Measure & Integration
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Lecture – 16 A
Measurable Functions on Measure Spaces

Welcome to lecture number 16 on measure and integration. If you recall in the previous lecture, we had started looking at the notion of measurable functions on measure spaces. We will continue that study of a measurable functions and their properties and then will specialize measurable functions on measure spaces. And then we will look at the space of measurable functions on when x is real line and the sigma algebras are other Borel sigma algebra or Lebesgue sigma algebra and if there is a time will start looking at the integration of non negative simple measurable functions.

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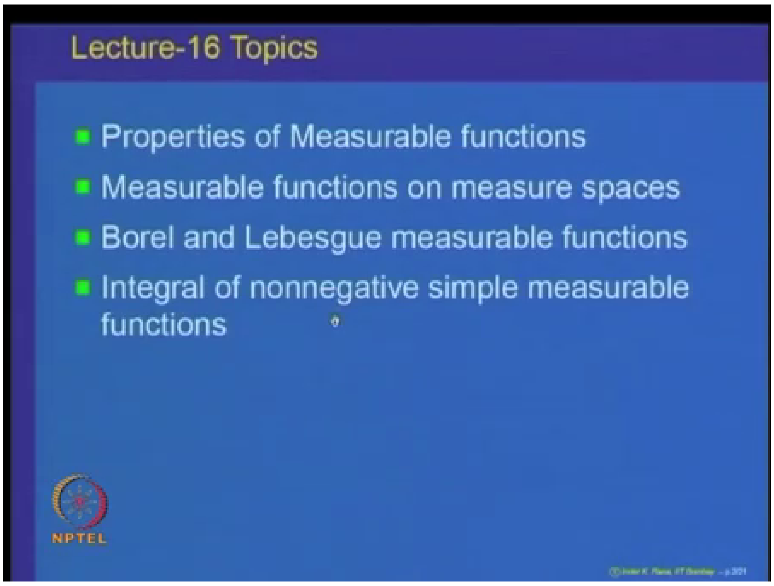

So, let us recall what we had been doing we had been looking at properties of measurable functions; so, that is what will be continue doing. Then we look at measurable functions on measure spaces; look at Borel and Lebesgue measurable functions and then look at integral of non negative simple measurable functions. So, let us just recall what we had done regarding measurable functions.

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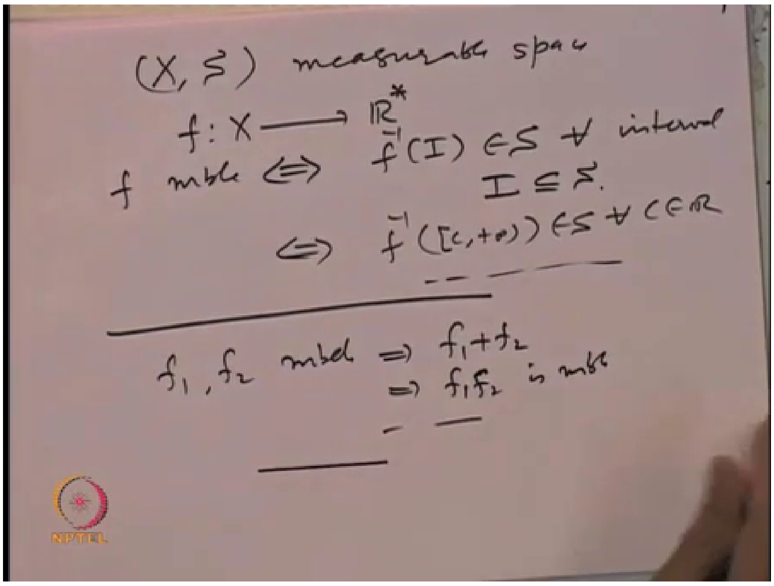

We had said that if X S is a measurable space and f is a function from S to r star; then saying that f is measurable; f measurable it is same as saying the inverse image of every interval; I belongs to S for every interval, I contained in S. And there were equivalent ways of defining measurability; in terms of special intervals like this is same as if and only if f inverse of for the intervals of type C to infinity belong to S for every C belonging to r and so on.

