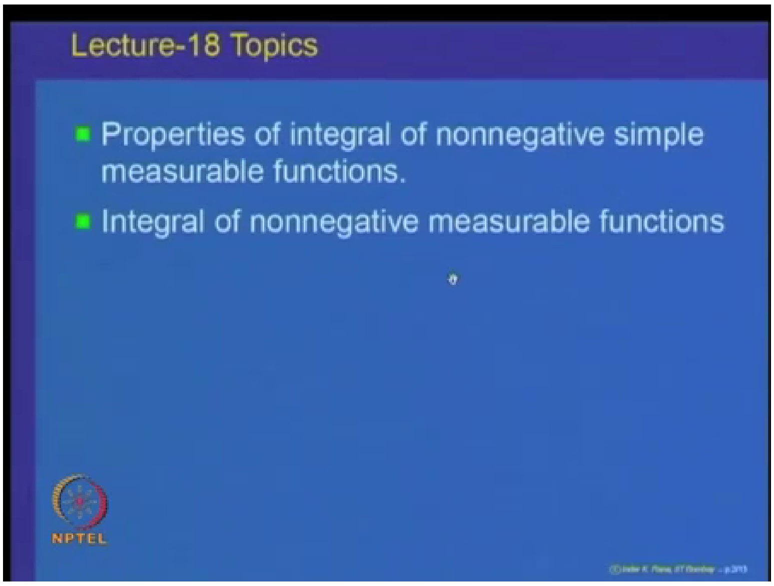
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 18A
Properties of Nonnegative Simple Measurable Functions

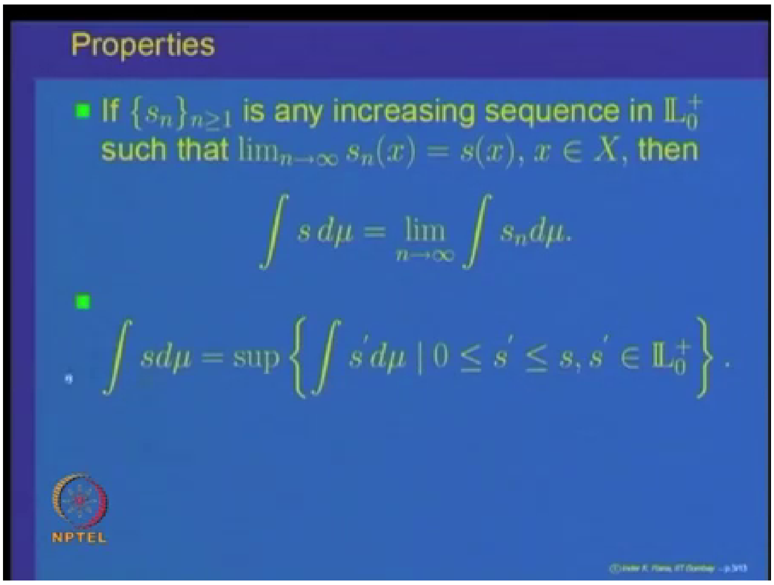
Welcome to lecture number 18 on measure and integration. If you recall in the previous lecture we were defined the notion of integral for nonnegative simple measurable functions and we are started looking at the properties of this integral. So, we will continue this study of the properties of the integral for nonnegative simple measurable functions, and then later on we will extend it to this integral to nonnegative measurable functions.

(Refer Slide Time: 00:51)



So, topics for today would be properties of integral for non negative simple measurable functions, continue the study of that and then define integral for non negative simple func measurable functions. If you recall me at the last property that we have proved in the previous lecture was that if sn is any increasing sequence of nonnegative measurable functions.

(Refer Slide Time: 01:05)

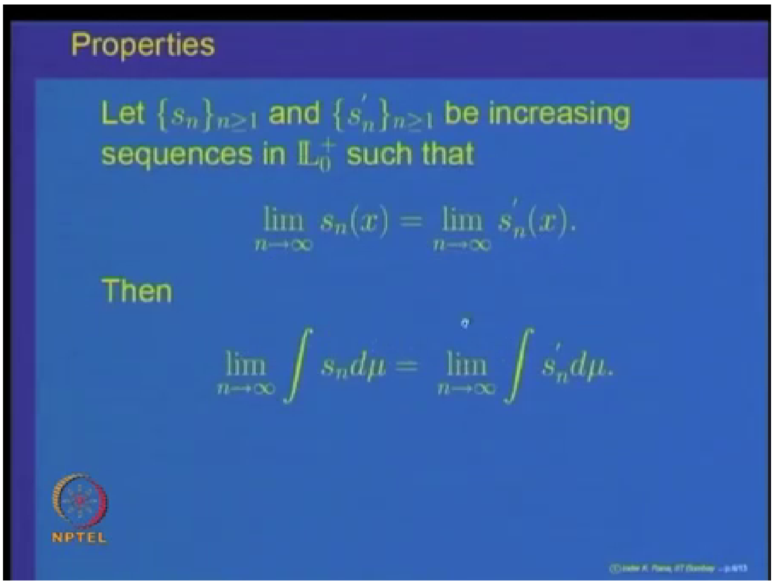


Such that they converge to a simple nonnegative simple function s of x then integral of s is equal to limit of integral of snd mu so; that means, under increasing limits if the limit is again a nonnegative simple measurable function then you can inter change the order of integration and the notion of limit. So, integral of sd mu which is s is the limit of a snd mu sn's is same as limit of n going to infinity of integral sn's d mu. So, integrals of sn's converged to integral of s.

