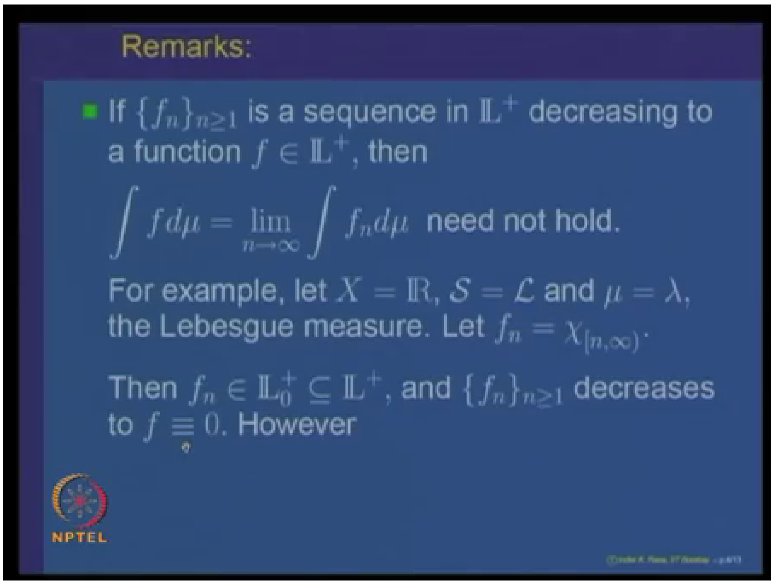

So, here is an example which says that if f n is a sequence of functions which are nonnegative measurable, and that decrease to a function f then integral of f need not be equal to integral of f n d mu, and the example is on the Lebesgue measurable space. So, look at x to be the real line, the sigma algebra to be the sigma algebra of Lebesgue measurable sets and mu to be the Lebesgue measure.

Look at the function f n, which is the indication function of the interval n to infinity. So, the claim is so, this is actually a nonnegative simple measurable function, each f n and f n is decreasing and decreasing to the identity function identically equal to 0, that is quite obvious to see so, what is f n. So, we are looking at, so here is n and we are looking at the interval n to infinity.

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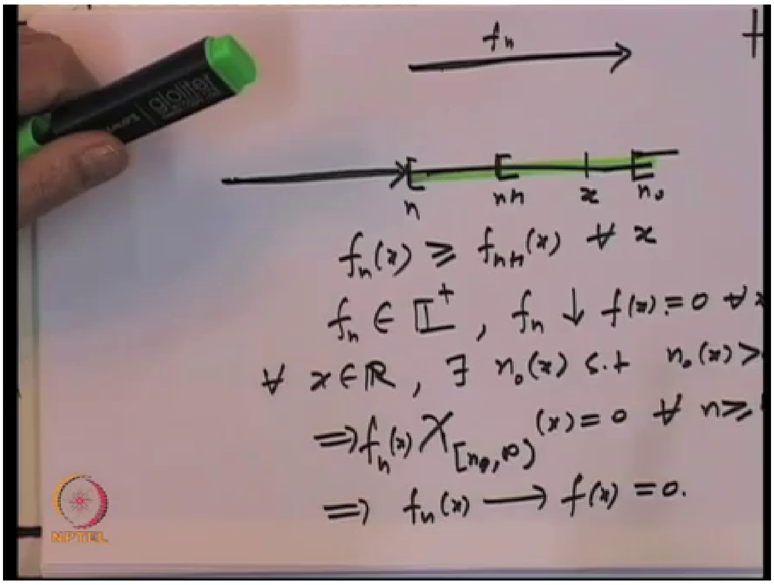

So, we are looking at this interval, and we are looking at the indicator function of n to infinity. So, the function is 0, and it is 1 here so, the function is this site is 1 so, this is the function f n it is 0 here, up to here and then it starts and goes so, that is the function f n. So, we take n plus 1 so, this is n plus 1. So, n plus 1 will be 0 here, but f n is equal to 1 here. So, clearly f n of x is bigger than or equal to f n plus 1 of x for every x.

So, f n is a sequence in l plus and f n is decreasing, and the claim is f n decrease to f of x which is identically equal to 0 for every x and that because if I take any point x on the real line then I can find some integer n say n naught which is on the right side of it then. So, for every x belonging to real line fix, I can find a point n not a positive integer n naught of course, it will depend on x says that n not of x is bigger the x. So, that will imply that the indicator function of n not to infinity, or n naught to infinity let us even n to infinity at x is going to be equal to 0, for every n bigger than or equal to n not and that is my f n of x so f n of x is equal to 0 for every n bigger than so; that means, f n of x convergence to f of x which is equal to zero.

So, f n is a sequence of nonnegative measurable functions which is decreasing to f identically zero, but we if you look at the integral of each f n, so what is the integral of each f n, so integral of f n, d lambda. So, there is integral of the indicator function n 2 infinity d lambda.