So, and then we looked at what is called the algebra of measurable functions; we proved that if f 1 and f 2 are measurable; then f 1, f 2 measurable implies f 1 plus f 2 is measurable implies; f 1 into f 2 is measurable and so, on. So, today we look at the properties of sequences of measurable functions.

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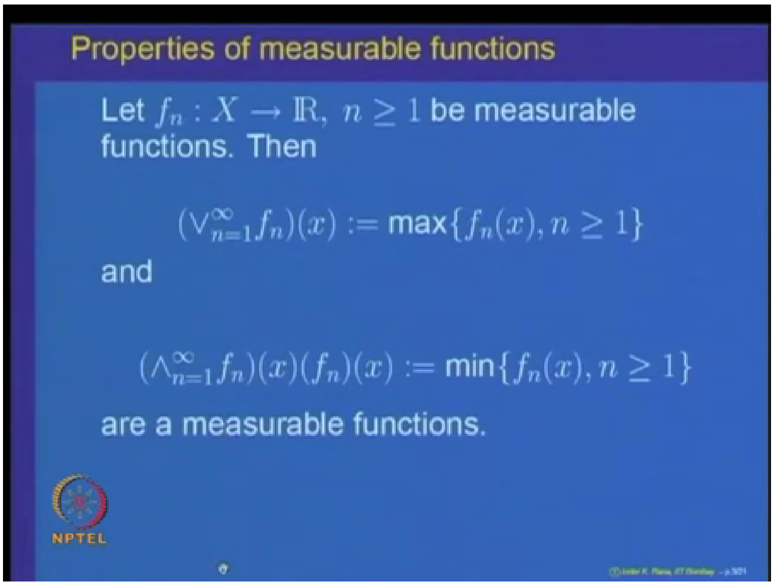

So, we want to prove the following namely; look at a sequence f n of measurable functions; then look at the function what is called the maximum of f n's. So, this is a function denoted by V n equal to 1 to infinity f n of x is defined the maximum of f n of x; n bigger than or equal to 1. So, this is called the maximum of the sequence f n and similarly we have the notion of minimum of f n's; which is denoted by wedge one to infinity f n x equal to minimum.

So, this is this is extra; so, that is a definition. So, claim is that if f n is a sequence of measurable functions; then the maximum and the minimum are also measurable functions. So, let us prove this.

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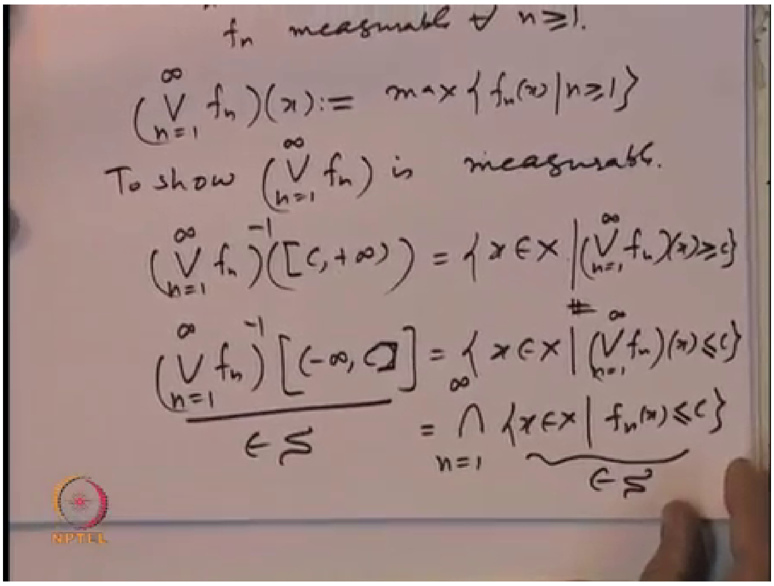

So, f n is defined on X to r star and f n is measurable for every n bigger than or equal to 1. And we define the maximum n equal to 1 to infinity of f n; of x to be equal to maximum of f n x; n bigger than or equal to 1. So, this is a definition of this maximum and we want to prove to show that this function V n equal to 1 to infinity f n; this is measurable. So to prove that we can use any one of those conditions which we had defined earlier for measurability. So, let us look at maximum n equal to 1 to infinity; f n of inverse of the interval.

