Measure & Integration
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Lecture – 16 B
Measurable Functions on Measure Spaces

Next, we are going to look at measurable functions which are defined on measure spaces. So, they play also a play role later on. So, we want to define, we want to look at functions F, which are defined on a set X taking extended real values and on X there is a sigma algebra S and a measure mu given on S. So, let us first define what is the meaning of the notion of almost everywhere. So, we say that

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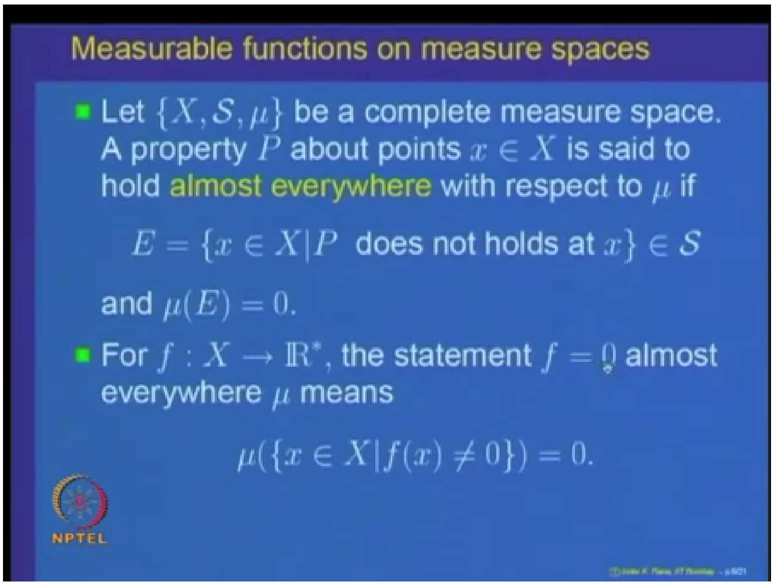

a property P about the points of X is said to hold almost everywhere with respect to the measure mu. If you look at set of points X for which the property P does not hold at X. So, look at all those points of X, such that the property P does not hold at the point X. So, this is a subset and we want this subset belongs to the sigma algebra and mu of E is equal to 0.

So, what we are saying is except for a set of measure the property holds. So, that is why we give it a name that the property P holds almost everywhere with respect to the measure mu. Let me illustrate this with some examples; let us take a function f and we look at the statement that f is 0 almost everywhere. So, f is a function which is extended a function defined on the set X and we want to say that, this function is 0 almost everywhere.

So, look at the set a points where f is not 0. So, what will the statement mean that set where f is non 0, should be a element in the sigma algebra and its measure should be 0. So, X belonging to X say that f of X is not zero, that should be a element in the sigma algebra and its measure is a measure of can be defined only when the set is in the sigma algebra. So, mu of that set is equal to 0. So, the statement that f is equal to 0 almost everywhere will mean mu of measure of the set. A points where fx is not 0 is 0. Let us look at another illustration of this.

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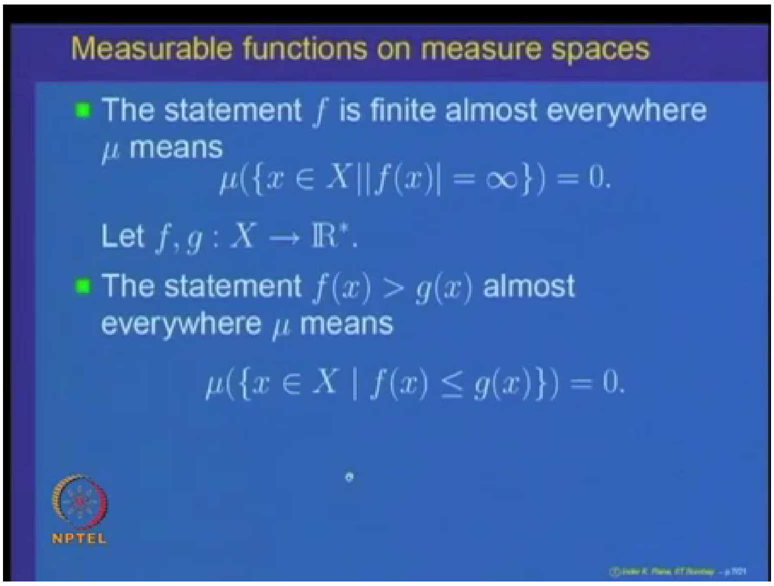

