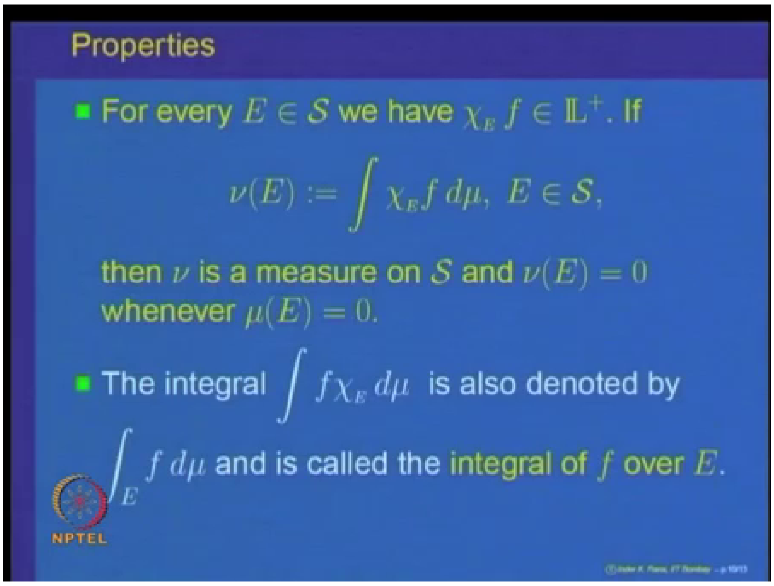
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 18 B
Properties of Nonnegative Simple Measurable Functions

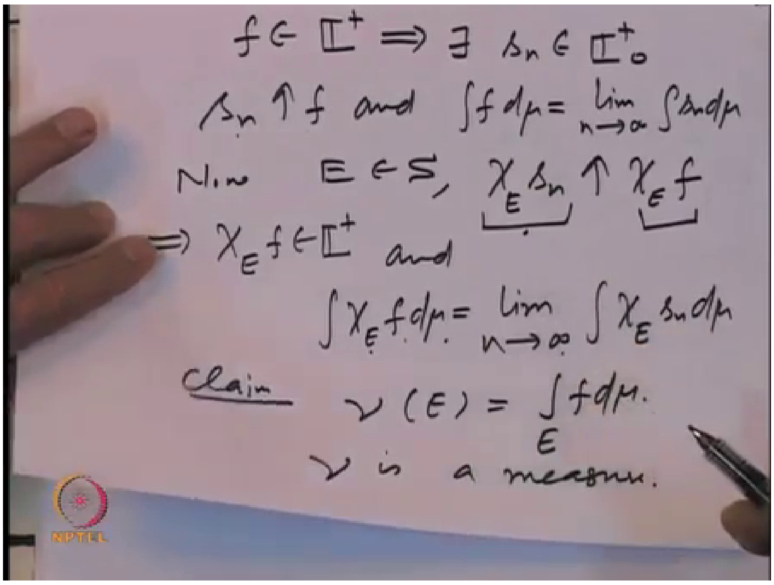
(Refer Slide Time: 00:17)



Next we prove an important property, and an extension of the earlier version for non negative simple functions; namely for every a measurable set E. If you look at the function, the indicator function of E times f, then that is also a nonnegative measurable function, and if its integral is denoted as nu of e. So, nu of E is the integral chi E fd mu, where E belongs to S, then this is a measure, this nu is a measure on the sigma algebra S, and has the property that nu of a set E is 0, whenever mu of the set E is equal to 0.

So, let us prove this property also, and this property again we are going to use the fact that the integral of a nonnegative simple measurable of a measurable function is given by, as a limit of the integrals of a increasing sequence of nonnegative simple measurable functions.

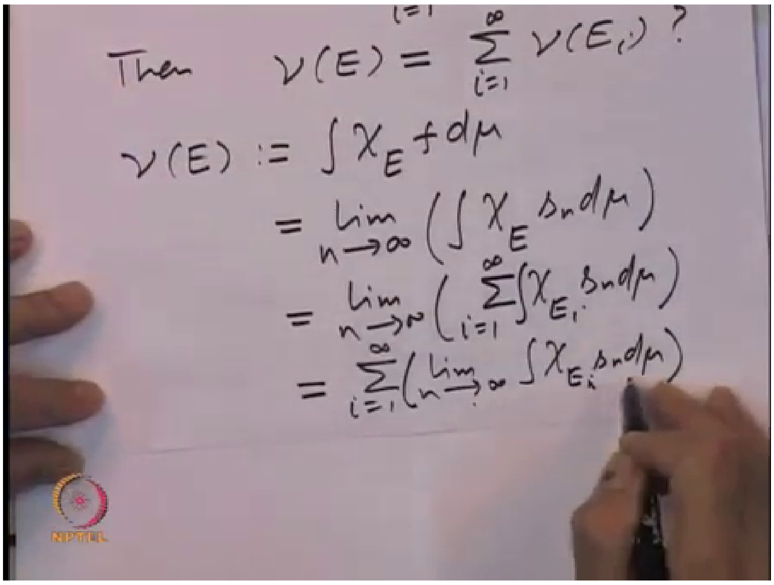
(Refer Slide Time: 01:21)



