Measure & Integration
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Lecture – 17 A
Integral of Nonnegative Simple Measurable Functions

Welcome to lecture 17 on measure and integration. Today, we will start the topic of integration. First I will explain the building blocks for the integration and how the process will be done.

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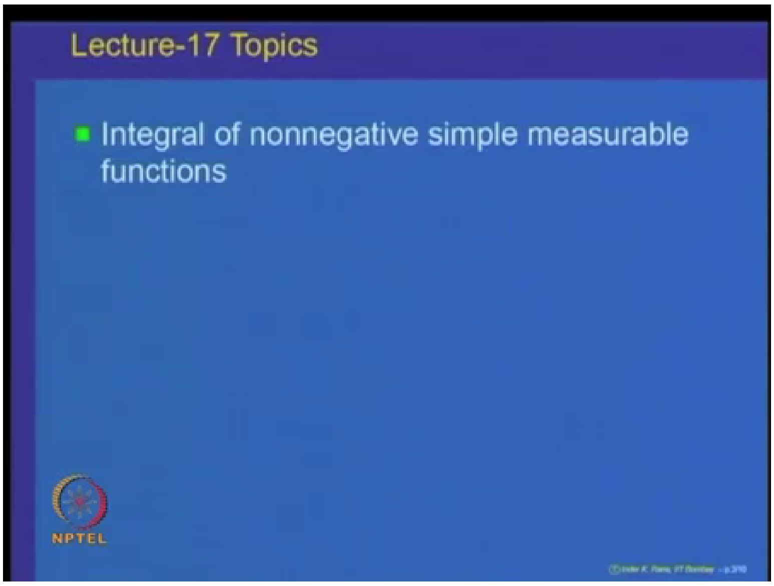

So the topic for today's discussion is integral of nonnegative simple measurable functions. See the basic idea is we want to define the notion of integral for a function f defined on a set x taking values in r star. So, now, for a function f it is we can represent a function f as the positive part minus the negative part.

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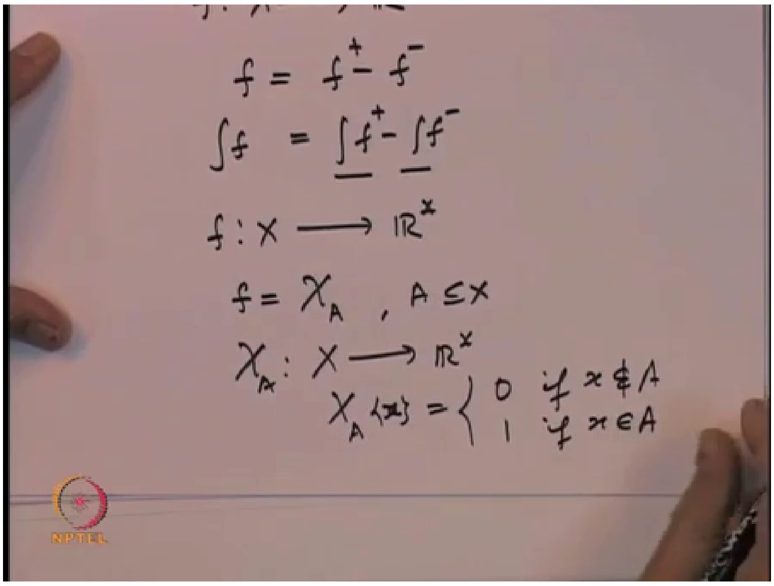

The advantage of doing this is that f plus and f minus both are nonnegative functions and so, it is a and integration being a linear process. So, integral of f is going to be equal to integral of f plus minus integral f minus. So, it is enough to define the notion of integral for nonnegative functions. And for nonnegative functions f on x to are star we recall that we can take it as we look at functions, which are first of all very simple functions for example, let us look at a function f which is a indicator function of a set x. Now of a set a contained in x this is a function which takes only 2 values.

So, indicator function of a is a function on x taking values in r star. So, chi of a at 0 of at a point x is equal to 0 if x does not belong to a and is 1 if x belongs to a. So, you can one can think of this function taking only 2 values right. Now the value where it is 0 the integral. So, we want to define the notion of integral.

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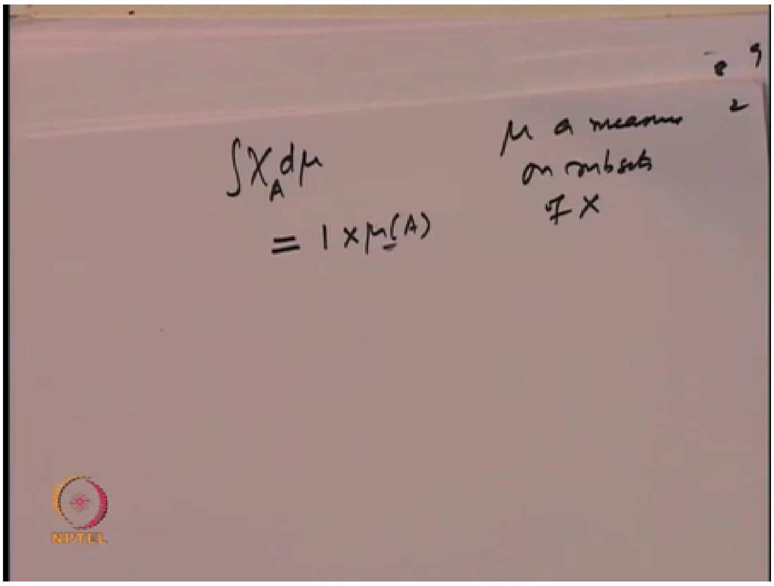

And this is going to be with respect to a measure mu on x. So, mu a measure on subsets of x. So, we are going to write it as xa d mu. So, what it should be on a the value is one. So, we like to put it as one times mu of a, in some sense mu of a is this size of a set and one is the height. So, this is in in a sense the area of the we know the graph of the function. So, let us look at functions which are going to be linear combinations of indicator functions. So, we start looking at the integral of nonnegative simple measurable functions. So, let us recall.