The statement that f is finite almost everywhere what will that mean? So, that will mean look at the set of points where f is not finite; that means, what f is a extended real valued function. So, it can take the value plus infinity or minus infinity. So, the set of points x belonging to x. So, that mod fx is equal to plus infinity. So, that is same as where, either f of x is plus infinity or f of x is equal to minus infinity that set is in the sigma algebra and mu of that set is equal to 0. So, saying a function f is finite almost everywhere means the set of points, where it can take the values plus infinity or minus infinity is the set of measure 0. So, let us look at two functions f and g and let us look at the statement that f is strictly bigger than g almost everywhere. So, f is strictly bigger than g almost everywhere what will that statement mean; that means, the set of points where fx is not strictly bigger than and that is the same as the set of points, where fx is less than or equal to gx.

So, the compliment of that statement is fx less than or equal to gx. These set of points have got measure 0. So, saying that fx is strictly bigger than g almost everywhere with respect to mu means mu of the set, where this statement is not true and that is fx less than or equal to gx is 0.

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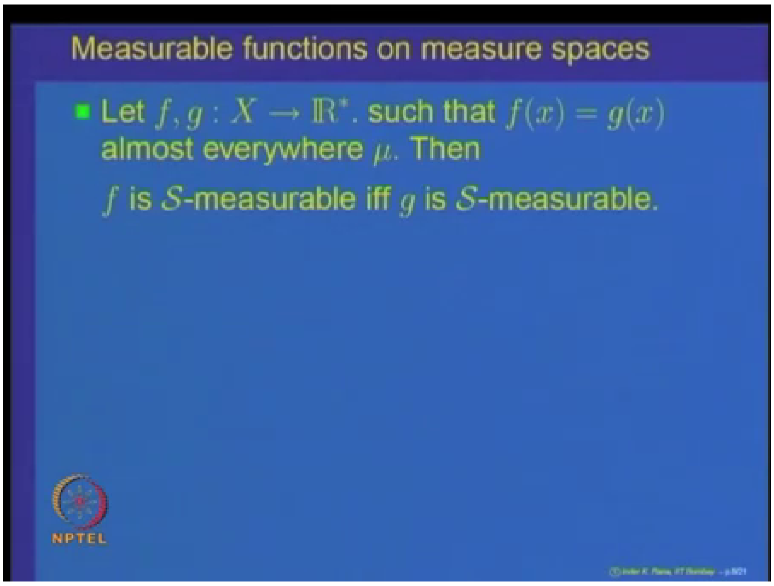

This concept of almost everywhere is quite useful when looking at measurable functions. So, let us prove the property that if f and g are too extended, a real value functions say that fx is equal to gx almost everywhere, mu then measurability of one of the functions f implies the measurability of the other function g. So, f is measurable, S measurable, if and only if g is S measurable. So, let us prove this property that if two functions are equal almost everywhere then the measurability is not change one is measurability of one. So, let us look at f.

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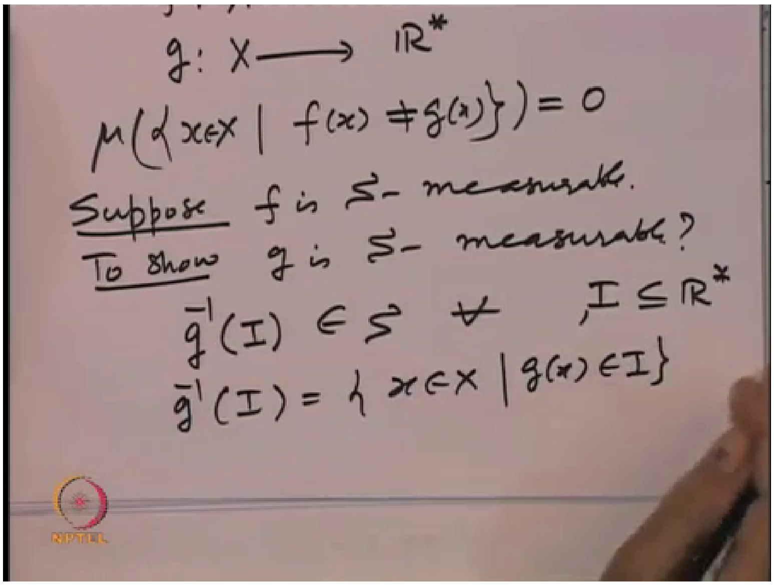

F is from X to R star g is also from X to R star and we know that the set of points X belonging to X. Such that fx not equal to GX. This set as mu measure equal to 0. So, let, suppose f is S measurable to show g is S measurable. So, to show that g is measurable. Let us look at g inverse of any interval I. So, g inverse for every interval I, I an interval in R star then g inverse of I, we want to show that.

