Measure and Integration
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Lecture – 15B
Properties of Measurable Functions

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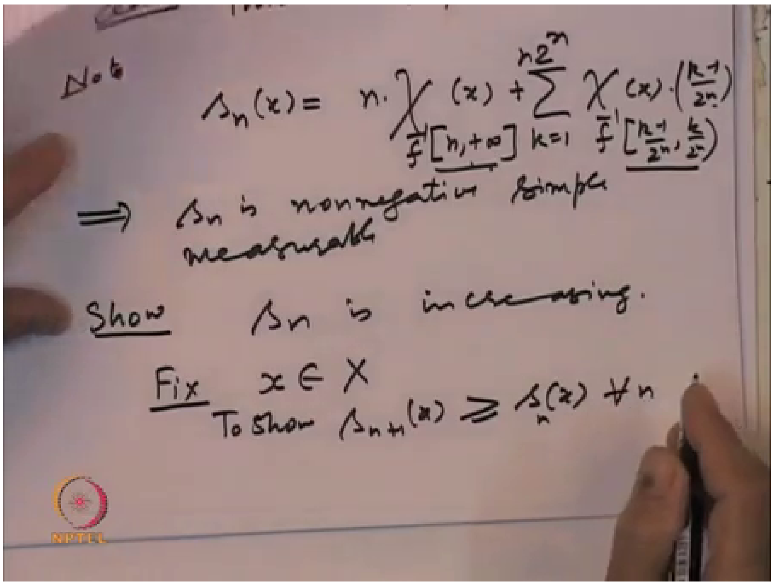

And now let us prove that this is sn is so, claim we want to show sn is increasing. So, let us fix x belonging to x, to show sn plus 1 of x is bigger than or equal to s of x sn of x for every n right. So, let us look at that. So, why is that true?

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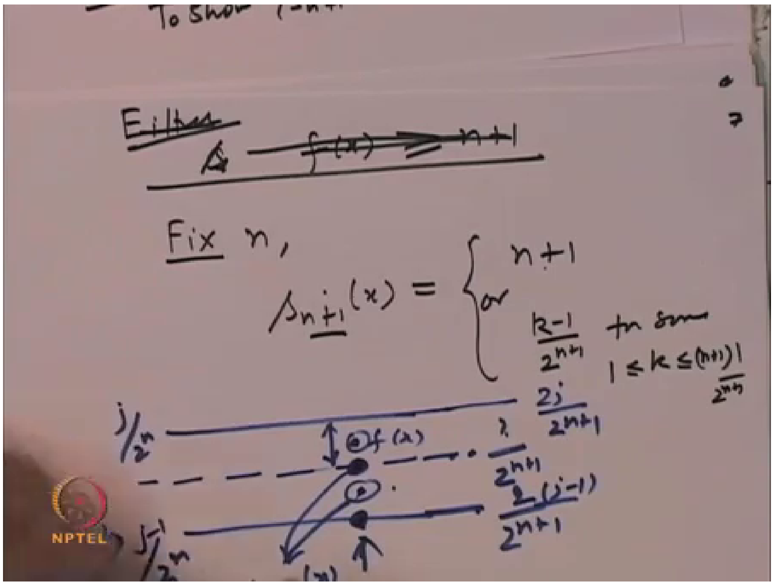

now look what is the value of either f of x is bigger than or equal to n plus 1, is no let us look at the slightly differently. So, I want to prove that sn is increasing. So, to prove the increasing part.

So, let us. So, fix n right. So, we want to look at. So, what is sn plus 1 of x, say that is going to be dependent upon whether. So, either it is n plus 1 or it is going to be some k minus 1 over 2 to the power n plus 1 for some k between 1 and n plus 1 times. So, 1 over 2 to the power n plus 1. So, what do I saying is sn plus 1 x either it will be bigger it will be equal to n plus 1 that will be the case if f of x is bigger than or equal to n plus 1. or it will be equal to 1 of lower values of one of the sub intervals at the n plus 1 th stage in the n plus 1 th stage we will be dividing the interval into 2 n plus 1 parts.

So, let me write this draw this picture slightly here to understand what is happening. So, here is so, the here is let us say j by j minus 1 by 2 to the power n. and that is j to the power 2 to the power n at the n th stage right f of x is somewhere in between. now at the next stage what we are doing we are going to divide this into 2 equal intervals. So, that this part is this part is something divided by 2 to the power n plus 1 and this part is something divided by 2 to the power.