Let us observe one more simple property of this integral for any nonnegative simple measurable function s the integral sd mu can also be represented as the supremum of the integrals of s prime d mu where s primes are nonnegative simple measurable functions less than x less than s this property is obvious because s is less than or equal to s. So, the supremum has to be at least integral sd mu, and it cannot be more because s prime less than s implies that integral of s prime is less than or equal to integral s. So, the supremum they cannot be bigger than or equal to sd mu also. So, this is obvious property, but we will see later on extension of this property later on.

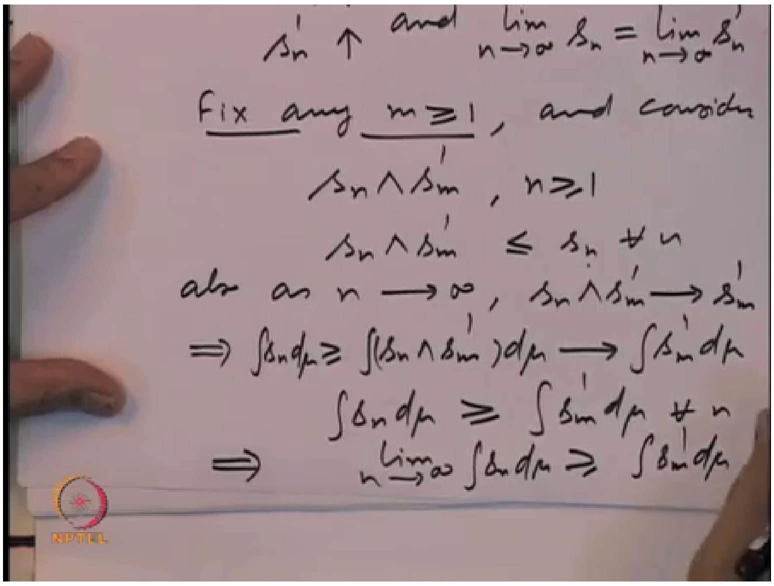
The next let us observe another important property about the integral of nonnegative simple measurable functions and that is the following.

(Refer Slide Time: 02:54)



Suppose S n and S n dash are 2 increasing sequences of nonnegative simple measurable functions, both converging to the same limit. So, limit of snx is same as limit of sn dash x. Then the claim is that the limit of integral snd mu has to be equal to the limit integral sn prime d mu; that means, if two sequences of nonnegative simple measurable functions have the same limit then there integrals also converged to the same values.

(Refer Slide Time: 03:38)

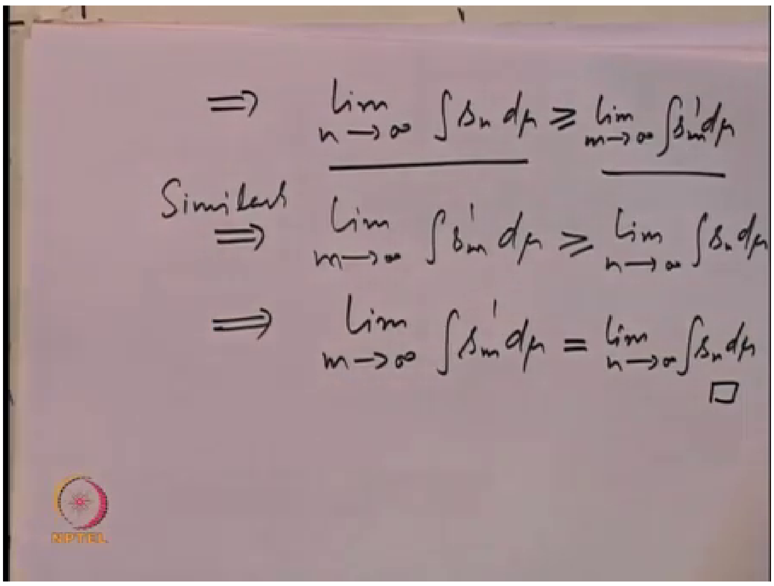


So, let us prove this result from we have got a sn is a sequence of nonnegative simple measurable functions, sn's are increasing and also sn dash is another sequence which is also increasing, and the limit n going to infinity of sn is equal to limit n going to infinity of sn dash. So, that is given to us.

So, now let us fix any integer a positive integer m bigger than or equal to 1 and consider the sequence sn minimum sm dash. Look at this sequence n bigger than or equal to one. So, look at this functions. So, this is these are the functions which have the property that sn which sm dash is always this is the minimum. So, it is less than or equal to sn for every n, also as n goes to infinity look at the sequence sn which ah s m dash. So, the minimum of sn and sm dash as n goes to infinity sn is going to increase to a limit. So, it will take over sm dash at some stage so that means, this is going to converge to sm dash. So, this is obvious because sn and sm dash both the sequences are the same limit. So, at some stage sn will cross over sm dash for every m fixed. So, a fixed and integer m.