So, claim is that this belongs to S for every interval I. Now, we have to transform this property, this set into something regarding g. So, let regarding F. So, let us look at g inverse of I is same as all X belonging to X such that gx belongs to I. So, this is a subset of the set X. So, what I can do? I can write this set S. So, g inverse of I.

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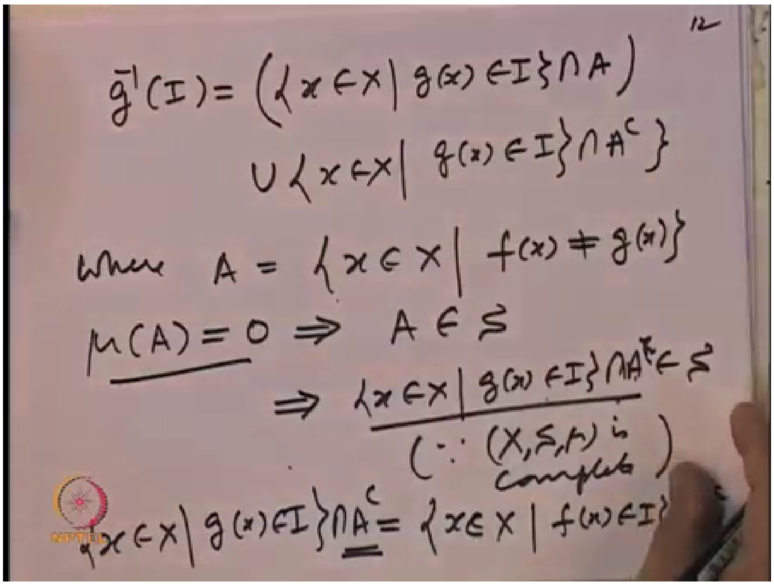

I can write it as intersection of. So, X belonging to X say that gx belongs to I and intersected with the set A also union. So, intersected with A compliment. So, g inverse of I. I have intersected with A and A compliments. So, it is a union of these two sets X belonging to X. Such that g of x belongs to I intersection A complement right. Now, let us look at the first set. So, this is gx belonging to I intersection A and what was the set A, what is set A, where A is the set x belonging to X where fx is not equal to gx FX. So, let us observe where given, we are given that mu of Aequal to 0. So, that automatically implies that A belongs to the sigma algebra S and that automatically implies that the set x belonging to X such that gx belongs to I intersection A also belongs to the sigma algebra. So, this set also belongs to the sigma algebra. Why because this is subset of A and A is a set of measure O. So, this is a set of measure 0 and we have already assumed our measure spaces are complete. So, because of.

So, they implies this is, so, because the measure space X S mu is complete. So, we have made the assumption that we are working on complete measure spaces. So, that gives relay that shows the important of complete measure spaces. So, this set belongs to A and the other part a complement on a complement f is equal to g and replace it by a complement. So, the set X belonging to X such that gx belongs to I intersection a complement is same as the set X belonging to X, where fx belongs to I intersection a complement, because on a complement f is equal to g so; that means, what; so, g inverse of I is written as g inverse of I intersection A. So, let us just rewrite this statement again.

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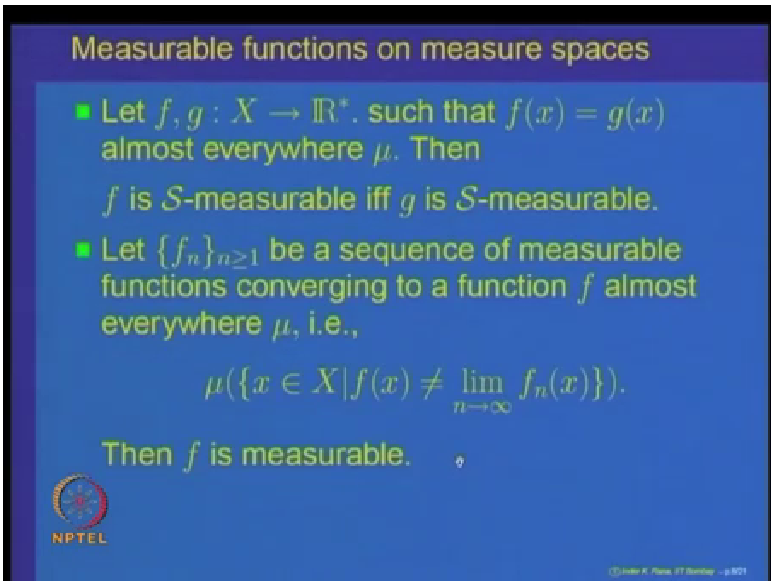

So, what we are saying is g inverse of.

