Measure & Integrate
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Lecture - 17B
Integral of Nonnegative Simple Measurable Functions

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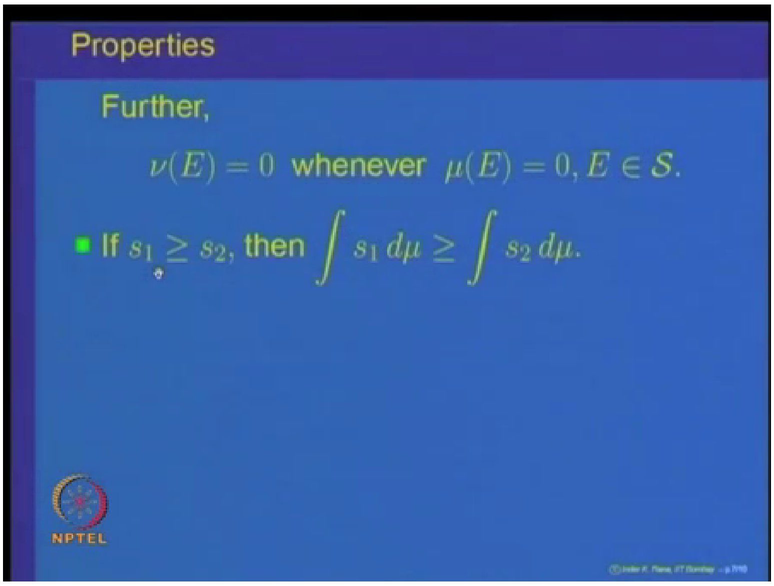

So, next property we want to check is that if S 1 is bigger than S 2, then integral S 1 is bigger than integral S 2. So, let us check that property.

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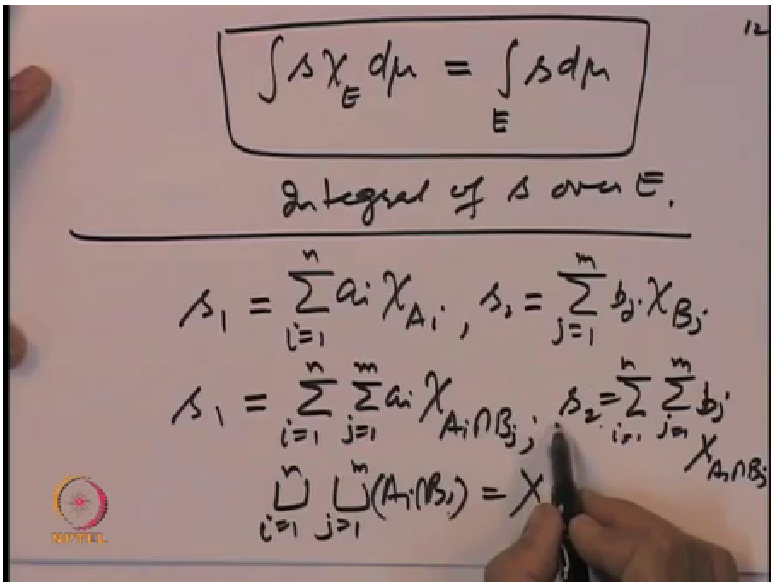

So, let us write S 1 which is non negative simple measurable function as sigma a i indicator function of A i and S 2 as sigma b j chi of B j j equal to 1 to m. So, as we had mentioned, whenever you want to do some analysis regarding two simple functions S 1 and S 2 bring them to a common partition.

So, we will write this is also equal to sigma over i 1 to n S 1 can be written as sigma over i sigma over j 1 to m a i times indicator function of a i intersection b j and so, S 1 can be written as this and we as can write S 2 as sigma over i sigma over j 1 to m of b j times the indicator function of A i intersection B j. So, now, and union of A i B j intersection b j j equal to 1 to m union i equal to 1 to n is a partition of X. So, now, they are common partitions and when you say S 1 is bigger than S 2 means what. So, let us take a point X. So, if x belongs to X, then it belongs to 1 of A i intersection b j S 1 as the value A i and S 2 has the value B j; that means, S 1 of x which is A i must be bigger than or equal to S 2 of x which is B j on A i intersection B j so; that means, if this is a representation.