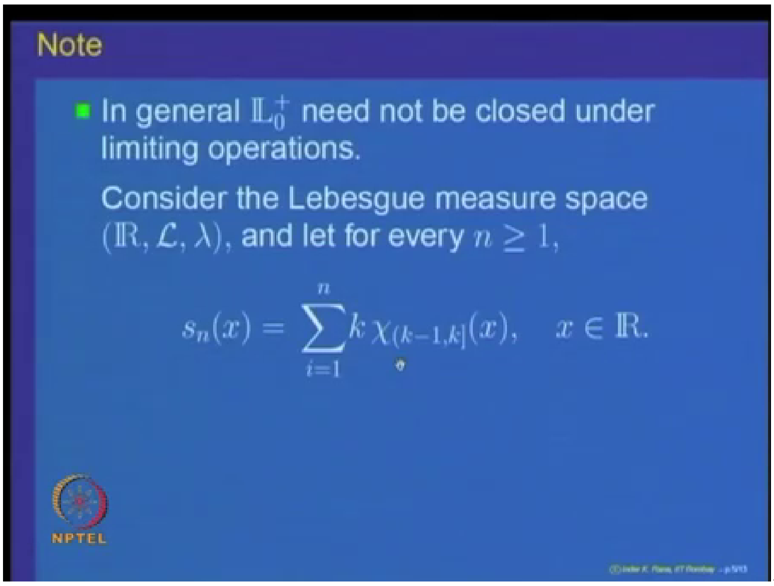
So, that implies that integral of sn wedge sm dash d mu will converge to integral of sm dash d mu. So, once and this is bigger than or equal to. So, this is less than or equal to integral of snd mu and that is because of this. So, we have got integral snd mu is bigger than or equal to integral sm dash d mu for every m so that means. So, this implies for all n large enough so that implies that limit of n going to infinity of integral snd mu is bigger than or equal to integral sm dash d mu. For every m fix and hence because this is true for every m fix; so, this implies that this is true for every m fix.

(Refer Slide Time: 06:35)



So, that implies that limit n going to infinity integral snd mu is bigger than or equal to limit m going to infinity of integral sm dash d mu. So, we have strong that limit of sn integral sn s is bigger than or equal to limit of integral sm's, and because now you can interchange that to so that implies similarly limit n going to infinity integral sn dash d mu is bigger than or equal to limit n going to infinity of integral snd mu. So, that proves that the two are equal. So, this will prove that limit m going to infinity of integral sm dash d mu is equal to limit n going to infinity of integral snd mu. So, that proves the result namely if sn and sn dash are to increasing sequences of nonnegative simple functions, having the same limit then their integrals also converged to the same limit this will be used soon.

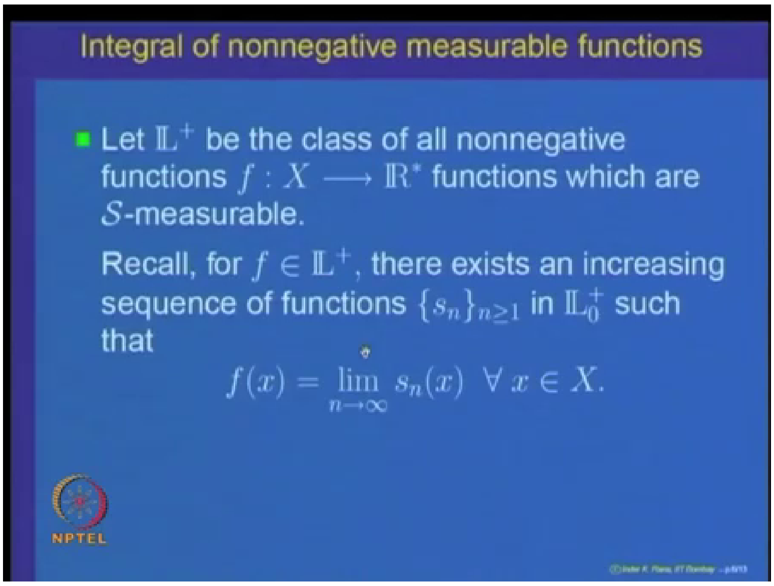
(Refer Slide Time: 08:05)



Let us look at and observation that in general the class of nonnegative simple measurable functions need not be closed under limiting operations. Way we showed that some of nonnegative simple measurable functions is a nonnegative simple measurable function, and we also just now observed that if a sequence sn is increasing to a sequence in L plus 0 that means, if a sequence of nonnegative simple measurable functions converges to a nonnegative simple measurable function then the integrals converge. In general for decreasing sequences in for example, this need not hold or for even the limit of nonnegative simple measurable functions may not be a nonnegative simple measurable function. So, to give an example of that let us consider the Lebesgue measurable space measure space R L and lambda R is the real line, L is the space of the sigma algebra of Lebesgue measurable sets and lambda is the Lebesgue measure.

So, let us define sn of x for every n o be equal to the indicator some of the indicator functions of k minus one to k where k goes from 1 to n. So, this is just a text the constant value k on the interval k minus 1 2 k. So, as is quite clear that as a this is the nonnegative simple measurable function on the real line, and as an increase is this is going to be increasing sequence that also is clear and the at let increases to a function which is equal to k on every interval k minus 1 to k so, that the limit function is not going to be a nonnegative simple measurable function of course, it will be a nonnegative measurable function. So, the aim what we are saying is that the class L plus 0 it is not closed under limiting operations. So, that says that we should go over to bigger class of functions namely the class of nonnegative measurable functions and define integral there also.

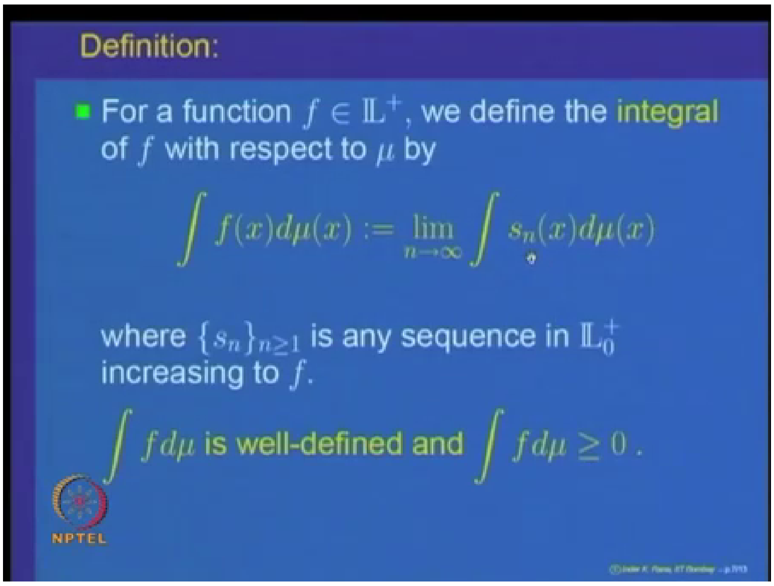
(Refer Slide Time: 10:31).