Let us look at say C to infinity. So, that means, what? That means, this is all x belonging to x; such that n equal to 1 to infinity; f n of x is bigger than or equal to c. So, to prove this now we have to convert some how this relation into individual f n's, because each individual f n is measurable. So, that seems this saying maximum is bigger than or equal to C; that means, at least one of them cross is over C.

So, that is one way of doing it, but let us look at the equivalent criteria namely let us look at the sets f n; inverse the maximum is less than or equal to C. So, that is equal to minus infinity, so to C let us look at that. So, this is same as x belonging to X; such that the maximum value f n x is less than C. Now, if maximum of something is less than C; then each one of them has to be less than C. So, that is the reason instead of using this kind of intervals; it is more convenient for the maximum to use this kind of intervals; because on this set can be written as intersection n equal 1 to infinity of x belonging to x such that; f n x is less than C. So, this is each may be this may not be exactly true.

So, let us take it this close because maximum less than or equal to C then every one of them will be less than or equal to C that is ok. So, we look at the intervals of the type minus infinity C close and that is intersection of the sets and each one of this sets belong to the sigma algebra S. Because each f n is given to be measurable; so, it is a intersection of elements in the sigma algebra. So, this set also belongs to the sigma algebra S; so that means, this proves the fact that the maximum of f n's is a measurable functions. A similar proof will work for the minimum.

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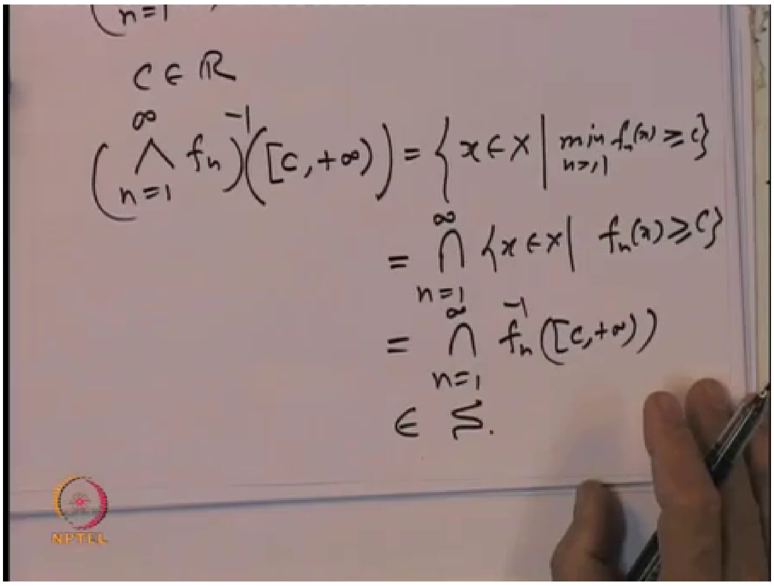

So, let us look at the wedge; n equal to 1 to infinity f n that is defined as the minimum of f n x; for n bigger than or equal to 1. So, claim is that this is a measurable function. So, once again for any C belonging to R; let us look at the minimum of f n; n equal to 1 to infinity, inverse of some type of intervals we want to show it belongs to S.

So, let us try looking at minimum of this is bigger than C, so, let us try this. So, this is all x belonging to X; such that the minimum of f n x; n bigger than or equal to 1 is bigger than or equal to C. So, if the minimum of some certain numbers is bigger than or equal to

C; then each one of them has to be bigger than or equal to C. Because, if even if 1 is smaller; then the minimum will become smaller.

So, this is equal to intersection of n equal to 1 to infinity of all x belonging to X; such that f n x is bigger than or equal to C. So, that implies and this is nothing, but intersection n equal to 1 to infinity of f n inverse of C to plus infinity. And each f n being measurable; this is a measurable set; so, implies that this is a set in S. So, we have proved that if; f n is a sequence of measurable functions, then we look at the maximum or the minimum of the sequence of this measurable functions; then that both of them are again a measurable functions. So, as a consequence of this we prove that the limit of measurable functions is also a measurable function.