So, f belonging to l plus implies, we have a sequence Sn belonging to l plus 0; such that Sn increases to f, and integral f d mu is written as limit n, going to infinity of integral S nd mu. So, that is by the fact that f belongs to l plus, and integral of f is defined as limit of integral Sn d mu for any sequences n which increases to f.

Now, for E a set in the sigma algebra S, because sn is increasing to f. So, clearly indicator function of E times S n will increase to indicator function of E times f, and observe we have done it earlier also, that this is a nonnegative simple measurable function, it is increasing to this function. So, that implies that indicator function of E times f. So, this implies that the indicator function of E times f is a nonnegative measurable function, and because we have got this sequence increasing to this non negative measurable function. So, integral of the indicator function of E times fd mu is nothing, but limit n going to infinity of integral indicator function of E times S nd mu times integral of Sn d mu. So, that is, this is how the integral is defined. And now we want to claim that if you call this number nu of E as integral over E fd mu, then nu is a, the claim is nu is a measure.

(Refer Slide Time: 03:31)



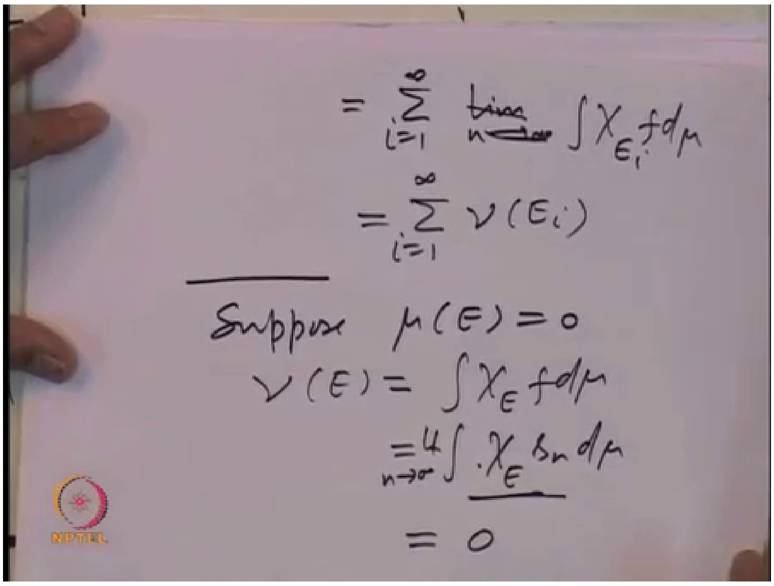
So, to prove this let us. So, what we have to prove is the following. So, to prove this is what you have to show that, if a set E is a countable disjoint union of sets Ei E i's in the sigma algebra S, then we want to show that nu of E is equal to sigma nu of Ei i equal to 1 to infinity. So, this is what we have to show. So, let us start looking at nu of E by definition nu of E. Just now we saw that nu of E is nothing, but integral of the indicator function of E times fd mu right. So, integral of E f d mu is nothing, but limit of integral indicator function of E S nd mu, by the fact that Sn is increasing to f. So, just now observed that.

So, this can be written as limit n going to infinity of integral indicator function of E S n d mu. So, is just from the fact that Sn is increasing to f. So, indicator function of E times S n will increase to indicate the function E times f. hence integral of indicator function of E times f is nothing, but the limit of the integrals of the indicator function E S n d mu, and now this E is a disjoint union of sets E i. So, that implies. So, let us write this limit n going to infinity of. So, this i can write as summation chi E i of Snd mu i equal to 1 to infinity. So, here we are using the fact that E is a disjoint union of sets E i and for a nonnegative measurable function Sn, if you integrated over a set E, then that is a measure.

So, that is the property. So, the corresponding property for non negative simple measurable functions which we had already proved is true. So, we are using that fact to bring it here, and now this is limit of a series i equal to 1 to infinity of. Sorry this is a integral here, integral of chi e. So, E is union. So, this is integral of the union. So, now, I am going to interchange this summation, and all limit and summation Y equal to 1 to infinity, and that is allowed, because all the quantity is involved are non negative.