So, let us define for let will denote by L plus the class of all nonnegative functions X or R star which are s measurable keep in mind we have got a fix measures space which is complete that is x s and mu.

So, look at all non-negative S measurable functions on the set X and let us denote that that this class of functions by L plus, and if you recall we had proved a theorem that for a nonnegative measurable function there is a sequence of nonnegative simple measurable functions which is increasing to f. So, for every f belonging to which is a nonnegative measurable function. So, the function in the class L plus, we know that there exists a sequence sn of nonnegative simple measurable functions, sn which increases to f. So, f is a limit of increasing sequence of nonnegative simple measurable functions, and we know we are now just we have defined the concept of integral for nonnegative simple measurable functions sn. So, it is natural to defined integral of f, to be nothing, but the limit of integrals of sn's

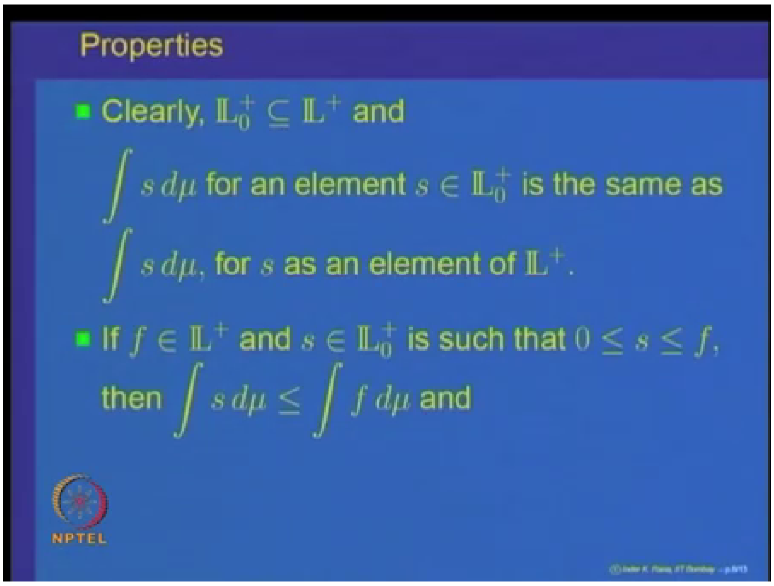
(Refer Slide Time: 11:57)



So, we define. So, for a function f in L plus, we define its integral with respect to mu. So, denoted by integral f x d mu x or simply by fd mu to be the limit n going to infinity of sn x, d mu x s where s n is any sequence in L 0 plus increasing to f and the first obvious claim is that this integral is well defined, it does not depend upon the choice of the sequence sn that we take which increases to the function f, that is because just now we have proved the result that if there are two different sequences sn and s n prime, nonnegative simple measurable both increasing to f then their limits are same.

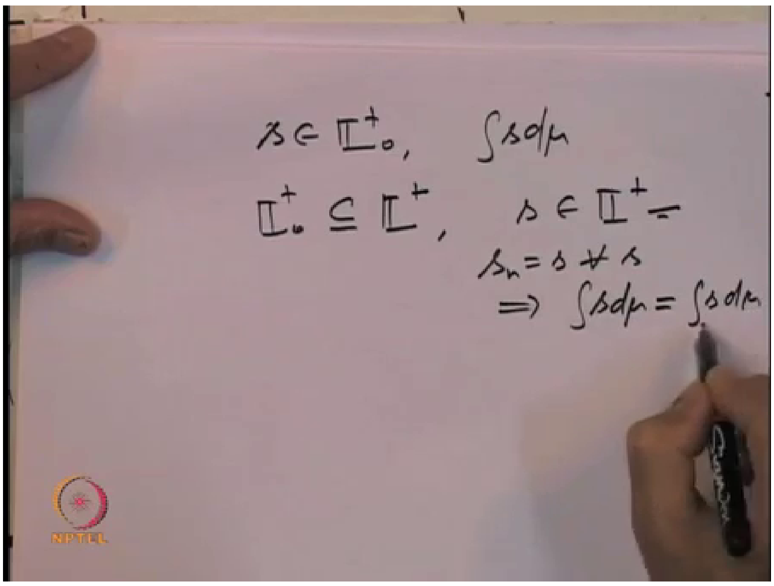
So, whichever sequence we take sn which increases to f, its limit is going to be the same extended real number and that extended real number is called the integral of fd mu. So, integral fd mu is well defined. So, well defined so; that means, whatever sequence sn increasing to f we choose it does not matter, that limit of that a sequence is same limit of integrals of sn's is same. So, that is called integral of fd mu; and because it is limit of integrals of sn's which are nonnegative simple measurable function. So, each integral sn is a nonnegative simple measurable function. So, as a result the limit also is non-negative. So, integral of a nonnegative measurable function f, by this process is a well-defined number and it is bigger than or equal to 0.