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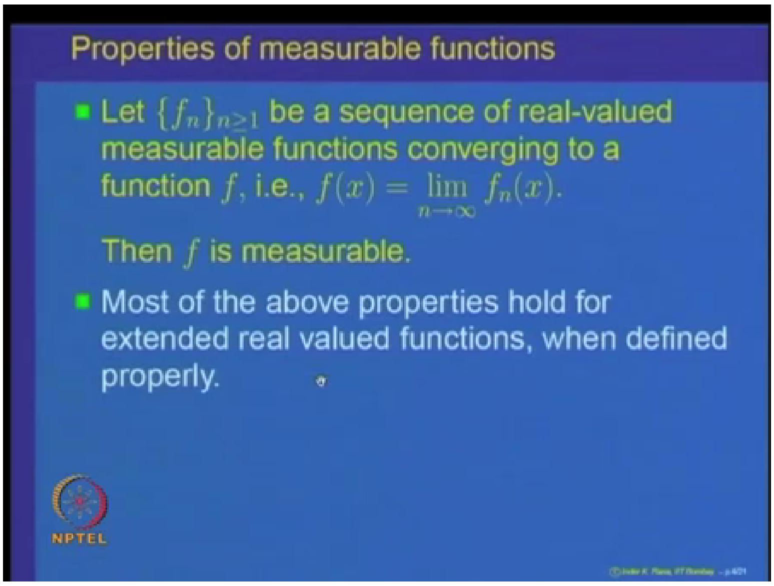

So, that is our next aim to prove that the limit of measurable functions is also a measurable function. So, for that let us understand what is limit of a function. So, let us look at a sequence f n.

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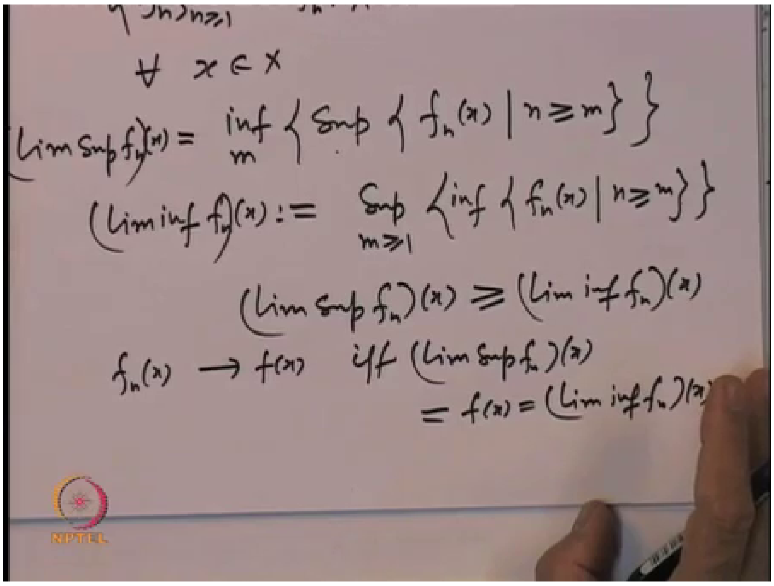

The sequence f n of functions each f n is defined from X to r star. So, to define the notion of limit of the f n at a point X; what we do is; for every x belonging to X; let us look at the maximum or the supremum of f n x; for n greater than or equal to some stage m. So, this number; the supremum that depends on m and then we take the infremum over all m’s. So, this is supremum; first take the supremum and then take the infremum.

So, this gives you a function; this is called limit superior of f n at the point x. So, this is called the limit superior of the sequence of functions f n x at the point x. Similarly limit inferior of f n x is; so limit inferior is defined as you fist take the infremums of f n x; for n greater than or equal to some stage m. And then look at the supremum for all m bigger than or equal to 1; this is called the limit superior.

And you must have seen in your elementary analysis classes that limit superior of f n x is always bigger than or equal to limit; inferior of f n x. So, this in the quality always holds and the sequence f n x converges to f x; if and only if limit superior f n x is equal to f of x; is equal to limit inferior of f n of x. So, these are elementary facts from basic analysis about when is the sequence of real numbers convergent.

So, it says that a sequence for any sequence of a real numbers or extended real numbers you can define the concept of limit superior and also you can define the concept of limit inferior. Limit superior is defined by looking at the supremums of the sequence a n; from some stage m onwards and then this supremum depends on m, so look at the infremum of all this supremums. So, that is called the limit superior. And similarly limit inferior is defined as first taking the infremums of the sequence f n x from some stage m onwards, and then looking at the supremums of these numbers which depend on m.