So, this interchange is possible. So, I can write it as summation i equal to 1 to infinity, limit n going to infinity of integral chi E i S n d mu. And now we simply observe that this last quantity is nothing, but summation i equal to 1 to infinity, limit n going to infinity, and the last quantity is nothing, but a nu limit of n going to infinity. So, that is that limit is nothing, but integral of chi E i f d mu, because Sn is increasing to f. So, chi E i times Sn increases to chi E i times f. So, this, this limit. So, this limit of n going to infinity integral of chi E i Sn d mu is nothing, but integral of chi E i times f. So, that value you put. So, and that is nothing.

(Refer Slide Time: 07:43)



But our definition of nu of E i so that proves that nu is a measure on E i. And once again observe that here we have used basically what we have done is. We have used the fact f is a limit of increasing sequence of nonnegative simple measurable functions.

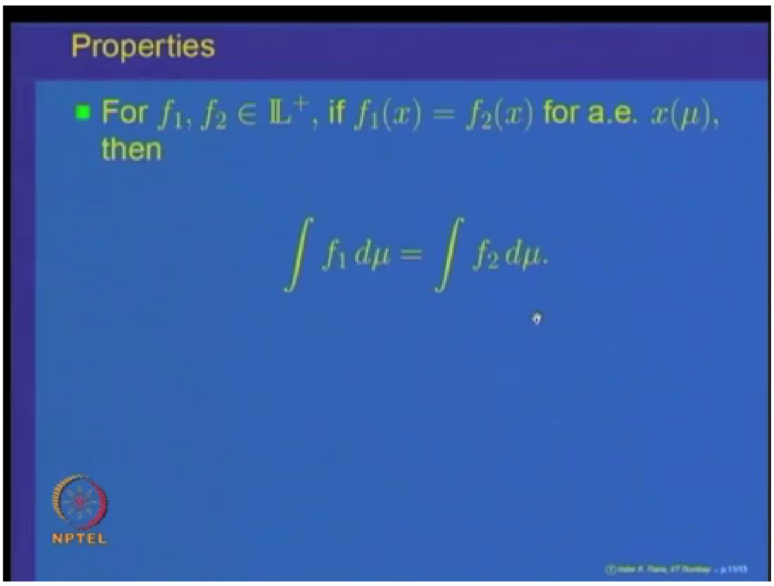
So, integrals of nonnegative simple measurable functions that sequence gives you integral of f and then; so, go to that sequence, use the property for non negative simple measurable functions, that property is true. So, and come back, and finally, to prove that nu of E equal to 0 implies mu of E equal to 0. So, that is. So, suppose mu of E equal to 0, then what is nu of e; so, nu of E which was defined as integral chi E of f d mu which was nothing, but integral of chi E times S n d mu limit of that. So, let us write limit n going to infinity of this, but this mu of E being 0, this. So, integral is 0. So, this is equal to 0.

So, once again for a nonnegative simple measurable function, its integral over E is 0, if mu of E is 0. So, that property is being used once again here. So, this proves the fact that, this measure nu of E which is constructed as integral of chi of E f d mu is a special measure, which has the property, that it is null sets nu of E is 0, whenever mu of E is 0.

So, till now what we have done is. We had define the integral of a nonnegative simple measurable function as a limit of integral of a nonnegative simple measurable functions, because if f is nonnegative measurable, it is a limit of nonnegative simple measurable functions, which increase to this function f. So, integrals of those non negative simple measurable functions is defined, take their limit and define integral of f to be limit of the integrals of nonnegative simple measurable functions, and using this, we have proved that this integration is linear.

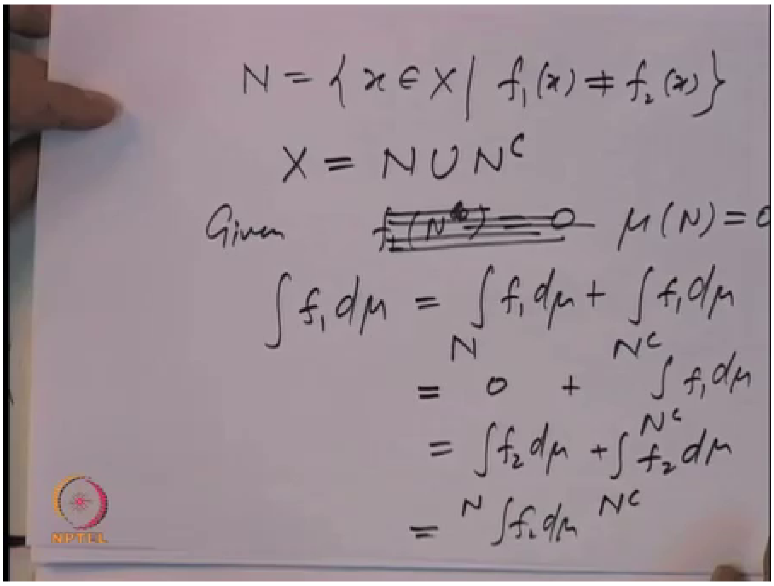
So, the next property we want to analyze is, this is this class. How does this class of non negative measurable functions and the operation of integral behave for sequences in the class helpless. So, here is the first important theorem that we are going to prove and that is ok.