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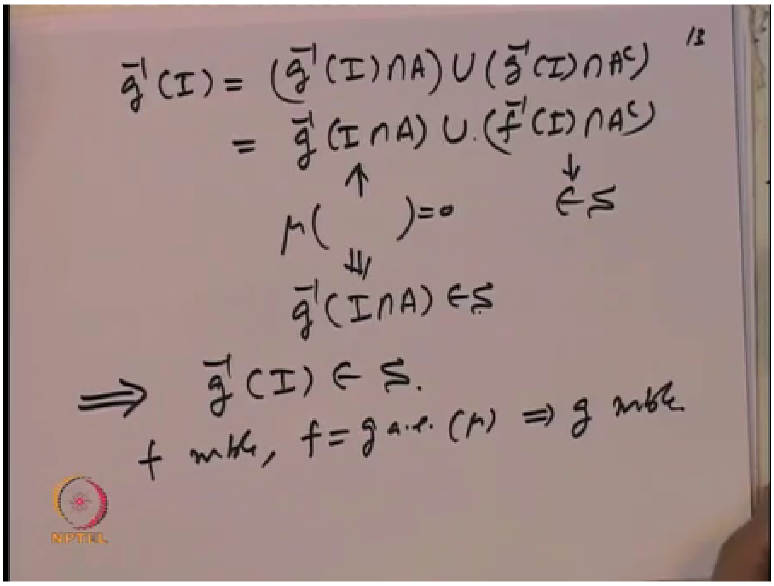

I can be written as g inverse of I intersection A union g inverse of I intersection a complement and that is same as g inverse of I intersection a union f inverse of I intersection a complement, because on a complement f is same as g and this is a set of measure 0 mu of this set is equal to 0. So, implies that this set g inverse of I intersection a belongs to the sigma algebra and this set f is measurable. So, implies this set is in the sigma algebra A is in the sigma algebra. So, a complement in the sigma algebra intersection in the sigma algebra. So, this element belong to the sigma algebra. So, and this is a union of two elements in the sigma algebra.

So, this implies that g inverse of I also belongs to the sigma algebra S. So, we have shown f measurable f equal to g almost everywhere mu implies g measurable. So, that is a importance of measurable functions equal almost everywhere, but keep in mind we have used the fact that underlying measure spaces A is a complete measure space. So, measurable functions; that means, this says that if f is measurable, you can change it's values on a set of mu measure 0 and still the function will remain measurable. So, another interpretation of this result is if f is measurable and is you change its values on a set of measures 0 and call that function as G. So, that is measurable. So, that is a quite important fact another application of this concept of almost everywhere is the following. Look at the sequence fn sequence of measurable functions converging to a function f almost everywhere that is the set of points where fx is not equal to the limit as got. This set has got measure 0, then the claim is, then the f is also measurable. So, just now we have proved that if a sequence fn of measurable functions converges to f than f is measurable and now we are saying that if FNS are defined on a complete measure space and fn converges to f almost everywhere even then this property remains true. So, basically, the idea is same as before. So, let us just look at how does one write a proof of this statement. So, we have got a complete