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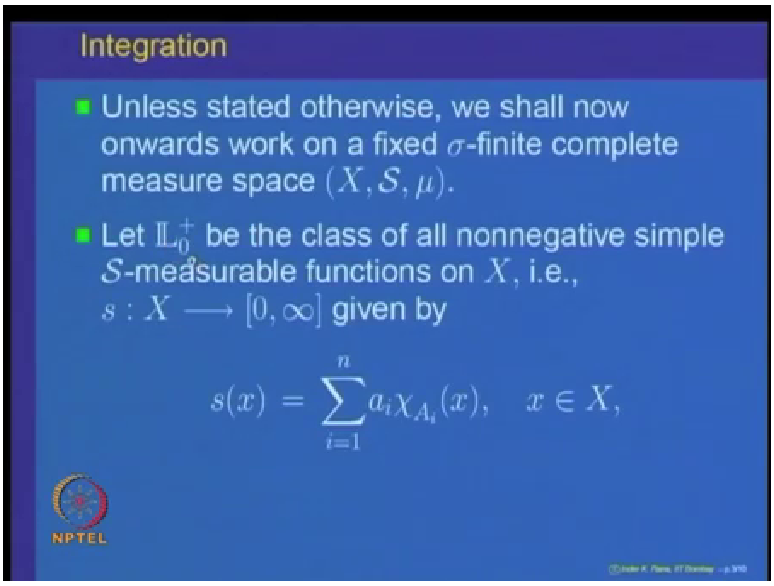

So, will fix our notation that from now onwards we are going to work on a measure space x S mu where x is a set s is a sigma algebra of subsets of x and mu is a measure on defined on s and this is a complete measure space; that means, that all sets a such that mu of a is 0 implies at a and all it is subsets are inside S.

So, let us denote by L lower 0 upper plus to be the class of all nonnegative simple s measurable functions on x. So, now, let us recall what was a nonnegative simple measurable function s, it is a function defined on x taking nonnegative values and it is it has a representation S of x is equal to sigma i equal to 1 to n A i times the indicator function of the set A i value evaluated at x x belonging to x. Where A i is a 1 a 2 a 3 a n are extended real numbers and the sets A i is are in the sigma algebra s. So, they are in the sigma algebra A i s are in the sigma algebra S.

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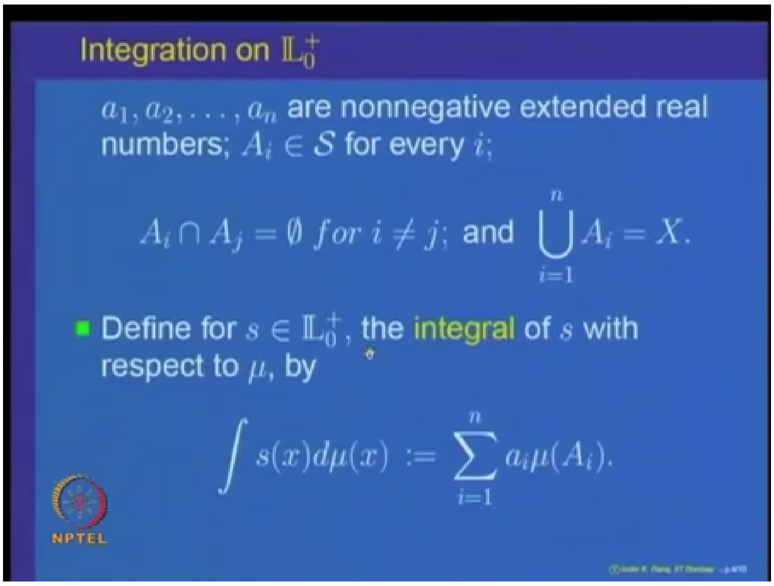

And they are pairwise disjoint; that means, A i intersection is empty for i not equal to j and the union of these sets is equal to X. So, this is going to be the class of nonnegative simple measurable functions. For such class of for functions in this class we are going to define the notion of integral.

So, for A functions S in this class if it is representation is S given before.

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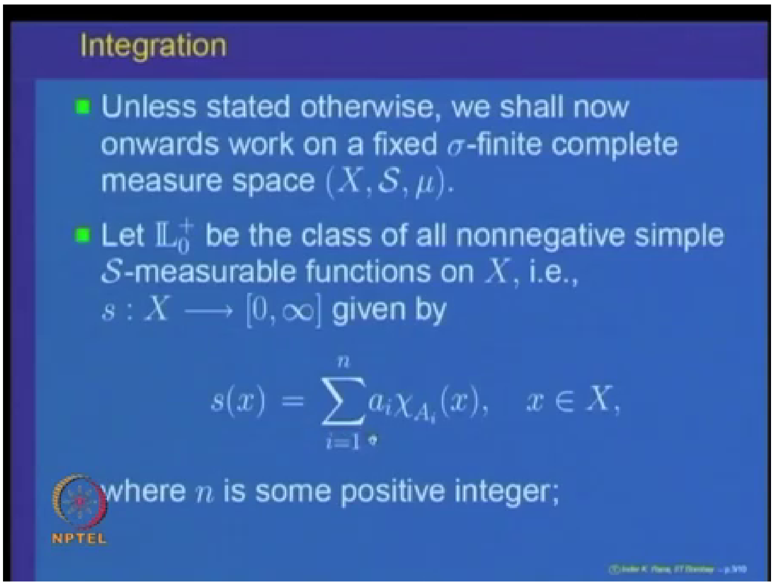