(Refer Slide Time: 10:44)



Before that let us just prove just a simple observation that if f 1 and f 2 are non negative simple, and f 1 is equal to f 2 almost everywhere, then integral of f 1 is equal to integral of f 2. So, that property is quite obvious.

(Refer Slide Time: 11:05)

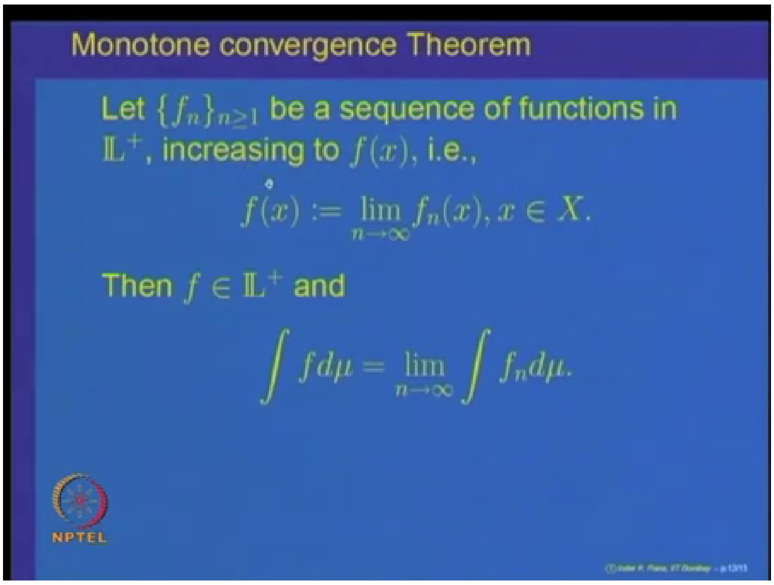


Because let us write the set n to be the set all X belonging to X, where f 1 X is not equal to f 2 of X, then the whole space can be written as n union n complement and we are given. So, we are given f 1 is equal to f 2. So, f 1 of n compliment, we are given f 1 is equal to f 2 almost everywhere.

So, where they are not equal; so, this set has got f 1. Sorry we are given that, sorry we are given that this set n f 1 is equal to f 2 almost everywhere. So, where they are not equal, that is a set of measure 0. So, mu of n is equal to 0. So, now, integral of f 1 d mu can be written as integral over n f 1 d mu plus integral over n complement of f 1 d mu, this is by the fact. Just now we have proved that integral over a set is a measure. So, this is integral over n, and integral over n complement that gives me integral over the whole space, and mu of n being equal to 0. So, this is, this first term is 0 plus integral over n complement f 1 d mu, but this is also same as integral over n of f 2 d mu, because measure of n is 0 and on n complement f 1 is equal to f 2.

So, I can write as n compliment f 2 d mu, and that once again is equal to integral f 2 d mu. So, integral f 1 d mu is equal to integral of f 2 d mu. So, that essentially says that the integral of a function does not change, if its change, its values on a set of measure 0 .

(Refer Slide Time: 13:04)



So, let us now come back to the property that I was trying to state earlier; namely if we have a sequence f n of nonnegative simple measurable functions, and f n's increased to f; that is f of X is equal to limit of f 1 X, then the claim is f belongs to l plus and integral f d mu is equal to limit n going to infinity integral of f n d mu. So, this is one of the important theorems in our subject, it is called monotone convergence theorem.

Monotone, because we are looking at sequence f n which is a increasing sequence is a sequence of functions, which is increasing. So, it is a monotonically increasing sequence, increasing sequence of a nonnegative measurable functions, increasing to a function f. We have already seen that f will be a measurable function, it is nonnegative, but the important thing is integral of f. So, f is the limit integral of the limit is equal to limit of the integrals. So, that is the important property we want to prove for integral for non negative symbol, non negative measurable function.

The proof of this requires a, some construction, and we do not have time today to complete the proof. So, we will do the proof of this theorem next time. So, we stop here today, by having stated the monotone convergence theorem, and look at the proof of this in the next lecture.

Thank you.