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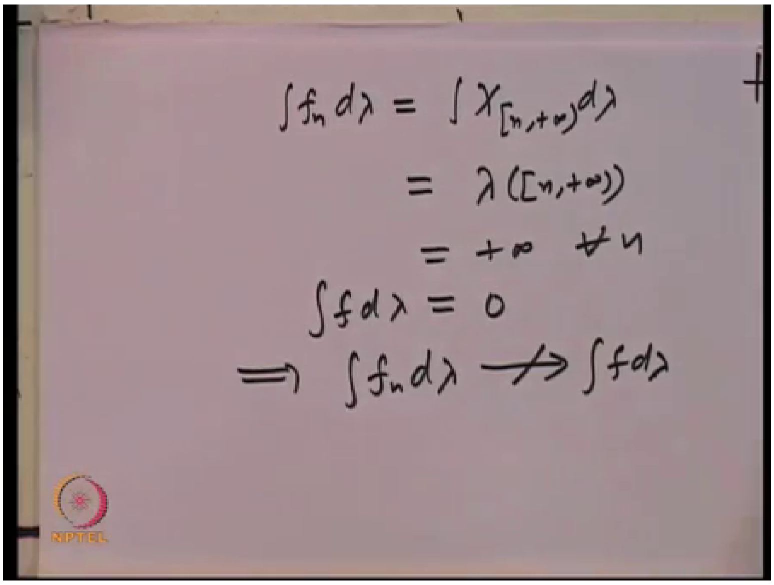

So, that is equal to lambda of n 2 plus infinity, and that is equal to plus infinity for every n. So, integral of f n is equal to plus infinity for every n and integral of f d lambda is equal to f is 0; so, it is 0. This implies that integral f n d lambda does not converge to integral f d lambda, whenever f n is a decreasing sequence of function nonnegative simple nonnegative even simple function we are given example here. So, for decreasing sequences this result does not hold. So, that gives a importance to monotone convergence there; that means, whenever a sequence f 1 of nonnegative measurable function is increasing, than integral f is equal to limit integral f n d mu for decreasing this n not hold. So, this is what we have shown just now by an example.

So; however, 1 can prove not n equality, but some kind of inequality for a sequence of nonnegative measurable functions, and that is also an important result.

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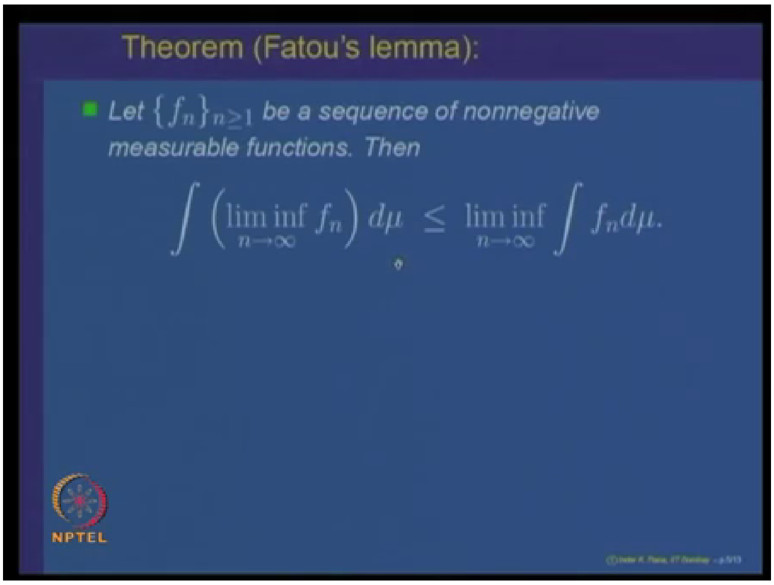

So, let us prove the result which is called fatuous lemma, it says let f n be a sequence of nonnegative measurable functions, then the integral of limit inferior of f n d mu is less than or equal to limit inferior of the integrals f n d mu. So, this is only on inequality and it need not be any quality, so what we are saying is if f n is a sequence of nonnegative measurable functions, then it is always true that the integral of the limit inferior of f n's is less than or equal to limit inferior of the integrals. So, let us give a proof of this theorem, so to prove this theorem.

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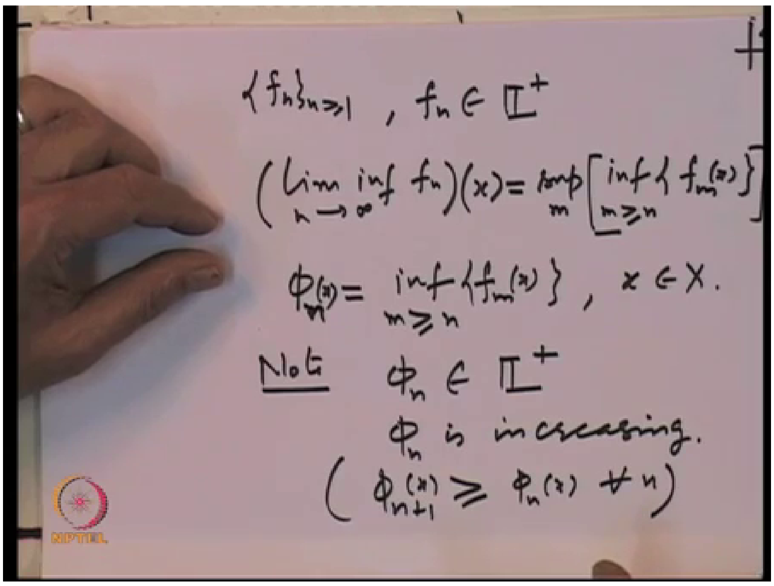