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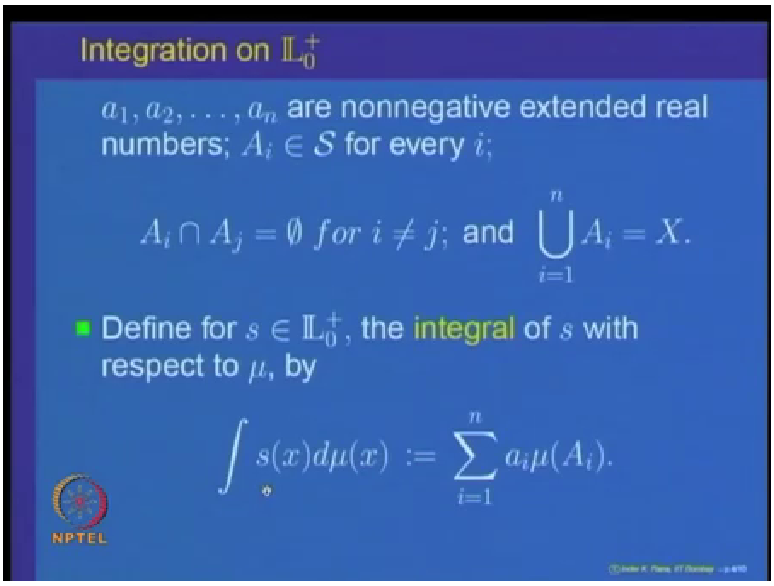

So, if S sub x is equal to sigma A i indicator function of capital A i, then it is integral is defined as S d mu. So, integral is noted by integral sign s of x d mu x to be A i that is the value of the function on the set A i times the measure of the set A i mu of Ai. So, integral of s with respect to mu as written here is defined as sigma A i times mu of A i, A i is the value taken on the set A i. So, A i times the size of the set A i. So, mu of A i sometimes we do not indicate the variable x, we just write as integral s d mu to be the integral of the simple function s nonnegative simple measurable function s with respect to mu and let us note here that our representation the integral is with respect to a representation of the function.

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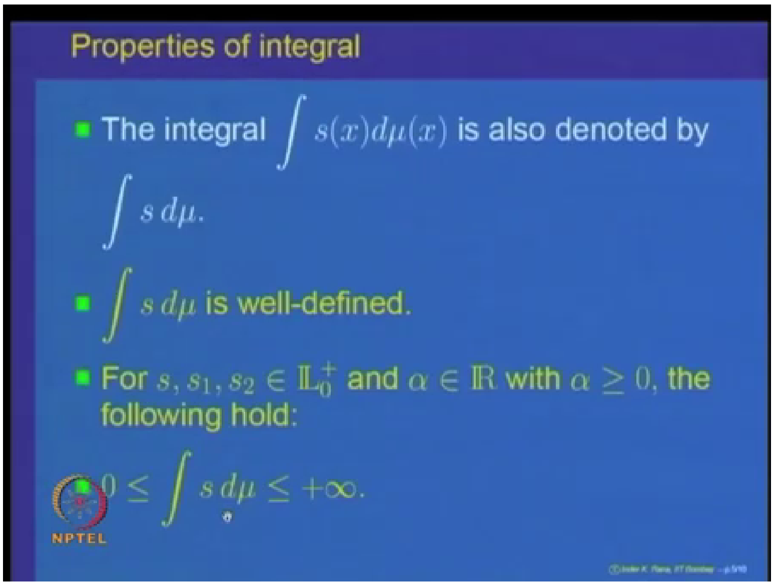

So, first of all we would like to show that integral s d mu is well defined.

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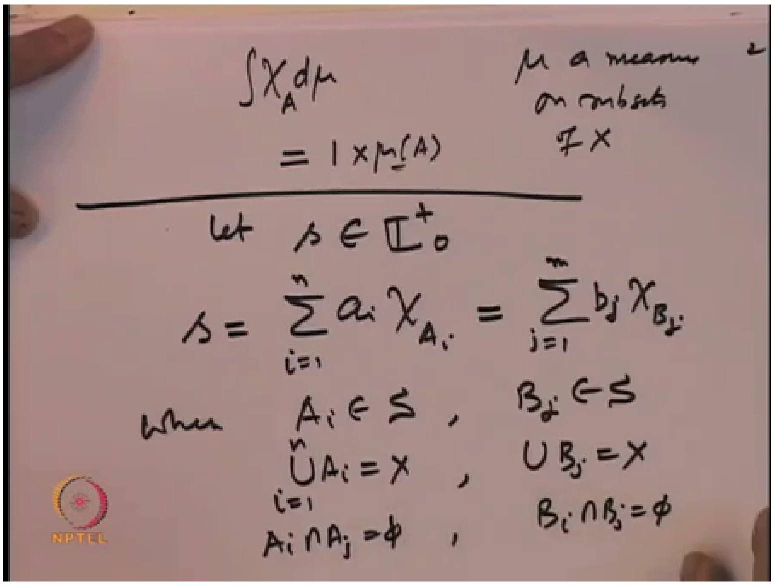

So, let us prove that the integral is well defined. So, let us take a function s belonging to L plus 0. So, it is a nonnegative simple measurable function. So, let us say s is written as sigma A i indicator function of A i i equal to 1 to n also representival as sigma j equal to 1 to m, of some b j chi of b j where the sets A i is belongs to the sigma algebra S, B js belong to the sigma algebra s. And union of ais is equal to x and union of B js is also equal to x and these sets are disjoint. So, A i intersection A j empty and bi intersection B j is empty for i not equal to j. So, let us say the set s a simple function s as got 2 representations possible. So, what we want to show. So, we want to show that the integral of S.