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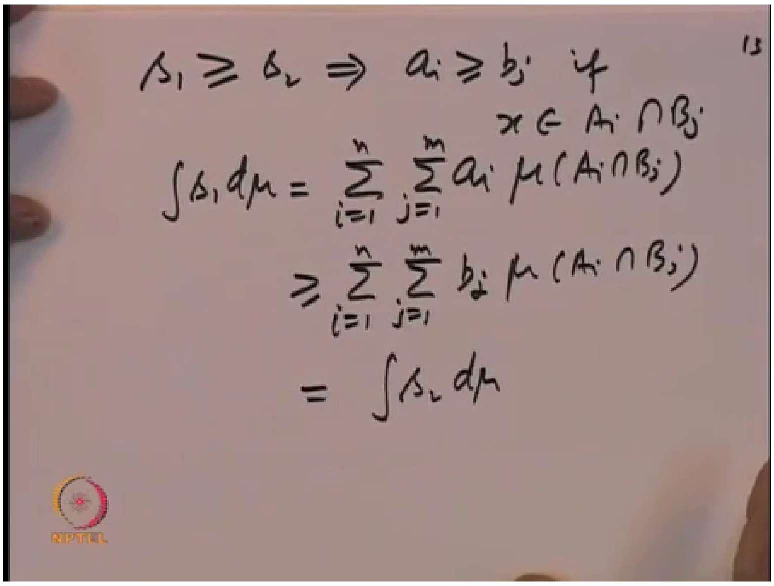

So, then S 1 bigger than S 2 implies that a i is bigger than or equal to b j, if x belongs to a i intersection b j, once we observe that now problem is solved. So, what is integral of S 1 d mu that by definition sigma over i 1 to n sigma over j one to m of a i mu of A i intersection B j and A i is bigger than B j if x belongs to this. So, this is bigger than or equal to sigma i equal to 1 to n sigma j equal to 1 to m of b j mu of A i intersection B j and which is equal to integral of S 2 d mu.

So, integral of S 1 is bigger than integral of S 2 if S 1 is bigger than or equal to S 2. So, that proves the next property.

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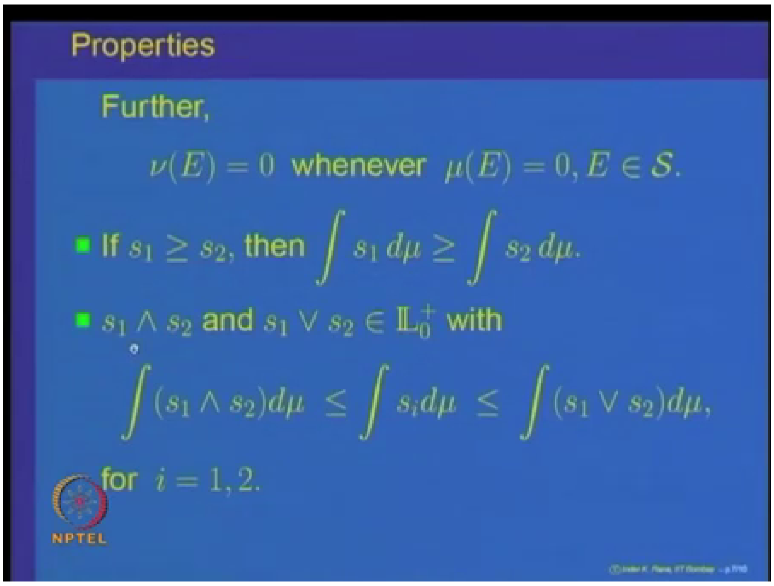

Now, we are going to look at a special functions if S 1 and S 2 are non negative simple measurable functions, then we want to look at S 1 V S 2 and S 1 wedge S 2, recall how was S 1 V S 2 defined S 1 V S 2 was defined as the maximum of S 1 and S 2 and similarly S 1 wedge S 2 was defined as the minimum of S 1 and S 2.

So, our; when we had shown that if S 1 and S 2 are non negative simple measurable functions, then the maximum of S 1 and S 2 and the minimum of S 1 and S 2 are also non negative simple measurable functions. So, we want to check this property. Now here that that integral of S 1 wedge S 2 is less than S i integral less than or equal to integral of the next one, but that is obvious because if S 1 and S 2 are simple measurable non negative simple measurable functions and you look at S 1 wedge S 2 S 1 wedge S 2 that is the minimum of S 1 and S 2 then clearly S 1 wedge S 2 is a minimum.