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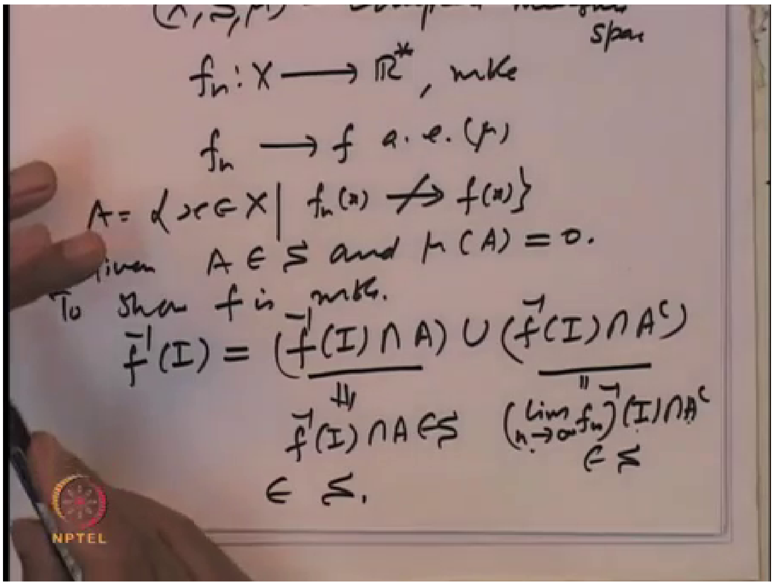

Measure space X S mu measure space. They got a sequence fn of functions FNS measurable FNS converge to f almost everywhere mu so; that means, look at the set A X belonging to X such that fn of X does not converge to f of X, then what is given to us that this set a belongs to the sigma algebra and mu of a is equal to 0. So, now, let us look at, so, we want to show that f is measurable to show f is measurable. So, once again look at for any interval I look at f inverse of I.

So, I can write it as f inverse of I intersection a union f inverse of I intersection a complement right now on A. So, we are given A, it is a set of measure 0. So, this is a subset of A. So, that has set of measure 0. So, this implies that f inverse of I intersection A belongs to the sigma algebra S, because this is a set of measure 0 and our underline measure space is complete and on this portion a complement fn is converging to A. So, this f I can write it as limit N going to infinity of fn inverse of I intersection a complement.

So, this set is same as this and now, we know that fn on a compete FNS converge to f on a complement. So, that is a measurable set. So, this is a element in the sigma algebra, because on a complement fn is converging. So, if we restrict ourselves to a complement, then that must be a element in the sigma algebra. So, both belong to a sigma algebra. So, this belongs to the sigma algebra S. So, the concept of almost everywhere when dealing with complete measure spaces, we can exploit that property. So, this implies that if fn is a sequence of measurable functions converging to a function f almost everywhere then the limit also is a measurable function.

So, this emphasizes the property of something holding almost everywhere. Now, let us specialize.

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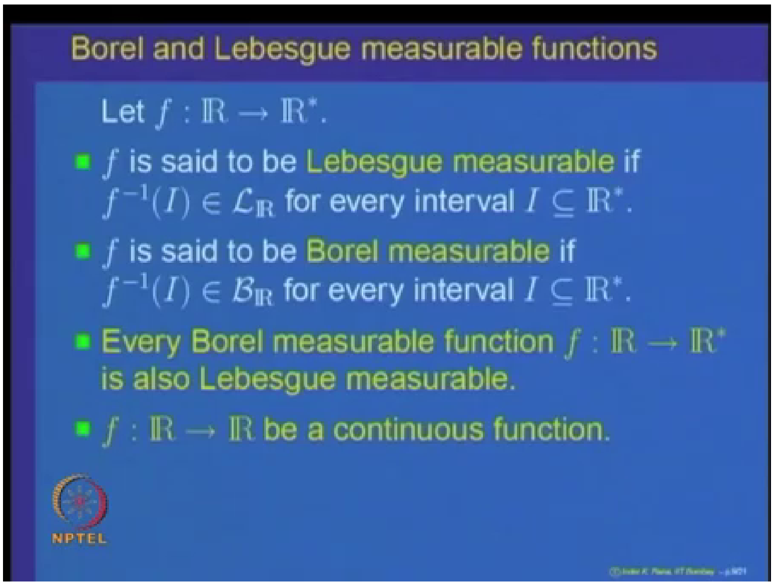

The case when our underlying measure spaces set is the real line, then we have got two sigma algebras, when X is equal to real line, then we have got two sigma algebras one is the Borel sigma algebra and other is the sigma algebra of Lebesgue measurable sets and we have shown that the sigma algebra of Borel subsets is a subclass of Lebesgue measurable sets. So, when we are looking at functions defined on real line look taking values as extended real numbers. There are two possibilities to analyze whether the function is measurable with respect to the Borel sigma algebra or measurable with respect to the Lebesgue sigma algebra.