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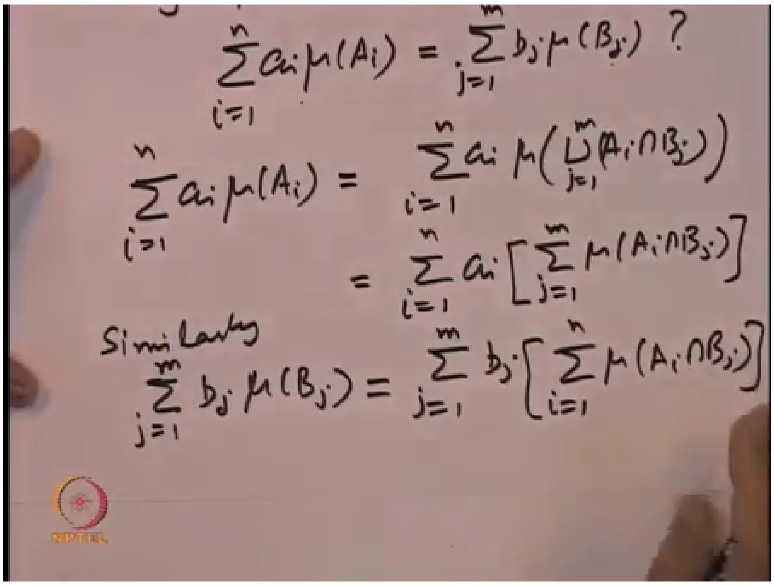

So, integral s d mu is well defined and; that means, what. So, mathematically; that means, we have to show that sigma A i mu of A i one to n is equal to sigma j equal to 1 to n B j mu of bj. So, this is what we have to show. So, let us start. So, sigma A i mu of A i i equal to 1 to n, I can write it as sigma i equal to 1 to n A i and then mu of this A i can be written as union of A i intersection B j j equal to 1 to m because union of B j is equal to x. So, A i intersection x and that is same as this now this is a bjs are disjoint. So, these sets are A i intersection B js for i fixed are disjoint.

So, by using finite tip property of the measure we have this is equal to i equal to 1 to n A i and this is nothing, but sigma j equal to 1 to m mu of A i intersection B j right. And similarly we can write the other side that is sigma j equal to 1 to j equal to 1 to m of B j mu of B j to be equal to sigma j equal to 1 to m B j sigma i equal to 1 to n mu of A i intersection B j. So, the left hand side here is written as this sum the right hand side is written as this sum. Now we want to show that these 2 sums are equal now let us observe that given that the function s has got 2 representations this equal to this. So, how is this function calculated at a point x, if x belongs to A i the value is A i and on the other hand it may belong to some B j the value will be bj. So, that in in. So, that forces one to say that if x belongs to A i intersection B j then A i must be equal to bj. So, this is the crucial thing to note here that if s a simp non neteg nonnegative simple measurable function is given 2 representations one is sigma A i, capital A i indicator function of A i and sigma B j indicator function of B j. Then for x belonging to A i intersection B j the value of s of x on one hand it is A i other hand it is bj. So, A i must be equal to bj. So, this is the crucial thing to note. So, let us make this observation and write it out. So, note that.

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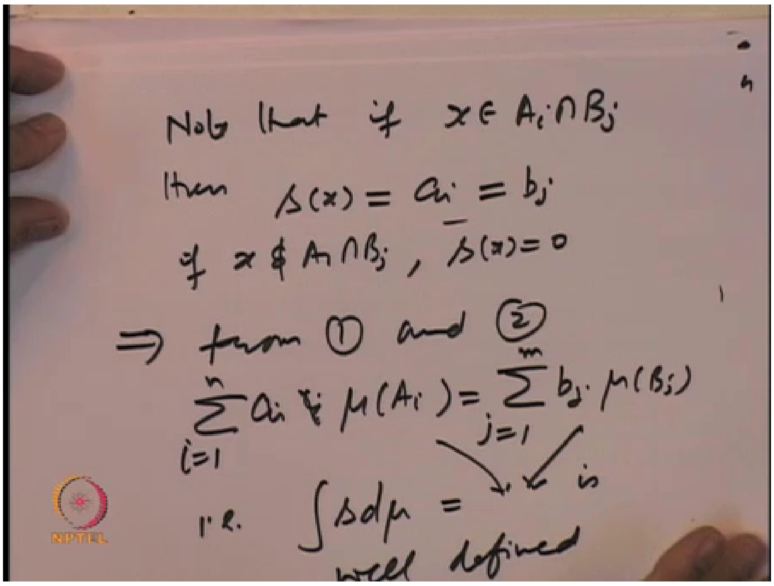

So, note that if x belongs to A i intersection B j. If x belongs to A i intersection B j, then s of x is equal to A i it is also equal to B j. So, A i is equal to B j and if x does not belong to A i intersection B j, then s of x is equal to 0. So; that means, in this summation in the summation whenever x belongs to A i intersection B j this A i is going to be equal to B j otherwise in the sum the term does not matter. So, that proves the fact that. So, that will imply from this 2 equations from equation one and equation 2. So, this implies from equation 1 and 2 that sigma A i i equal to 1 to n of mu A i is equal to sigma j equal to 1 to m B j mu of B j.

So, that is integral s d mu can be defined as either of this sums. So, is equal to either this or this is well defined. So, the integral of a nonnegative simple measurable function. So, we can choose any representation of we can choose any representation of the nonnegative simple function and define it is integral in terms of that. Next let us look at properties of this integral. So, we are going to look at functions s 1 s 2 which are nonnegative simple measurable functions alpha will be a real number alpha bigger than or equal to 0 then we are going to look at what happens to various properties of.

So, first observation is that integral s d mu is a nonnegative number. It could be equal to plus infinity. So, integral s d mu is an extended nonnegative real number. That is obvious because what is sd mu integral of s d mu is summation of A is times mu of A i all the terms are nonnegative. So, this is a nonnegative number. So, this is an obvious property the second property.