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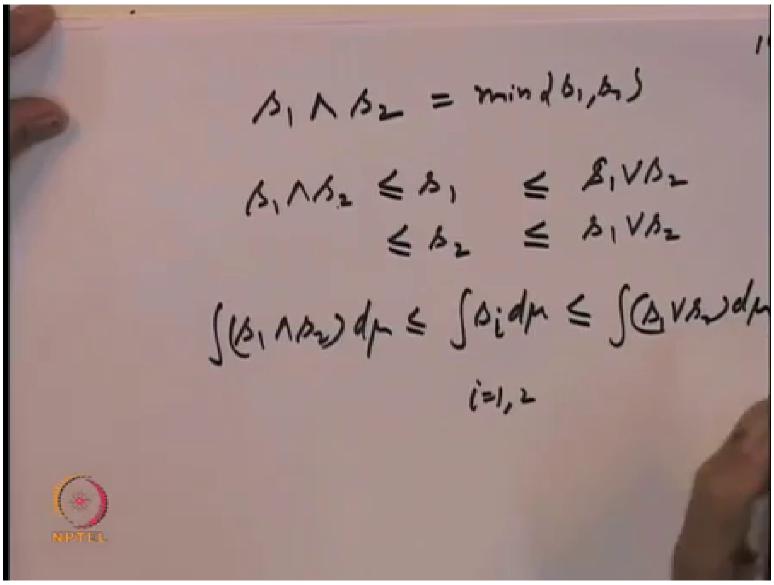

So, it is going to be less than or equal to S 1 and also going to be less than or equal to S 2 and S 1 V S 2; S 1 V S 2 the maximum is going to be bigger than S 1 and S 2 both. So, it is going to be less than or equal to S 1 maximum S 2.

So, what we are saying is S 1 wedge S 2 is less than or equal to both S 1 and S 2 and both S 1 and S 2 are less than or equal to maximum of S 1 and S 2 and just now and all are simple functions. So, what we have proved just now. So, that will say that the integral of S 1, S 2; the minimum of S 1 and S 2 d mu is less than or integral of S 1 also less than integral of S 2. So, less than or equal to equal to integral S i d mu i equal to 1 and 1 and 2 and both these integrals are less than or equal to integral of S 1 wedge S 2 d mu. So, that proves the required property and that follows from the earlier property on that if S 1 is less than or equal to or bigger than or equal to S 2, then integral of S 1 is bigger than or equal to integral S 2.

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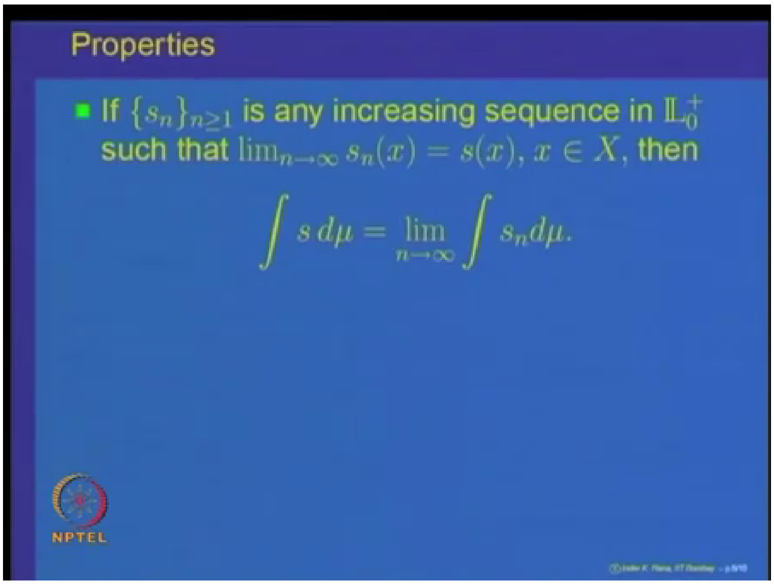

And now let us look at a property how does this integral behave with respect to limiting operations. So, we want to claim that if S n is a sequence increasing sequence in L 0 plus. So, it is a increasing sequence of nonnegative measurable functions increasing to a simple function S of X, then integral S d mu is limit n going to infinity integral S n d mu. So, this is the first in a sense non trivial argument required.