So, let us just once recall, what is so, f n is the sequence of nonnegative measurable functions. So, each f n is a nonnegative measurable function, and we want to look at limit inferior of f n as n goes to infinity this is a function so, let us observe how this function is defined limit inferior of f n at a point x is defined as, you take the infimum from some stage on words. So, m bigger than or equal to n of f n of x. So, look at the numbers f n of f x f m of x for m bigger than or equal to n so, I am looking at the tale of the sequence f n of x from a m onwards so, this number infimum will depend on m. So, let me take the supremum of this overall m so, first take the infimum from some stage on words and then take the supremum of these infimums.

So, let us observe that this infimum let us put a bracket here so, observe so, let me call it as phi m to be the infimum from the stage n on words, so, infimum of m bigger than or equal to n of f n of x phi m of x to be defined as the infimum from the stage n onwards of fm of x. So, then because it is a infimum of a sequence of functions which are nonnegative measurable, so clearly, so note so, observation is that each phi n is also a nonnegative measurable function.

So, it is a nonnegative measurable function that is 1, and secondly we are taking the infimum from some stage n onwards. So, if you increase so, the claim is this phi n is increasing this is a increasing sequence because phi. So, phi n from the infimum from the stage n onwards is going to be less than or equal to the infimum from the stage n plus 1 onwards because we will have more numbers for which you have taking infimum; so, infimum can the infimum when you take infimum or more numbers then infimum when decrease. So, infimum from the stage n on wards and the infimum from the stage n plus 1 n plus 1 onwards, so that says that the infimum from the stage n plus 1 onwards, will be bigger than or equal to the infimum so increasing that is phi n plus 1 is bigger than or equal to phi n of x, for every n. So, it is a increasing sequence of nonnegative measurable functions and its limit is nothing, but the limit inferior.

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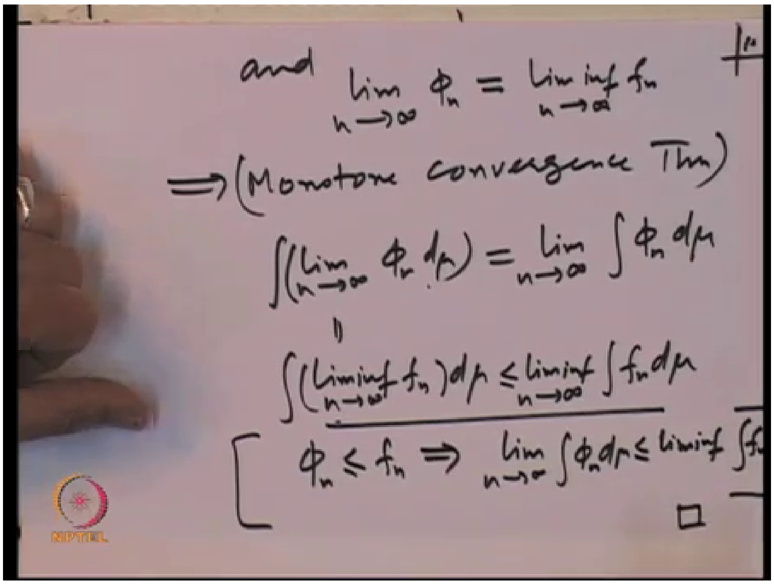

So, it is increasing and limit n going to infinity of phi n is equal to limit inferior of f n n going to infinity, so just stage is set perfect for an application of a monotone convergence theorem, phi n is a sequence of nonnegative measurable functions phi n s are increasing.

So, by monotone convergence theorem, so, we can apply implies by monotone convergence theorem, by monotone convergence theorem that integral of limit n going to infinity of phi n d mu, is equal to limit integral phi n d mu n going to infinity. So, this is nothing, but so, this side is nothing left hand side is nothing, but integral of limit inferior n going to infinity of f n, d mu. So, that is equal to limit of integral phi n's of integral phi n's. Now let us look at what is phi n; phi n is the infimum from the stage n onwards, so each phi n is less than or equal to f n. So, that is the observation from here by the definition of phi n we have that each phi n is less than or equal to f n so, integral of phi n will be less than or equal to integral of f n. So, it will be less than or equal to limit inferior of n going to infinity.

So, what we are observing here is because each phi n is less than or equal to f n. So, this implies. So, this is what we are using here that is phi n is less than or equal to f n, then the n the limit n integrals of phi n s are increasing so it limit exist, so the limit n going to infinity integral phi n d mu is less than or equal to; however, integral of phi n s f n's may not exists. So, we can say it will be less than or equal to limit inferior of integral f n's d mu. So, this is what is being used in this conclusion, and that proves the theorem what is called the fatuous lemma.