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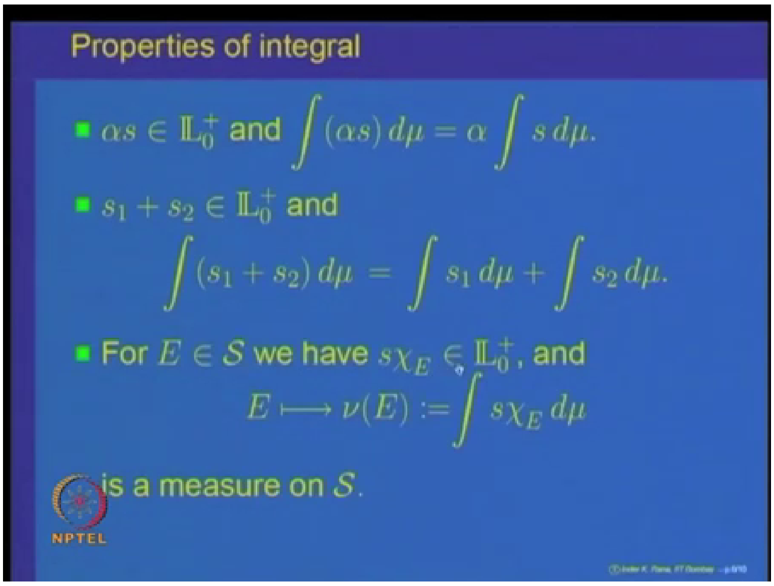

We want to check that for a nonnegative simple function s alpha s belongs to L plus 0 plus and the integral of alpha s d mu is same as alpha times the integral of s d mu. So, let us check that.

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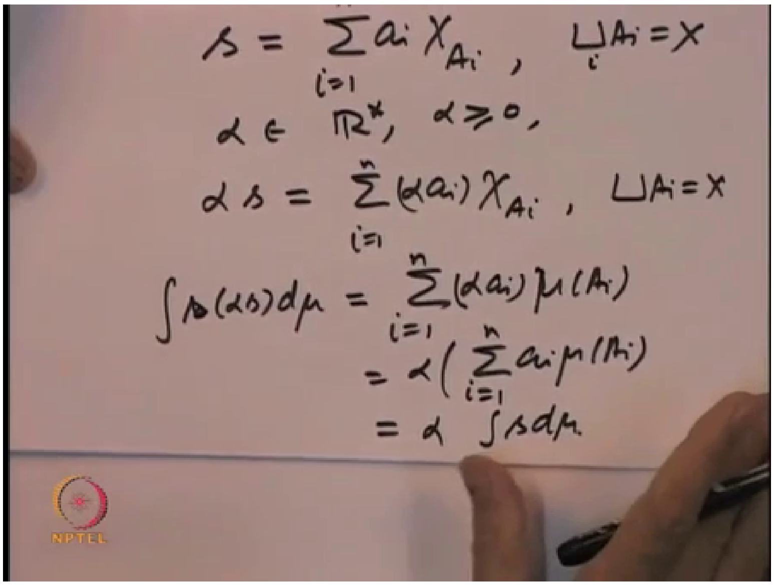

So, s belongs to L plus 0 is a nonnegative simple measurable function. So, let us write let us write s is equal to sigma A i, indicator function of A i where union A i is equal to x. So, whenever it is a partition we will write as this square bracket union over i equal to x. And alpha is a nonnegative alpha belonging to r star alpha bigger than or equal to 0, then alpha of s. So, has the representation it is alpha A i, chi of A i and A is are still a partition of x, but; that means, if this is the representation.

So, integral alpha s d mu integral of alpha s with respect to mu is going to be equal to by our definition, i equal to 1 to n alpha A i times mu of A i and this is a finite sum nonnegative everything. So, alpha comes out alpha times the summation of i equal to 1 to n of A i mu of A i and that is nothing, but alpha times integral of s d mu. So, that proves the property that the integral of nonnegative simple functions is if you multiply it by a constant alpha then the alpha comes out. So, integral of alpha is d mu is equal to alpha times s d mu. Next we want to show that it is a linear operation. So, we want to check that if s 1 and s 2 belong to L 0 plus then s 1 plus s 2 belong to L 0 plus that we have actually we have already checked, but will check it again today also and the integral of s 1 plus s 2 d mu is integral of s 1 plus integral of s 2. So, for such things we have.

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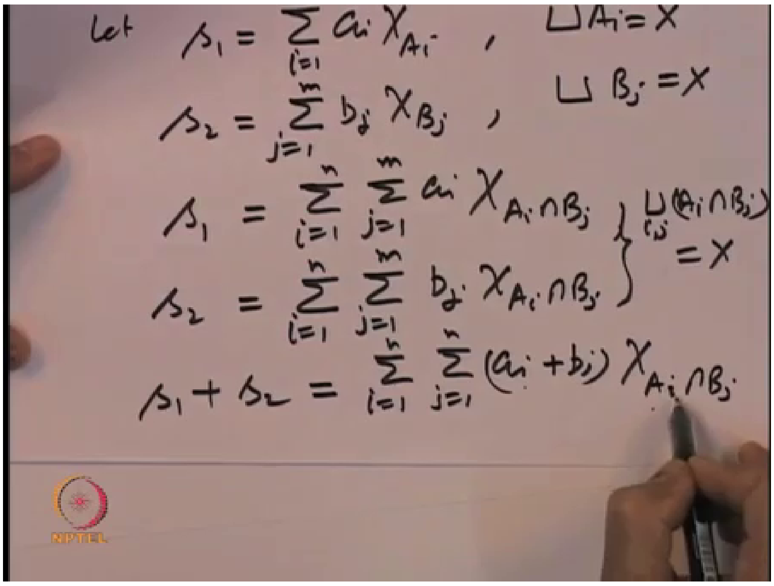

So, let us take a function s 1 s 2 belonging to L plus 0. So, nonnegative simple measurable function. So, let us write let s 1 be equal to sigma A i chi of A i where A i is form a partition of x and later write s 2 a sigma j equal to 1 to m B j chi of B j union B js partition of x. So, if you recall we had said that we can bring the both s 1 and s 2 a common partition and what is that common partition A i intersection B j.