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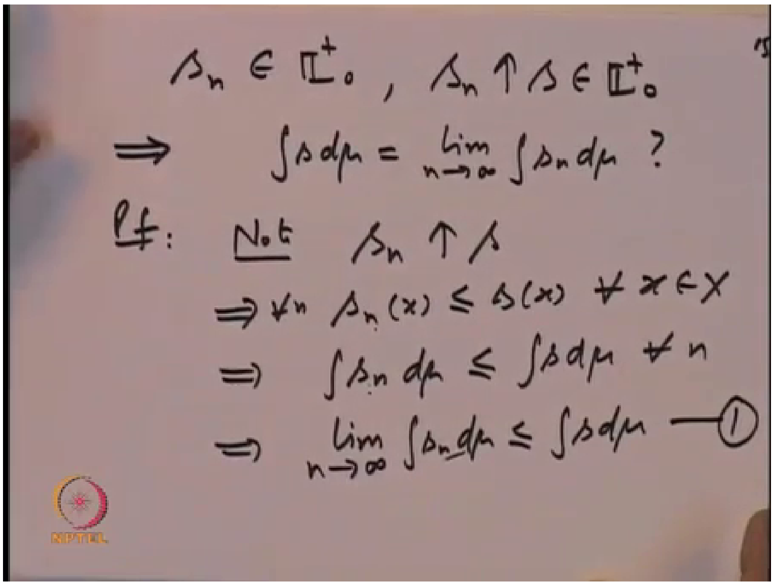

So, S 1 S n are functions in L 0 plus non negative simple measurable functions S ns increasing to S; S belonging to L plus 0; non negative simple measurable, we want to show this implies that integral S d mu is equal to limit n going to infinity of integral S n d mu. So, this is what we want to show. So, now, let us start observing. So, what is the proof of this? So, note what we are given is S n is increasing to S so; that means, what if S n is increasing to S; that means, that S n of x is going to be less than S of x for every x belonging to X; right. So, that is obvious from this if this, then this implies S n is increasing to S implies each S n x is less than or equal to S of x.

Now, S n is a simple function S is a simple none negative simple measurable function S n is less than or equal to this for every n. So, that implies that integral of S n d mu is less than or equal to integral S d mu for every n. So, integral S n d mu is less than or equal to integral S d mu and integral S n d mu is an increasing sequence of extended non negative extended real numbers. So, implies that the limit of that which are just may be equal to plus infinity S n d mu is also less than or equal to integral S d mu. So, here is a; that an is a sequence of non nonnegative extended real numbers a n less than or equal to a implies a n S are increasing. So, limit of a n it will be less than or equal to a.

You are in extended real numbers keep in mind. So, we have proved. So, let us call it as 1. So, we have proved in the required equality, we have proved that a right hand side limit n going to infinity integral S n d mu is bigger than or equal to integral as below we want to prove the other way round inequality also. So, to do that here is a here is a small manipulation that we have to do.

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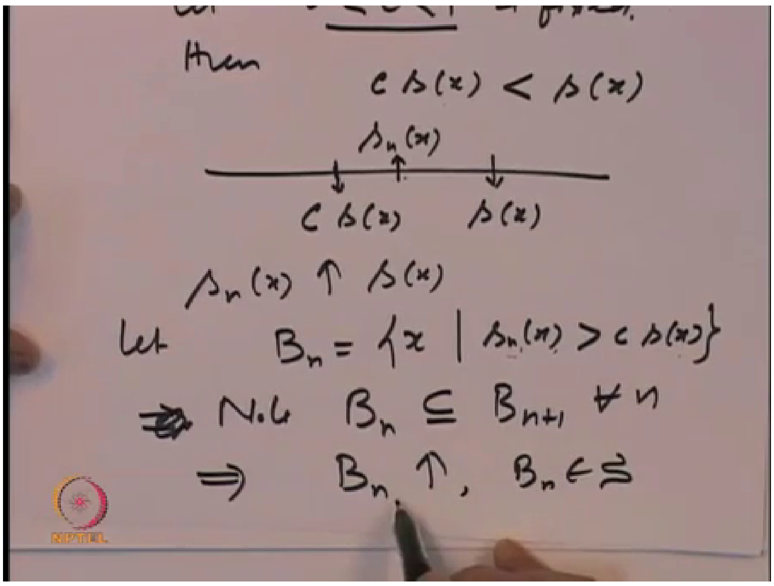

So, for that what we do is the following. So, let us fix let a number c between 0 and one be fixed then c times S of x for any point x c times S of x is going to be strictly less than S of x. So, here is c times S of x and here is S of x, right. So, let us fix c between 0 and one and look at the n for any point x, let us look at c times S of x, then the first observation because c is between 0 and 1 c is strictly less than one. So, c times S of x will be less than S of x. So, it will be somewhere here and now S n x is increasing to S of x; S n x.