(Refer Slide Time: 13:47)



So, now let us look at the properties of this that the class L0 plus is a subset of L plus. So, that is obvious and we want to claim that integral of sd mu as an element of s is same as an element of L plus. So, that also is obvious because of the following fact.

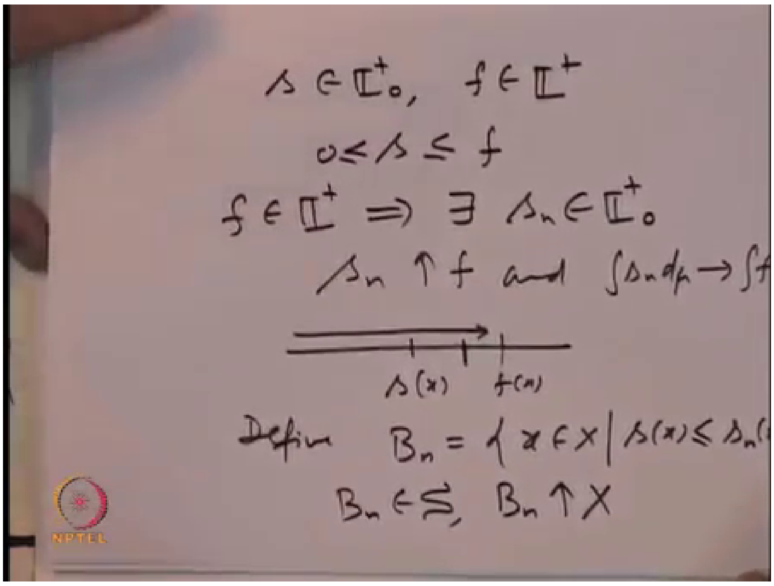
(Refer Slide Time: 14:17)



So, we have got. So, let us take s belonging to L plus 0, and then we have its integral sd mu as a integral of a nonnegative simple function and L plus 0 we are now treating it as a subset of L plus. So, when, if you treat s as an element in L plus, then we can take the constant sequence sn is equal to s for every s and that will imply that integral of sd mu as an element in L plus is same as limit of integral sns which is same as integral sd mu as an element in L plus 0. So, as if you take a nonnegative simple measurable function and as a element of L plus, and look at the integrals as a element of L plus then that integral is same as an element of the non-negative simple measurement; that means, that the new integral that we are defined is in fact, an extension of the motion of integral from non-negative simple measurable functions to nonnegative measurable functions.

Now, next lest us look at the property that if f is a function in L plus then s is a function in l zero plus such that 0is less than or equal to s less than or equal to f, then integral of sd mu is integral less than or equal to integral fd mu. So, let us prove this property.

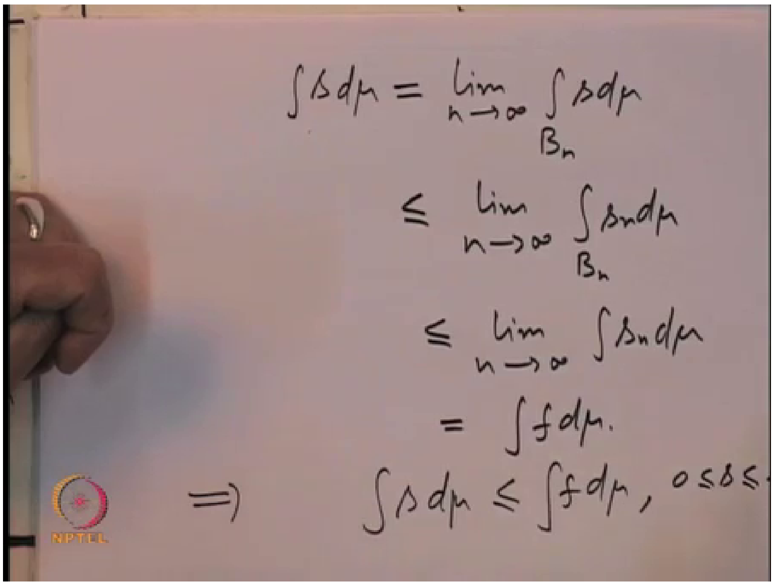
(Refer Slide Time: 16:03)



So, we have got s a nonnegative simple measurable function, and f is a nonnegative measurable function and we are given that s is less than or equal to f. Now since f belongs to L plus implies there is a sequence sn of nonnegative simple measurable functions such that sn increases to f right. So, now, let us look at sn increase to f and the integral of snd mu converges to integral fd mu; now next let us look at. So, consider. So, observe here is s and s of x for any point and here will be some f of x, and sn is going to increase to f. So, sn x is going to cross over s of x for some n. So, let us define Bn to be the set of all those points x belonging to x such that s of x is less than or equal to sn of x.

So, observations that this said Bn is in the sigma algebra s, and because sn is increasing this sequence Bn of sets is also increasing to the whole space x; because sn is converging to f of x. So, b n is going to increase to s of X. So, these are obvious properties because if sn is bigger than or equal to a s of x, then sn plus one is also bigger than; that means, Bn is inside Bn plus x and as we have observed that for every x there will be some n s such that sn of x will cross over s of x. So, every x belonging to x belong to some Bn. So, Bn is going to increase to x. So, now, we observe the property that look at integral of the non-negative simple measurable function sd mu.