So, what we are saying is you can write s 1 as sigma i equal to 1 to n sigma j equal to 1 to m A i chi of A i intersection B j and also similarly s 2 can be written as i equal to 1 to n sigma j equal to 1 to m of B j chi of A i intersection bj. Now here note that union over i and j A i intersection B j that is a partition of the whole space that is equal to x. So, this is the this is the point to be sort of noted that whenever you are given 2 functions s 1 and s 2 with 2 representations, which involves some partitions A i and partition B j then we can bring them to a common partition namely A i intersection B j and now we can define what is s 1 plus s 2. So, s 1 plus s 2 is going to be equal to sigma over i 1 to n sigma over j equal to 1 to n A i plus B j chi of A i intersection B j. That is clear because on A i intersection B j s 1 is A i and on A i intersection B j s 2 is bj. So, s 1 plus s 2 will be equal to A i plus B j on A i intersection B j. So, once we have got on a representation of s 1 plus s 2. We can define what is the integral of s 1 plus s 2.

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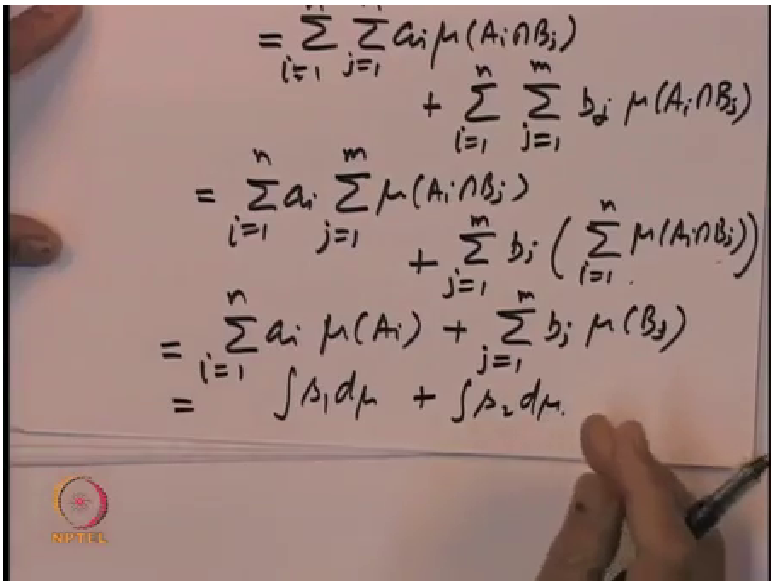

So, this representation gives us that integral of s 1 plus s 2 d mu is equal to summation over i 1 to n summation over j 1 to m of A i plus B j into mu of A i intersection B j right. So, because this is a representation. So, A i plus B j is a value on the set A i intersection B j. So, the integral is going to be equal to summation over i summation over j of A i plus B j the value on the set A i intersection bj. Now the right hand side we can write split. So, that is equal to 2 terms, one is summation over i summation over j of A i times mu of A i intersection B j plus the second term summation i equal to 1 to n summation j equal to 1 to m of bj. So, A i second term is B j mu of A i intersection B j. And now these are all finite sums.

So, we can write the first term as sigma i equal to 1 to n take A i outside and this is summation of mu of A i intersection B j because this is summation over i only. So, you can take it out over j equal to 1 to m of A i intersection B j plus here summation over j and summation over i. So, will write it as summation over j first B j and inside is summation over i equal to I have interchanged the order of summation there finite terms only finite sums only. So, that is allowed. So, that is one of mu of A i intersection B j and now we observed that the first sum by the finite addectivity property of the measure is nothing, but mu of A i and this summation over I this sum is nothing, but mu of B j because A i is form a partition of x and here bjs forms.

So, first term is equal to summation i equal to 1 to n A i mu of A i plus summation j equal to 1 to m B j of mu of B j. And now clearly this is integral of s 1 d mu plus the second term is integral is 2 d mu. So, that proves the fact that integration is a linear process namely all integral, if s 1 and s 2 are in L 0 plus then s 1 plus s 2 also is in L 0 plus and the integral is of s 1 plus s 2 is equal to integral of s 1 plus integral of s 2. Next property we want to check is the following that for a set if E is a set in the sigma algebra s and we multiply s nonnegative simple measurable function by the indicator function of e, then that function also belongs to a L 0 plus that again we had checked earlier when we defined nonnegative simple measurable functions. So, it is integral is defined and we want to check that E going to nu of E which is integral of a s indicator function of a d mu is actually a measure on S.

So, this gives a method of generating more measures on the sigma algebra e. So, let us prove this property. So, let us take a function.