And one proves that the limit superior of a sequence is always bigger than or equal to limit inferior and the sequence is convergent if and only if the limit superior is equal to limit inferior. So, in case you have not come across this concepts; I strongly suggest that you pick up a book on elementary analysis and revise a concepts of limit superior and a limit inferior. So, we are going to use that fact now here to prove that f is measurable. So, what is f? f of x is nothing but limit superior and limit inferior.

So, only thing to show is that the limit superior and limit inferior are both measurable functions.

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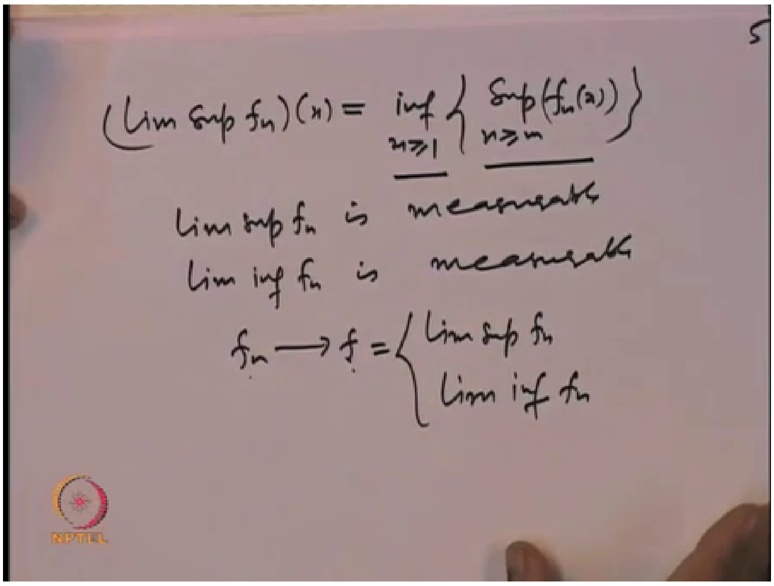

But limit superior of f n is nothing, but first taking the supremums of f n x; from n bigger than equal to m and then taking infremums m bigger than equal to 1. And just now I have shown that if f n is a sequence of functions, then the supremums; the maximums are also measurable functions. So, this is a measurable function; infremums of measurable functions that is a measurable function.

So, this implies that limit superior f n is measurable and similarly limit inferior; f n is measurable. And saying that f n converges to f is same as saying; this f is equal to limit superior f n or also equal to limit inferior of f n's. So, that proves the fact that f n converges to f for every point x; implies and f n's measurable implies f is a measurable function. So, limits of measurable functions are also measurable functions.

So, this proves a theorem that the class of all measurable functions is nice; it is closed under taking point wise limits. Now, let us observe that most of these properties hold for extended real valued functions also when properly defined.

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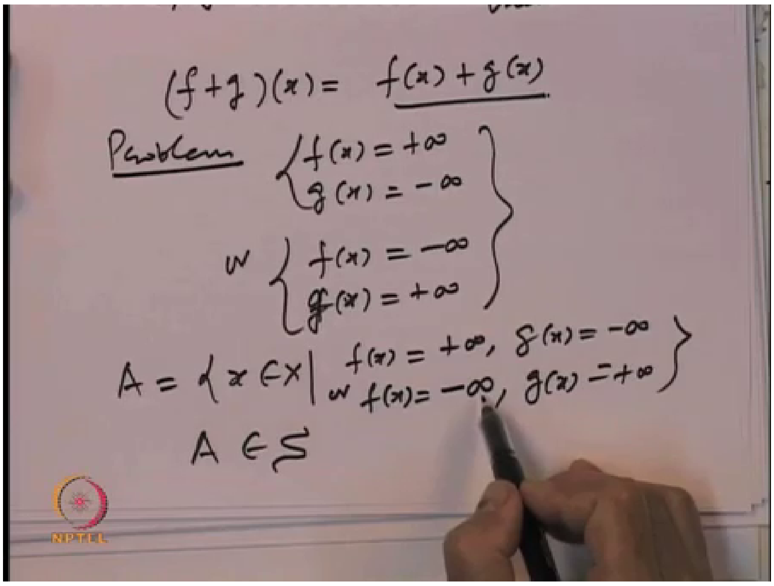

So, because only thing to observe is the following that if f and g are extended real valued function. And you want to define f plus g; a care has to be taken because f of you will like to define it as f of x plus g of x, but the problem comes if f of x is equal to plus infinity and g of x is equal to minus infinity.