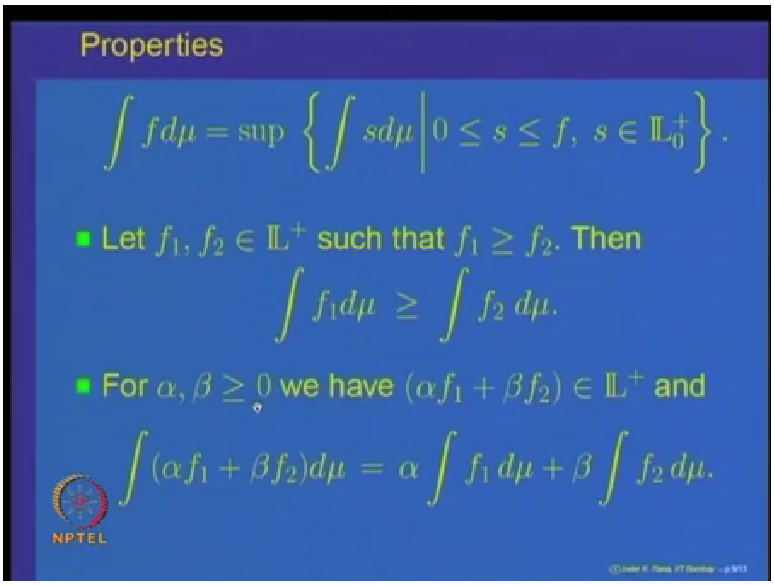
(Refer Slide Time: 18:12)



So, that we can write it as limit integral n going to infinity integral over Bn of sd mu. So, this is because sn is increasing sequence sn an increase is to Bn is an increasing sequence of sets Bn incases is to x and the integral over a set is a measure. So, keep in mind that the integral of a nonnegative simple measurable function over a set e gives you a measure. So, that measure mu of that measure at Bn we will go to that value at x. So, that is same as a in that integral s d mu is limit of integral s over b and d mu.

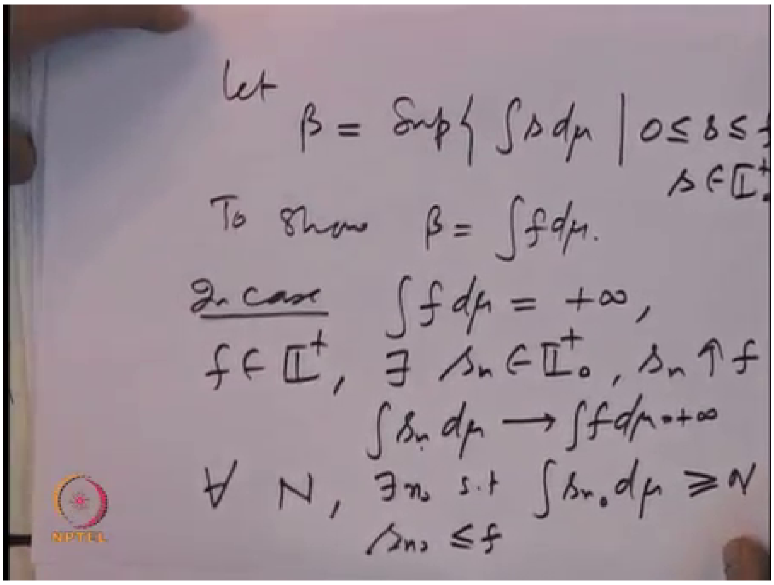
Now on Bn we know that on Bn sn is bigger the x. So, let us use that fact. So, this is less than or equal to limit n going to infinity integral over Bn of sn d mu. So, that is the nonnegative simple function one or non-negative simple measurable function is less than other than the integral of one will be less than the other. And now this is integral sn over Bn. So, if you replace that set Bn by the whole space this will still be less than or equal to limit n going to infinity integral over the whole square x of sn d mu. So, and that is equal to integral fd mu. So, that proves that integral of sd mu is less than or equal to integral of. So, implies that integral sd mu is less than or equal to integral fd mu whenever s is less than f and s is nonnegative simple measurable function. So, that proves this property that if f is a nonnegative simple measurable function, f is a nonnegative measurable function and s is a nonnegative simple measurable functions such that s is less than or equal to f, then the integral of s is less than or equal to integral of f.

(Refer Slide Time: 20:31)



And now as a consequence of this let us observe the property that integral of fd mu which we defined as the limit of integrals of snd mu for any sequences sn, can also be represented as supremum over of integrals sd mu where s is less than or equal to f, and f is s is a nonnegative simple measurable function. So, let us prove this property that. So, let us call.

(Refer Slide Time: 21:01)

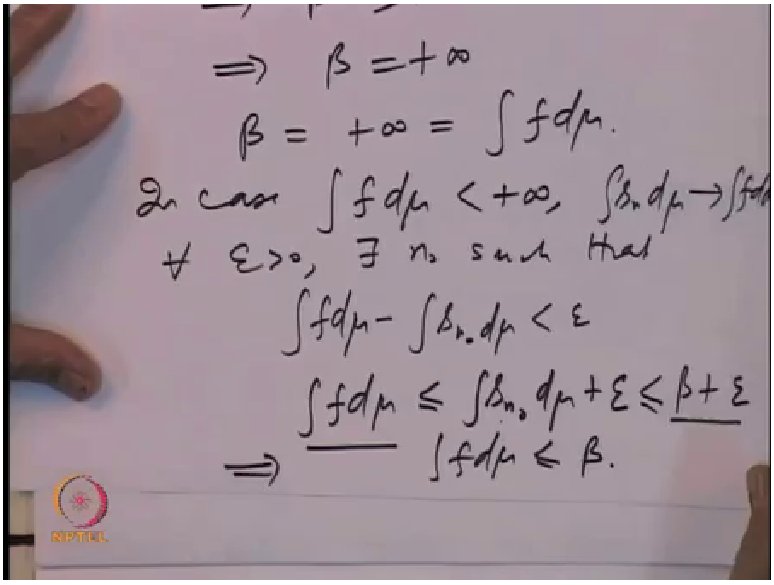


Let us define beta to be the supremum of integral of nonnegative simple measurable function sd mu, where 0 is less than or equal to the nonnegative simple measurable function s less than or equal to f. So, let us call this. So, we want to show that integral fd mu is equal to beta.