So, after some stage S n x must be on the right side of c times S of x. So, after some stage it must be on the right side of. So, this is the picture that you happened. So, let write. So, let us defined B n to be the set of all x such that S n of x is bigger than c times S of x. So, collect all those points write where this is going to happen where S n of x is bigger than see this stage will depend upon n. So, then now so; that means,. So, implies that first of all. So, let us note that B n plus if S n x is bigger than c of S n then S n plus one is any way bigger than S n of x because S n is increasing. So, S n plus one x is going to be bigger than; so, B n. So, if so; that means, this B n is inside B n plus 1 for every n; that means, that is B n is an increasing sequence. So, implies B n is an increasing sequence.

So, that is the first observation because all B ns; S n is increasing. So, if x belongs to B n, then S n x is bigger than c times S of x, right, but S n; S n is increasing. So, S n plus 1 x is going to be bigger than S n of x. So, if S n x is bigger than c times S of x, then S n plus 1 also is going to be bigger. So, x belonging to b an implies x belongs to B n plus 1; that means, B n is a subset of B n plus 1; that means, B n is an increasing sequence of sets and also observe that each B n is an element in the sigma algebra S, right, each B n is an element in the sigma algebra S because B n is where S n is bigger than c times S all are simple measurable functions and we have observed that such sets are in the sigma algebra.

So, B n is an increasing sequence of sets in the sigma algebra S and let us observe what is the union of this B ns.

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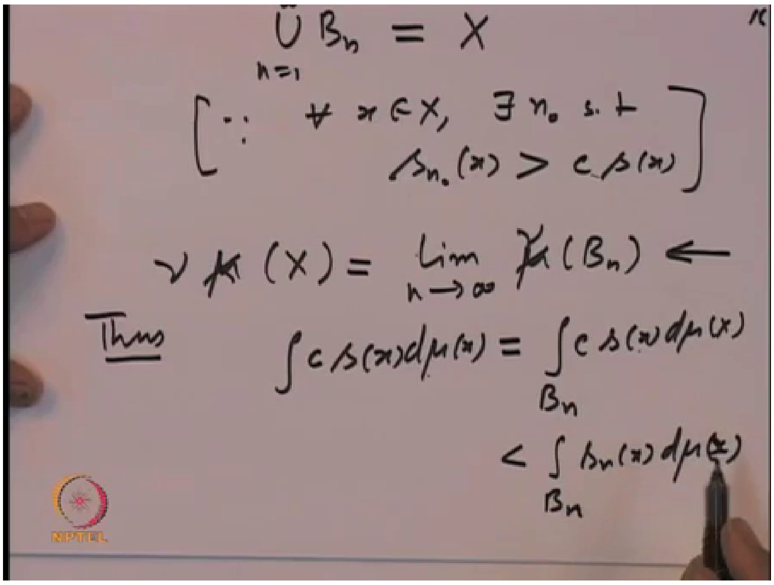

So, union of B ns n equal to 1 to infinity; obviously, it is contained in x because all are subsets of x, but by the fact that for every x this picture that we observed here for every x, there is going to be some stage after which S n is going to be bigger than x because c times S x is strictly less than this. So, that fact implies that this union is equal to x because. So, observation here is because for every x belonging to x there is a stage n naught such that S n naught x is bigger than c times S of x right that is because S n x is convergent to S of x; right because S n x is going to increase to S of x. So, it has to cross over this point c times S of x otherwise it cannot reach that point.

So, B n is an increasing sequence of sets in the sigma algebra and their union is equal to x and mu is a measure countable additive and we had proved a equivalent way of saying that mu countable additive is same as saying whenever a sequence of sets an is increasing then mu of an S must increase to mu of a. So, by that fact mu of x must be equal to limit n going to infinity mu of B x. So, that must be true.