Then what will be this number? That is not defined or f of x is equal to minus infinity and g of x is equal to plus infinity; even then the problem comes this number is not defined. So, what one does is to meaning say suitably defined means look at all the points call this that as a; where all x belonging to x where either of these things happen, where f x equal to plus infinity; gx is equal to minus infinity or f x equal to minus infinity and g x equal to plus infinity.

Now, one observes that this set A is in the sigma algebra; because f x is equal to plus infinity belongs to sigma algebra intersection of that belongs to sigma algebra. So, all these sets; all this set A is in the sigma algebra. So, A is the set on which the problem can come.

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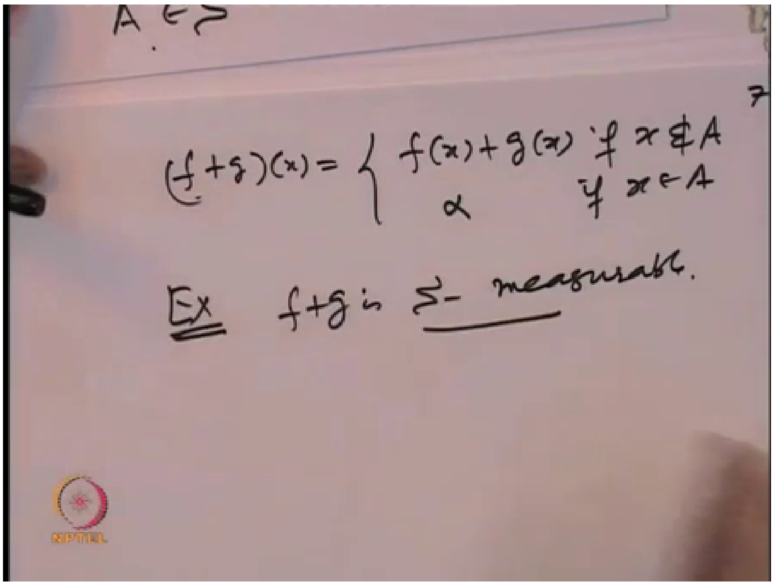

So, what one does is; one defines f plus g of x to be equal to f x plus gx; if x does not belong to A and if x belong to A; you can define it any number alpha if x belongs to A. So, with this definition it is easy to observe. So, let me leave it as exercise for you to show; that if I define it this way with alpha any value on the set A; then f plus g is S measurable. So, that is what I mean by saying that the above results; most of this properties hold for extended real valued functions also; when this functions are appropriately defined.

So, we will not go much into detail of this; one can easily verify these things.

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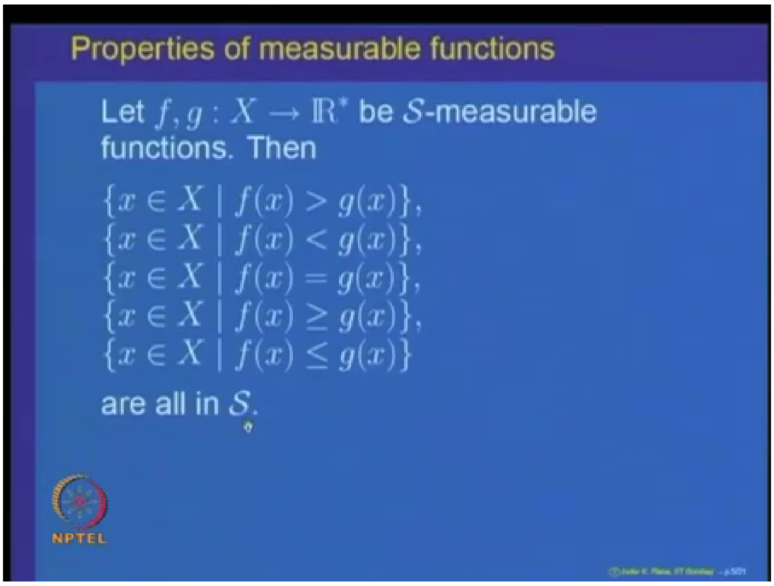