So, that two notions of measurability as where as real line is concerned and, so, will separate them out. So, will say a function is Lebesgue measurable, if the inverse image of every interval in R star is a Lebesgue measurable set. So, if the inverse image of every interval in R star is a element in is a Lebesgue measurable set, then will say that function is Lebesgue measurable and will say a function is Boral measurable if for every interval in R star. You will pull back its image is a pre image in R is a Boral set in R. So, here is the difference Lebesgue measurable requires that the inverse image is in the Lebesgue sigma, algebra sigma, algebra of Lebesgue measurable sets and f inverse of I in BR says, it is inverse image, is everywhere as always a Borel set in R. So, obvious because Borel subsets for a subset of R.

So, it; obviously, clear that every Borel measurable function is also a Lebesgue measurable function, because inverse image of every interval. If it is in BR and BR is a subset of LR. So, every Borel measurable function is also a Lebesgue measurable function, for example, let us look at function which is continuous. So, if R to R is continuous function, then it is going to be a Borel function. So, let us prove that every continuous function is a bore measurable function and hence, also Lebesgue measurable. So, f is a function,

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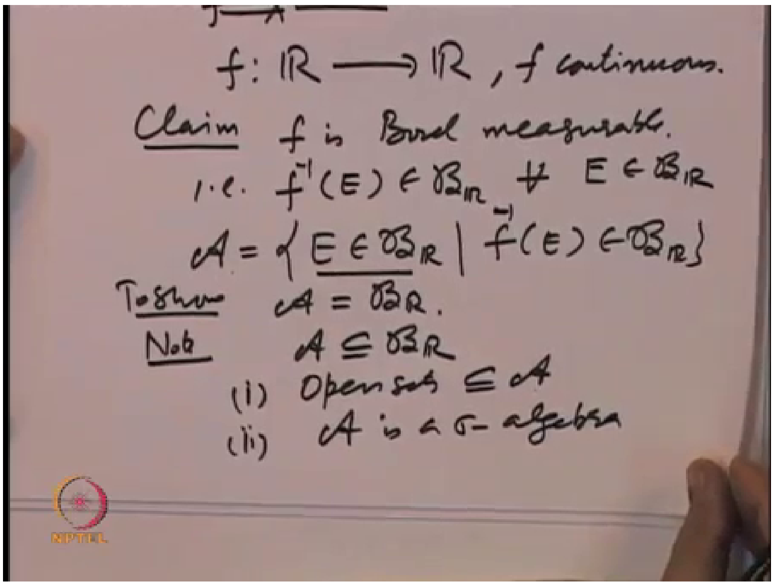

Which is defined from X, sorry X is real line. So, f is a function defined from R to R and f is continuous claim that f is Borel measurable, that is f inverse of any set E belongs to BR for every, for every set. Say E belonging to BR and continuity of a function can be expressed in terms of open sets. So, let us look at the class A of all sub sets E belonging to BR such that, this property is true f inverse of E belongs to BR. So, what you have to show to show saying that f inverse of E belongs to BR for every E in BR. It is on a equivalent to saying to show that this A is equal to BR and that is were we are going to use our sigma algebra technique. So, to show that A is equal to BR. Note that A is already a subclass of BR, because of we are picking up sets in BR. So, to show that A is equal to BR we have to show that BR is inside A.

So, for that will show two steps; one open sets are contained in A and second will show that A is a sigma algebra, because once A is a sigma algebra and include open sets, it must include the smallest sigma algebra generated by open sets that is BR. So, BR will be inside A and will be through. So, to prove this two facts will call it obvious because of the given condition. So, fun open sets belong to A. So, let

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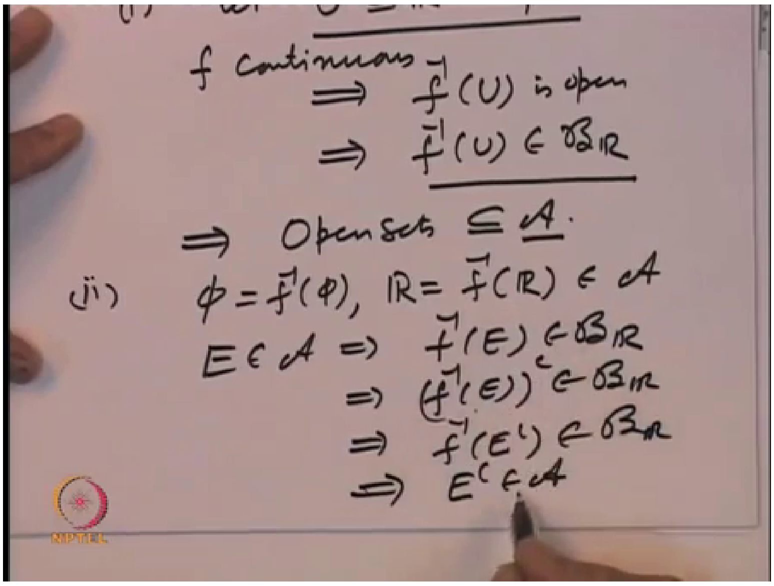