So, now let us use all these facts and look at. So, now; so, thus if you look at integral of c times integral of c times S of x d mu x you look at this integral we. So, first of all we claim that this is equal to integral of c times S of x d mu x over B ns. So, first observation we want to make at that and that is because if we look at this as a measure right if we look this as a measure mu of B n. Just now we have proved integral over sets of simple functions over sets is a measure.

So, look at that measure nu n u and n b and is increasing to x. So, nu B n must go to nu of x. So, this fact we are using for this is the fact we are using for in not mu, but we are using for nu and where what is nu n u is integral of c s x over B n. So, that is the fact we are using here so; that means, this is equal to. So, now, on B n what is happening on the set B n on B n S n of x is bigger than c times so; that means, c times S of x is less than. So, it is less than integral over B n of S n of x d mu x because that is the definition of the set B n.

So, we are replace c S x, we are using the fact integral of S 1 is less than integral of S 2 whenever S 1 is less than or equal to integral of whenever S 1 is less than or equal to S two. So, this is less than or equal to this now B n is a set subset of x. So, this integral, I can replace and say that this is less than. So, this is less than or this is less than or equal to integral over the whole space x S n x d mu x.

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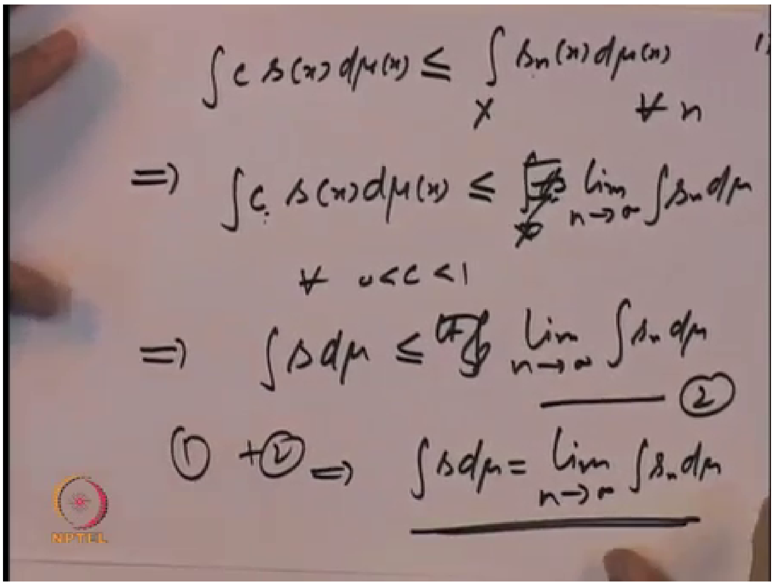

So, what we are saying is by this analysis what we have shown is that the integral c times S of x d mu x is less than or equal to this for every n and because this happens for every n and S n this integrals are the increasing sequence of numbers.

So, this implies that integral c times S of x d mu x is also less than or equal to integral over x of S limit. So, is less than or equal to limit n going to infinity of integral S n d mu and now this holds for every c between 0 and 1. So, I can take the limit as c goes to 1. So, implies that integral of S d mu is also less than or equal to integral limit less than or equal to limit n going to infinity of integral S n d mu. So, that is my other way round inequality 2. So, we have proved both ways inequalities 1 and 2.

So, one if you recall we had already shown one that integral S n d mu is less than or integral S d mu that was one we proved and now we have proved integral S d mu. So, 1 plus 2 imply that integral S d mu is equal to limit n going to infinity integral S n d mu. So, that proves a the result the required result namely that integral of S n; if S n is increasing sequence in L plus, then you can interchange. So, what is S; that is a limit?

So, integral of the limit is equal to limit of the integrals whenever S n is increasing non negative simple functions. So, is a nice properly for increasing sequences; so, at this stage one ask the question that we have proved that if S n is a increasing sequence of non negative simple measurable functions increasing to S then integral of S n S convergent to a integral of S will this property hold for decreasing sequences.

Namely if S n is decreasing non negative simple functions decreasing to S can we say that integral of S n S will decrease to integral S, we do not know that right at present we cannot prove at present this fact right. In fact, many more properties of such things will explore as we extend the notion of integral.

So, we will stop here today and analyze next time another way of representing integral of non negative simple measurable functions and then go were to define integral of nonnegative measurable functions, we will extend the notion of integral from non negative simple measurable functions to non negative measurable functions we will do it next lecture.

Thank you.