So let us look at f and g to be another property of measurable functions as following; let us look at two functions f and g which are measurable; then the following holds; then look at the sets x belonging to X; say that where f x is bigger than g x or the set x belonging to X; where f x is strictly less than g x or x belonging to X; where f x is equal to gx and similarly, where f x is bigger than or equal to or f x is less than or equal to. So, all these type of sets are claim is; are in the sigma algebra S. So, let us look at proof of one of them and others will follow similarly.

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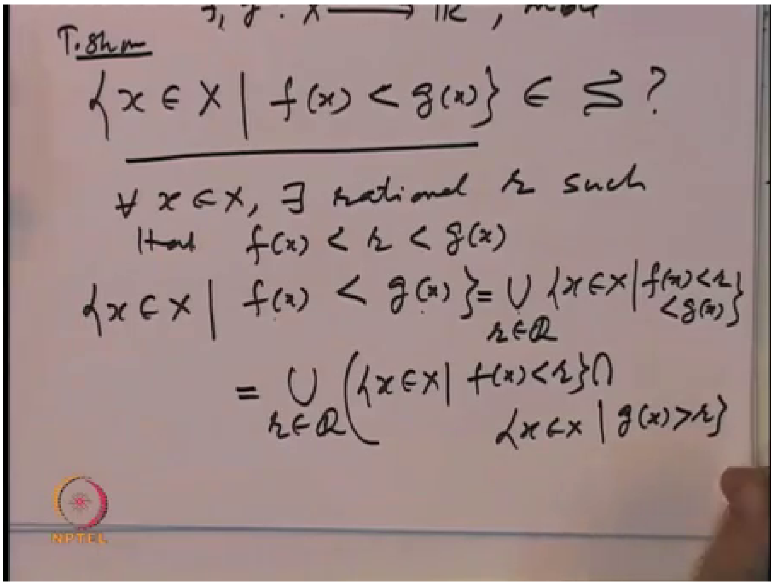

So, f and g are functions X to r star measurable; let us look at the set x belonging to X; such that f of x is strictly less than g of x. And so, our aim is to show that this belongs to the sigma algebra S and since we are given f and g are both measurable; we are given the property that f x less than or equal to some real number is belongs to the sigma algebra. And similarly g x less than or equal to a real number belong to the sigma algebra.

So, objective is try to interpret the set in terms of union intersections of something of sets of the type, where f x is less than something and g s is less than something. So, for that; we observe that for any x; if f of x is less than g x, then there must be a rational number in between them. So, for every x belonging to X; there exist a rational r such that f x is less than r is less than g x.

So, here we are using the fact that rationals are dense on the real line. So, with this property, you can write x belonging to X say that f x; is less than g x. So, this implies that x belonging to X such that for some rational; f x is less than or is less than g x; for some. And conversely; if for some r this is true then; obviously, f x is equal to g x. So, claim is this is equal to union over r belonging to rational numbers.

So, this is the only crucial point in this that the set f x less than g x can be written as a union over all rationals; such that f x is strictly less than R; strictly less than g x. And now observe this set is a intersection; so, I can write it as r belonging to Q; this set is where f x is less than r and g x is bigger than r. So, it is x belonging to X such that f x less than r intersection; with the set x belonging to X; such that gx is bigger than r.

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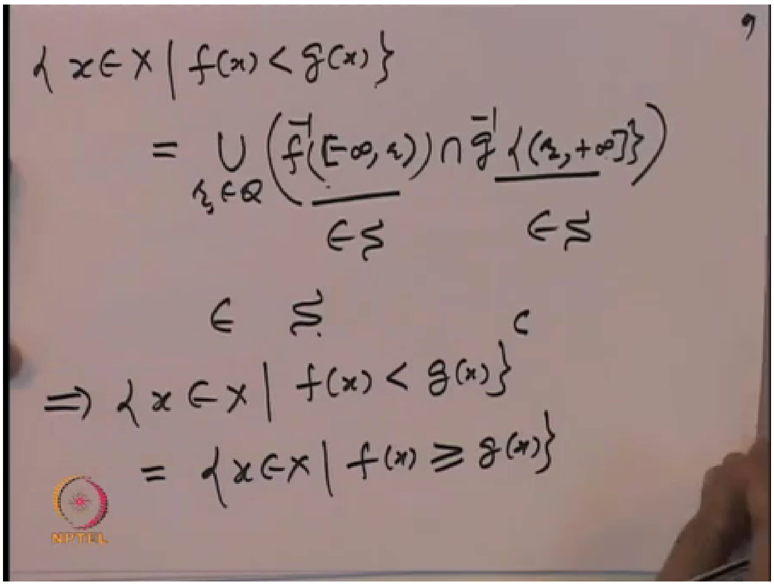