So, let us prove that integral of fd mu can also be written as supremum of integral sd mu where s is less than or equal to f. So, to prove this property let us define. So, let beta b equal to supremum of integral sd mu, where 0 is less than or equal to s is less than f and s is a non-negative simple measurable function. So, we want to show that beta is equal to integral fd mu. So, now, one possibility is in case integral fd mu is equal to plus infinity then we know that f belongs to l plus. So, there is a sequence sn in L plus 0 nonnegative simple measurable functions sn increasing to f and integral sn d mu converging to integral fd mu, but now in this case we know this is equal to plus infinity; that means, for every positive integer n there exists some n naught such that integral sn nuaght d mu will be bigger than or equal to n because this number is going to converge to infinity.

So, this must exceed every n. So, there is n naught and this sn naught is less than or equal to f. So, we have found a nonnegative simple measurable function sn naught less than or equal to f such that its integral is bigger than or equal to n so that implies that the supremum beta must also be bigger than or equal to n.

(Refer Slide Time: 23:26)

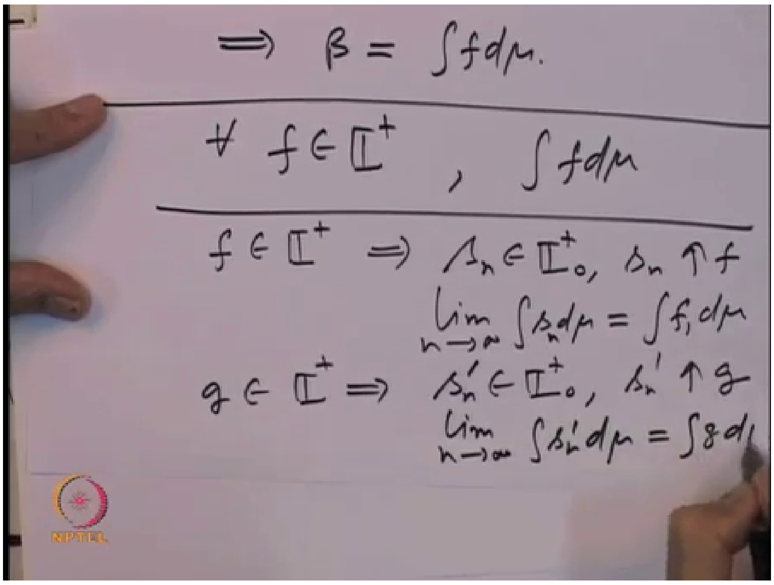


So, this implies that the supremum beta is bigger than or equal to n for every n and hence that implies beta is equal to plus infinity. So, in the case integral fd mu is equal to plus infinity that implies beta is also equal to plus infinity. So, that is beta is equal to plus infinity is equal to integral fd mu.

So, now let us look at the case when this integral is finite. So, incase integral fd mu is finite; that means, and we know that integral s n d mu convergence to integral fd mu. So, for every epsilon bigger than 0, there is some n naught such that integral fd mu minus integral sn naught d mu is less than epsilon and; that means, integral fd mu is less than or equal to integral sn naught d mu plus epsilon; and sn naught is one function which is less than or equal to f. So, this is less than or equal to beta plus epsilon. So, integral fd mu is less than or equal to beta plus epsilon and this holds for every epsilon. So, implying integral fd mu is less than or equal to beta.

So, once again ah we have proved that integral fd mu is less than or equal to beta and clearly.

(Refer Slide Time: 25:01)



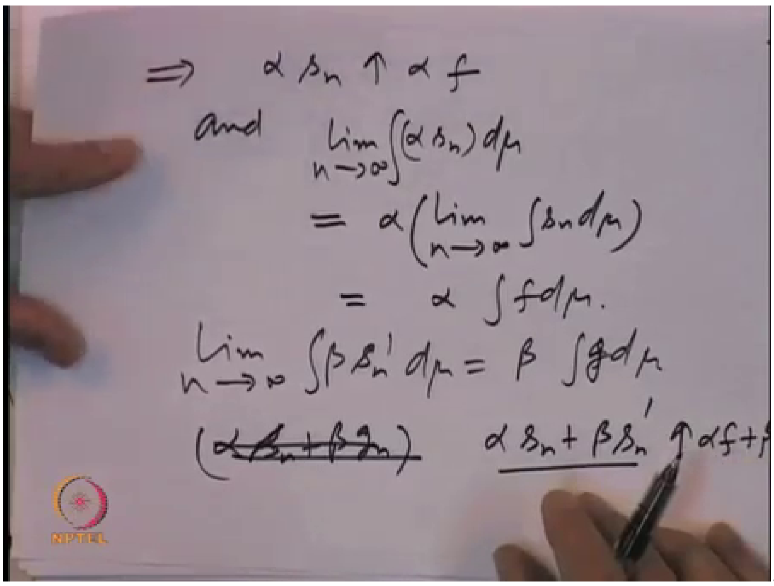
Beta is less than or equal to integral fd mu, that is obvious because beta is a supremum overalls nonnegative simple functions sd mu less than or equal to integral of. So, beta is the supremum. So, definition beta is the supremum of integral sd mu where s is less than or equal to f. So, so integral sd mu is less than or equal to integral f. So, beta is always less than or equal to integral of fd mu. So, hence this implies beta is equal to integral of fd mu. So, that proves another way of defining the integral of a nonnegative simple measurable function, that if f is a nonnegative simple measurable function then its integral can also be defined as the supremum overall integral sd mu where s is a nonnegative simple measurable function. So, using these two definitions let us prove that various properties of the integral. So, for every function f in L plus we are defined integral fd mu, and now we are going to look at the properties.