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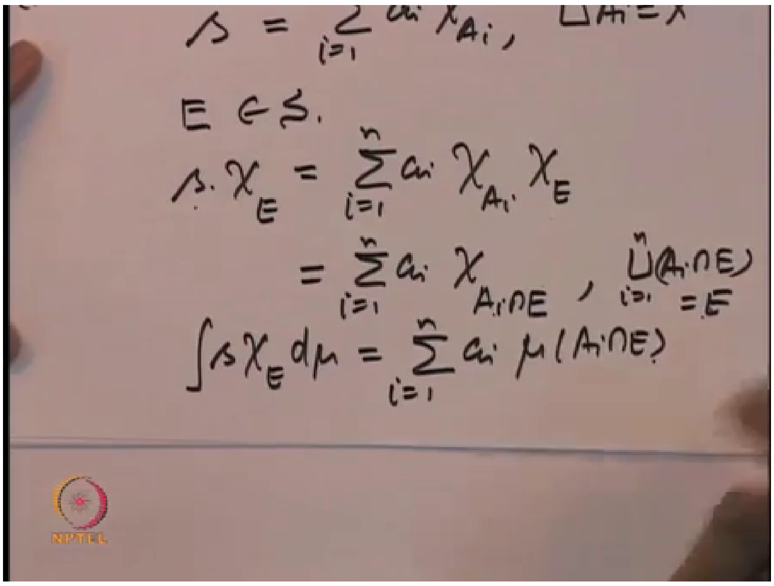

So, let us take a nonnegative simple measurable function L plus 0 s of given by sigma i equal to 1 to n, A i indicator function of A i where union of A i is equal to x. And E is a fixed set in the sigma algebra s. Then s times the indicator function of e. So, multiply this equation on both sides where indicator function that is i equal to 1 to n A i chi A i multiplied by chi of e. And now here is the observation that the product of indicator function of 2 sets is nothing, but the indicator function of the intersection. So, this can be written as i equal to 1 to n A i this product indicator function of A i into indicator function of each can be written as the indicator function of A i intersection E.

So, that is only observation one has to make and now. So, s times indicator function of E is given by this. So, where union of ais intersection E what will be that that is the disjoint union giving you the set e. And on E complement this functions is0. So, if you like you can add one more term here 0 times the indicator function of E complement, but that is not. So, normally whenever the that kind of a set that term will not mention it here. So, a automatically on the complement it is 0 and that gives a partition of the set.

So, this means s of indicator function of E is A i times indicator function of A i intersection E where these things form a partition. So, that implies s times the indicator function of E is a nonnegative simple measurable function and what is the integral of that. So, integral of s chi of E d mu is equal to sigma i equal to 1 to n A i mu of A i intersection E. So, that is a integral of this function. So, we want to prove that, if we call this as nu of E, that is a measure. So, let us check that property to check it is a measure what we have to check.

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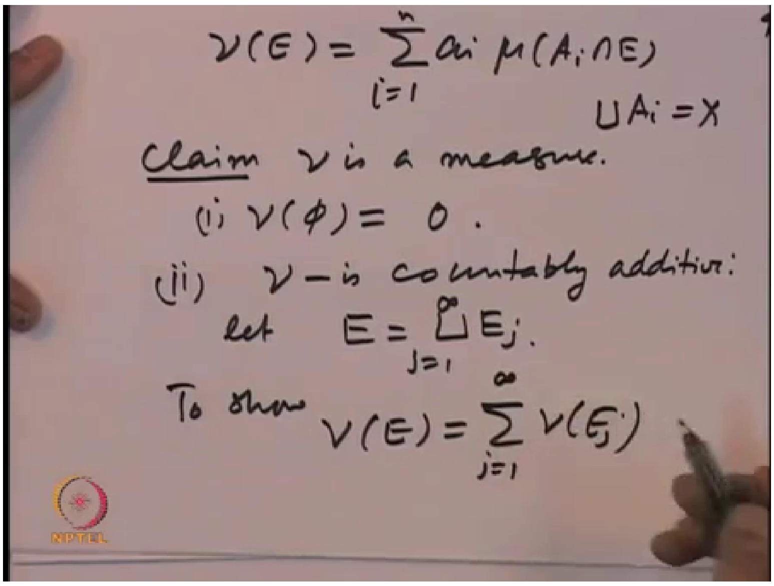

So, nu of nu of a set E is defined as sigma by our previous calculations A i times mu of A i intersection E, i equal to 1 to n where A i is are partition of x and ais are in the sigma algebra always. So, claim nu is a measure right. So, what is to we checked nu of empty set equal to E is empty set. So, mu of A i intersection E that is empty set. So, that is 0. So, it is equal to 0 what is the second property you want to check nu is countably additive. So, for that. So, let us write let E be equal to union of E j, j equal to 1 to infinity where all the sets are in the sigma algebra. So, you want to show that to show nu of E is equal to sigma nu of E j j equal to 1 to infinity. So, that is what we have to show. So, let us compute both sides and show the required property. So, let us look at nu of E.

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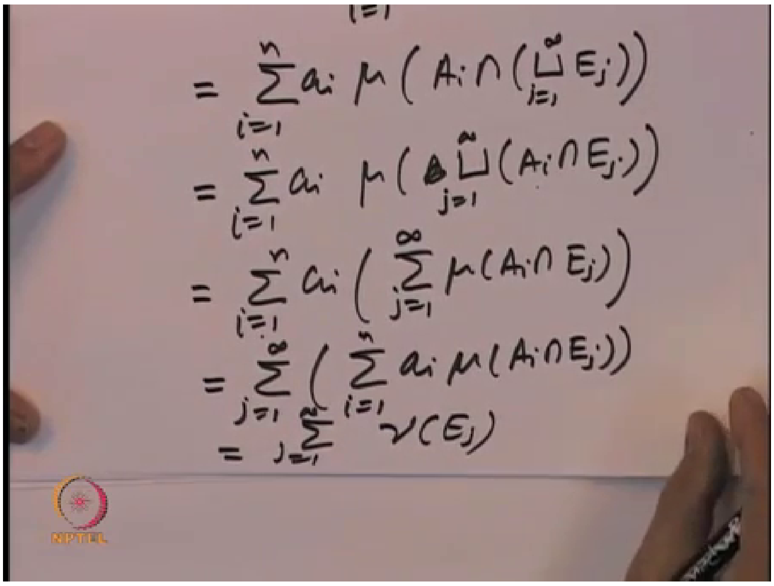

So, nu of E is equal to sigma i equal to 1 to n A i mu of A i intersection E by definition right nu of E is defined as this things in what is E let us put the prop value of e. So, it is i equal to 1 to n A i mu of A i intersection union disjoint union ej j equal to 1 to infinity right that is by the definition of by the fact that E is at disjoint union of ej, but that we can write it as summation i equal to 1 to n A i mu of so this is nothing, but. So, we can write it as disjoint union over j 1 to infinity of A i intersection ej right by the distributive property of intersection over union. So, this is a countable disjoint union of sets in the sigma algebra. So, by the countable additive property of the measure mu this term is equal to summation i equal to 1 to n A i summation j equal to 1 to infinity of mu A i intersection ej right. And now note that we have got 2 sums here one is summation A i another is summation j equal to 1 to infinity and all are non negative extended real numbers. So, we can interchange the order of integration without any problem.