So, what we have done? We have interpreted the set x belonging to X such that f x less than gx as; so, let us just look at the set again. So, this is union over r belonging to Q; what is this set? This is f inverse of f x less than r; that is less than r means it is minus infinity to r. And the second set is nothing but g inverse of gx bigger than r, so it is r plus infinity.

So, the set f x less than gx is written as union over rationals intersections of these two sets. Now f and g being measurable; this set belongs to the sigma algebra; g being measurable, this set belongs to the sigma algebra it is intersection. So, the whole set belongs to the sigma algebra intersection belongs to the sigma algebra and rationals are countable. So, this is a countable union of elements in the sigma algebra. So, this belongs to the sigma algebra; so, what we have shown is the set x belonging to X. So, here f x strictly less than g x; belongs to the sigma algebra S. So, that proves the first property of the theorem; now if you take just the complement of this set. So, also implies that x belonging to X; such that f x less than g x the compliment of this set what will be that.

So, that is all x belonging to X such that f x is bigger than or equal to g x. So, that set also belongs to the sigma algebra and similarly the argument; similarly you said that f x less than gx belongs to it.

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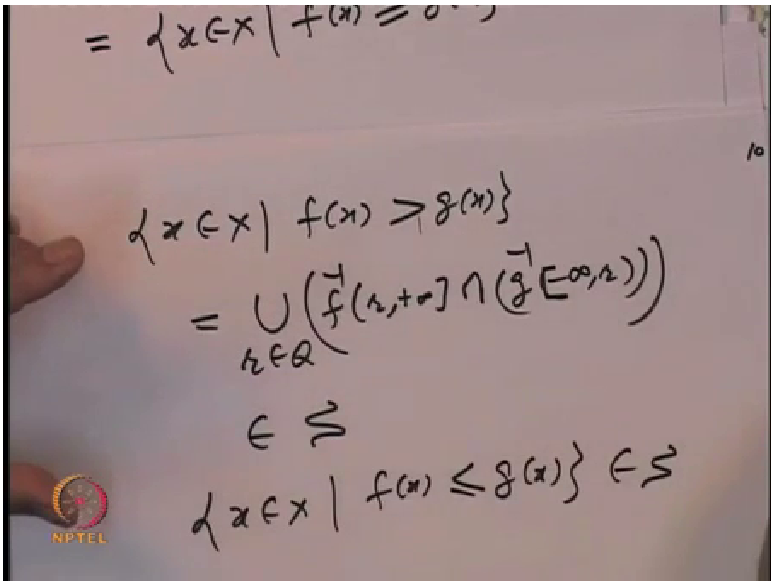

So, let us write x belonging to X; such that f x is strictly bigger than g x is also in the sigma algebra. Because by similar arguments, I can write this as the union over all rationals of f inverse of; so f x bigger than; so that will be r plus infinity in intersection with g inverse of minus infinity to r; so, by similar argument, where we had f x less than again we can inter f x is bigger than g x. So, there must be a rational in between. So, that must be true and that will imply that this belongs to the sigma algebra. So, saying that f x bigger than g x belong to the sigma algebra is ok and if you take the complement of this that is nothing, but x belonging to x such that f x less than or equal to gx; so, that also belongs to the sigma algebra because this is the complement of the set in the sigma algebra.

So, measurable sets have nice properties; namely if f and g are measurable; then operations involving measurable sets, measurable functions give you again sets in the sigma algebra. So, these are nice properties and will see a use of these properties soon. So, with this we complete the study of measurable functions on measurable spaces.