U contained in R, BE open f continuous implies that f inverse of U is open and hence, this means f inverse of U belongs to BR. So, what we have shown is, if U is open then f inverse of U is in BR. So, that proves. So, implies that the open sets are inside A and A is a sigma algebra that is more set forward. So, let us observe, empty set is equal to f inverse of empty set, so, and R is equal to f inverse of R. So, both belong to A. So, because empty set and the whole space they are equal to this. So, this is obvious empty set and the whole space belongs the second property. If E belongs to A that implies f inverse of E belongs to BR and that implies f inverse of E complement belongs to BR and that implies, because this set is same as f inverse of E complement that belongs to BR. So, what you have shown is E belongs to A, then f inverse of E complement belongs to BR and that implies that E complement belongs to a.

So; that means, E complement. So, the class a includes empty set includes the whole space that is closed under complements. So, finally, let us show that it is also closed under countable unions so; that means,. So, let us take sets EN

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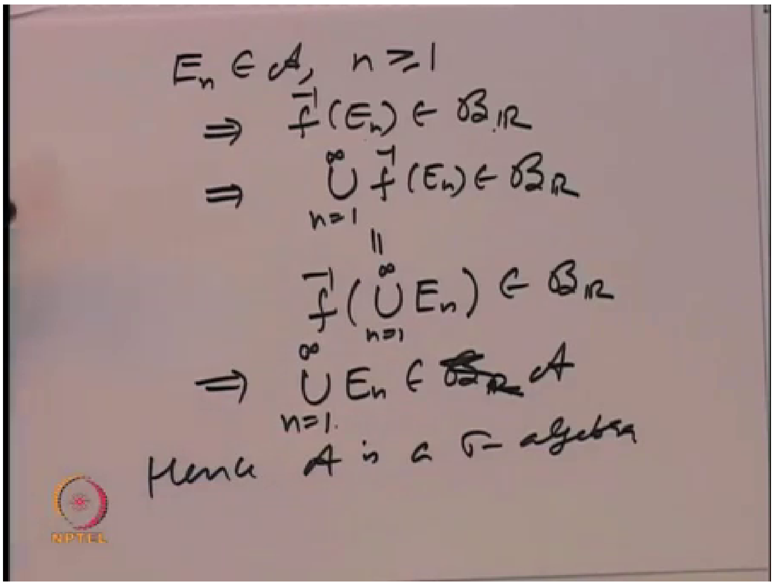

Belong to A N bigger than or equal to one then look at; that means, what we are given that f inverse of EN belongs to the sigma algebra BR, because property EN belongs to a means, then inverse image is in BR, so, but that implies BR is a sigma algebra that implies union 1 to infinity, f inverse of EN belongs to BR and now a simple observation that this set is same as f inverse of union EN, N equal to 1 to infinity that belongs to BR. So, we if EN is belong to A then f inverse of the union belong to BR.

So; that means, union of N equal to one to infinity ENS belong to BR belongs to A, if ENS belongs to A, then f inverse of the union belongs to BR; that means, the union belongs to A. Hence, we have shown that A is a sigma algebra of subsets of A and it includes open sets. So, it must include BR and hence, this is equal. So, that proves that every continuous function from R to R is Borel measurable and hence, it is also Lebesgue measurable. So, all topologically nice functions, continuous functions, become Lebesgue measurable on the real line. Let us look at some more properties.