So, we can write this as summation over j first then summation over i 1 to n A i mu of A i intersection ej. So, we write this as this. So, now, note that this the term summation over i A i mu of A i intersection ej is nothing, but the nu of ej. So, by definition this is summation over equal to 1 to infinity. So, this is nu of ej. So, we have shown that nu of E is summation nu of ejs whenever E is equal to union of disjoint by (Refer Time: 30:00) disjoint sets ej. So, that proves that nu is a measure. So, we have proved this property also that for a set E in s the integral s times indicator function of E is a nonnegative simple measurable function, and if it is integral is denoted by nu of E then nu of E is a measure as E varies over measurable sets and this measure has a very nice property. So, this nu measure nu of E has a very nice property that nu of E is 0 whenever mu of E is 0.

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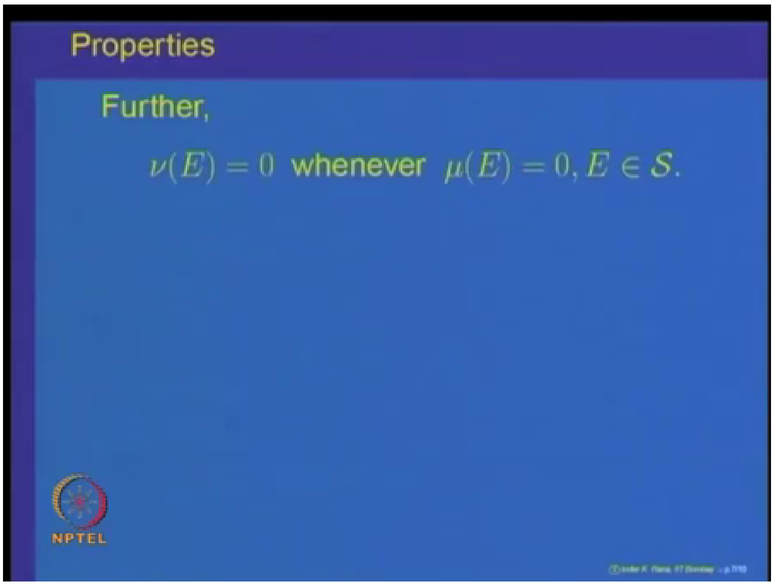

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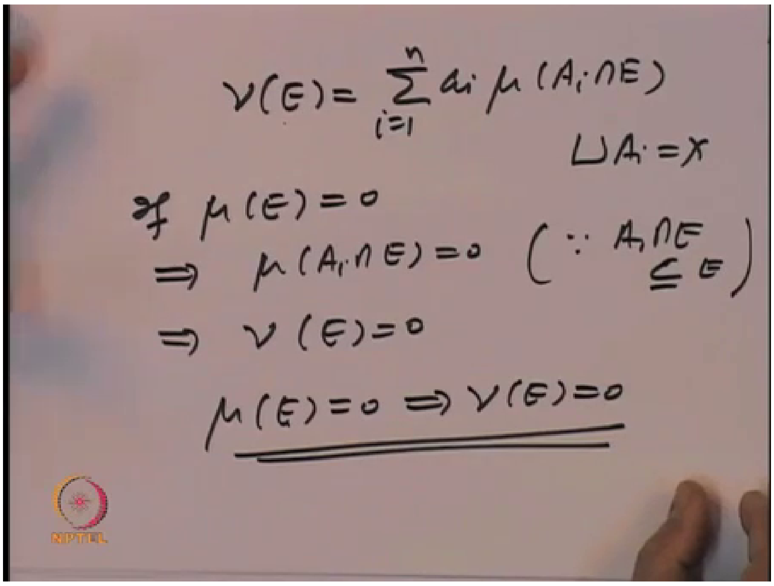

So, let us just check that property again check that property then nu of E is defined as summation i equal to 1 to n A i mu of A i intersection, E where union of A is is equal to x. So, if mu of E is equal to 0 that will imply that mu of each A i intersection E is also 0 because A i intersection E is a subset of E and mu is a measure. So, mu is also monotone right. So, mu being monotone mu of a intersection E is less than or equal to mu of E which is equal to 0 so; that means, this is equal to 0. So, implies nu also. So, each term in the definition of nu of E is zero; that means, nu of E is 0.

So, this nu measure which is defined where integration of nonnegative simple functions has the property that mu of E equal to 0 implies nu of E equal to 0. This say is a very special property. So, it relates 2 measures mu and nu; that means, it says whenever E is a set of measure 0 for mu it is also a set of measure 0 for nu and later on almost in the end of the course, will characterize such measures whenever 2 measures are related by this there is a theorem which says that nu must be representable as integral with respect to mu.

So, will come to that theorem bit late in our course when we are finish integration and some other more properties of it. So, this nu of E which is written as which is a integral is having a special property and let us also mention that integral of s indicator function of E d mu is also written as integral E of s d mu. So, this is another way of writing. So, this is called integral of s over E. So, this. So, you say this is integral of s over the set E. So, that is the notation will follow because outside E s is 0 in this representation.