So, the first property that we are saying is if f 1 is bigger than f 2 then integral f 1 is bigger than integral f 2, and that follows from the above definition itself because integral f 1 d mu integral f 1 d mu is going to be the supremum over integrals of all nonnegative simple measurable functions s such that s is less than or equal to f 1 and similarly for f 2 it is going to the supremum overall nonnegative simple measurable functions s less than or equal to f 2, but if s is less than or equal to f 2 and f 2 is less than or equal to f 1. So, s is going to be less than or equal to f 1. So, this supremum for f 1 is taken over a larger class and then that of f 2. So, that supremum is going to be for integral f 1 the supremum is going to be bigger than or equal to integral over f 2. So, that follows directly from here.

From the above definition that this integral is supremum over integral of nonnegative simple measurable functions below f. So, as a consequence of this we immediately have this theorem that if f 1 is bigger than f 2, then integral f 1 d mu is bigger than or equal to integral f 2. Next let us look at the linearity property of this integral namely, if alpha and beta are nonnegative real numbers extended real numbers, then and f 1 and f 2 are in L plus then alpha times f 1 plus beta times f 2 belongs to L plus, and integral of alpha f 1 plus beta f 2 is equal to alpha times integral f 1 plus beta times integral f 2. So, to prove that lets. So, f belongs to L 1 L plus implies there is a sequence sn of nonnegative simple measurable functions sn increasing to f, and limit n going to infinity integral s sand d mu giving us the integral of 1 d mu and similarly g belongs to L plus implies that there is a sequence let us call it as sn prime of nonnegative simple measurable functions sn prime increasing to g and its limit n going to infinity integrals of sn prime d mu, giving us the integral of g d mu.

So, now from these two let us just simply observe that if sn increases to f then alpha sn will increase to alpha f and similarly beta sn prime will increase to bit of g and integral of alpha sn for nonnegative simple measurable functions is equal to alpha times integral sn. So, combining all these properties will have the required result.

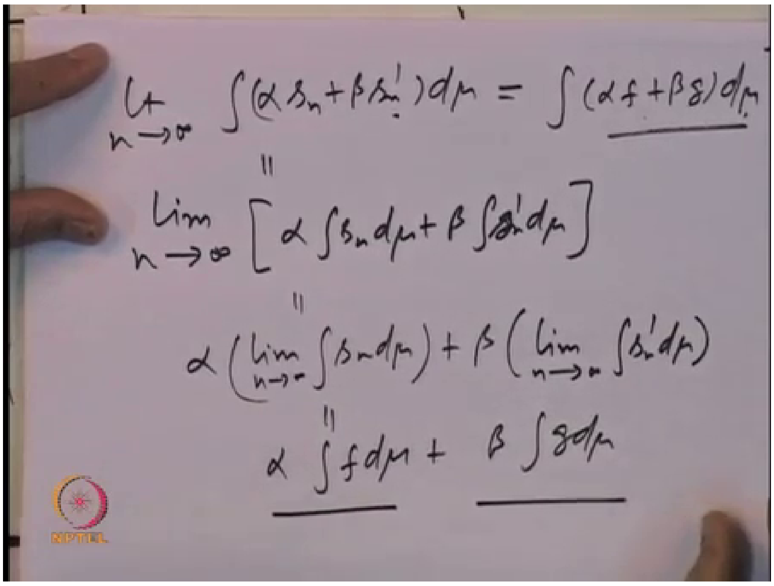
(Refer Slide Time: 29:43)



So, let us write just write it out. So, implies that alpha sn increases to alpha of f and limit alpha sn n going to infinity integral d mu will be equal to alpha times limit n going to infinity of integral snd mu, because for nonnegative simple measurable functions alpha times sn is same as alpha times the integral, and that is equal to alpha integral fd mu and similarly limit of n going to infinity of beta sn prime integral d mu will be equal to beta times integral of gd mu.

On the other hand if you look at the sequence alpha sn plus beta gn sorry if you look at the sequence alpha sn plus beta sn dash, then this is a sequence of nonnegative simple measurable functions and that increases to alpha f plus beta g by the properties of sequences. So, as a result we will have that the integral s alpha sn d mu. So, let us write.

(Refer Slide Time: 31:23)



So, because of this we have the property that the integral alpha sn plus beta sn dash d mu limit n going to infinity will be equal to integral of alpha f plus beta g d mu, but integration is linear for non-negative simple measurable functions.

So, this side this nothing, but limit n going to infinity of alpha integral sn d mu plus beta integral gn d mu. And now by the properties of limits of sequences this is equal to alpha times limit integral of a snd mu plus beta times sorry this not this is a sn dash. So, beta times limit n going to infinity of integral s n dash d mu. And this we know is alpha times integral fd mu plus beta times integral gd mu.

So, integral of alpha f plus beta g is equal to alpha times integral f plus beta times integral of g. So, that proves the property that when alpha and beta are nonnegative extended real numbers, then alpha f 1 plus beta f 2 belongs to L plus that we are already shown also and now we are claiming that the integral of alpha f 1 plus beta f 2 is equal to alpha times integral of f 1 plus beta times integral of f 2 written for g. So, that is same as for this.