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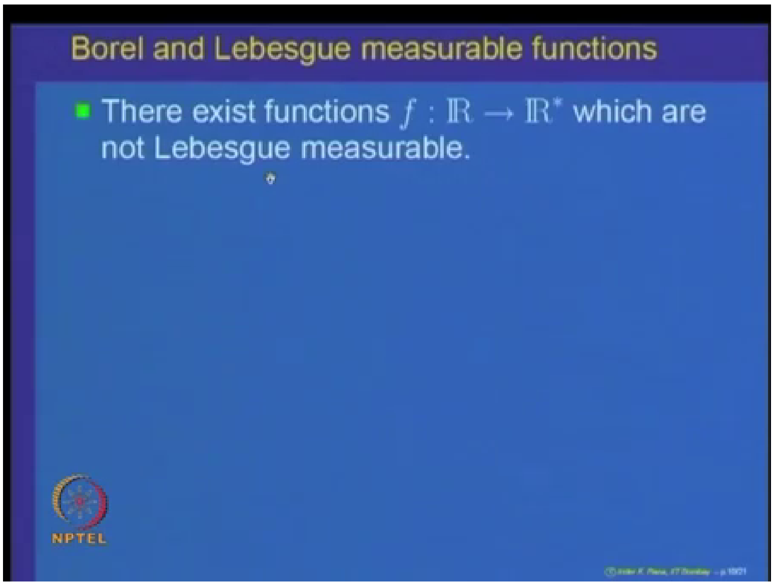

So, we showed that every Borel function is Lebesgue measurable.

So, there exists functions first of all R to R star, which are not Lebesgue measurable. So, to prove that we have to only simply observe that there are sets, let us go back and recall that for f function.

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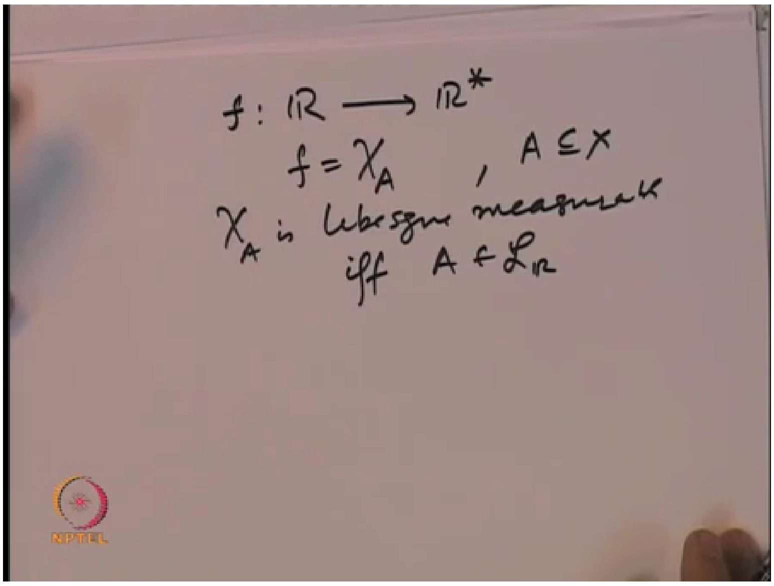

F from R to say R star. Let us f equal to indicator function of a set A, where A is a subset of X. So, recall KI of A is Lebesgue measurable. If and only if A belongs to l of R. So, if you can produce the set which is not Lebesgue measurable, then the indicator function will not be Lebesgue measurable. So, the answer to this question does like this the non Lebesgue measurable functions depends upon whether there is non Lebesgue measurable sets. And if you recall we had proved the fact that the non Lebesgue measurable sets exists, not that question is related to basic set theory. So, if we assume axiom of choice, then we construct it non Lebesgue measurable sets. So, assuming axiom of choice on can claim that there exist functions which are not Lebesgue measurable and by the same reasoning one can ask the question, do there exist functions which are Borel measurable, but not which are Lebesgue measurable, but not Borel measurable, because every Borel measurable is Lebesgue measurable.

So; that means to, for that by a same logic again. If we pick up a set A which is Lebesgue measurable, but not a borel, set then the indicator function of that set is going to be a function, which is going to be Lebesgue measurable, but not Borel measurable. So, these two questions that whether there exists non Lebesgue measurable non Lebesgue measurable functions and whether there exists functions which are Lebesgue measurable, but not Borel get tied up with the fact that the Lebesgue measurable subsets is a proper subset of power set of R and BR is a proper subset of the Lebesgue measurable sets. So, with that we conclude the study of property of measurable functions.

So, basically, let me just recall the measurable functions are functions defined on the underlying sets X with properties that the inverse image of every set E in the Borel sigma algebra of extended real numbers. The inverse image is again in the sigma algebra on the domain space that is S. So, this is a property about the inverse image is of sets being in the sigma algebra S and will see that how this property plays a role in our further study of study of integration. So, will do the in the next lecture will start the notion of integration for measurable functions.

Thank you.