So, what will be this. So, this will be 2 times j minus 1 divided by 2 to the power n plus 1. and this part would be 2 j divided by 2 to the power n plus 1. and that is a mid-point middle line in between. So, now, my sn plus 1 x depending on fx, if f of x is here if this is f of x then sn plus 1 is this is a value of sn plus 1. So, this is the value of sn plus 1 and if f is here, then this is the value of sn plus 1. So, either f of x will be here or it will be here ok.

So, if f of x is here it lies in that interval of length 2 to the power n plus 1, the value is a lower end point. So, value of sn is here, but in that case what is the value of this is the value of sn plus 1. So, this is the value of sn plus 1 x and what is the value of sn, that is always going to be equal to this value. and if f of x is this then this is the value of so, sn plus 1 x either will be here or it will be here and sn plus 1 of x will always be here. So, this value is less than or equal to this value. So, that analysis let us just write. So, it is equal to this. So, in either case.

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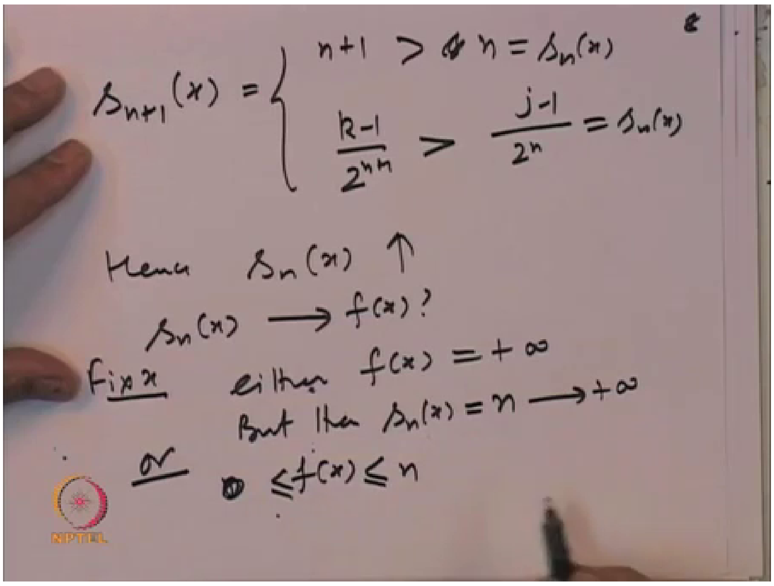

So, sn plus 1 of x either it will be n plus 1. So, either it will be n plus 1 or it will be which is bigger than n which is equal to sn of x. if not if it is bellow right then the value is if it is in one of those intervals then the value is going to be some k minus 1 over 2 to the power n plus 1 right. which is always going to be equal to 2 to the power n the lower value here. So, let us it is difficult to write those symbols. So, that lower value is k minus 1. So, this is k minus 1. So, that is less than or equal to k minus for some j and that will be equal to j minus 1 over 2 to the power n which will be equal to sn of x geometrically it is quite clear what is happening.

So, either if f of x is here in between here and the value of sn plus 1 is this value. and if f of x is here sn if plus value is this one which is a value of sn also. So, in either case. So, this implies hence sn x is increasing. and let us prove that snx converges to f of x that the limit is equal to f of x. So, fi fix s if x is fixed. So, the either f of x is equal to plus infinity that is one possibility, but then if this is plus infinity, what is sn of x that is always f of x is always bigger than n for any n. So, snx is going to be equal to n which goes to plus infinity sn goes to infinity. So, if f of x is plus infinity then sn x is equal to n for every n and hence it goes to plus infinity. or what is second part of building f of x is not infinity; that means, it is a real number. So, it will lie between some n it will be less than or equal to less than or equal to some n.

So, it will be some 0 less than or equal to n. So, then in that case, there exist a n such that this is happening. So, also will be less than n plus 1 and so on. So, and what is sn of x in that case in that case sn of x is going to be some k minus 1 over 2 to the power n right if this is less than this then f of x will belong to one of the intervals k minus 1 or 2 to the power n and k by 2 to the power n right. So, implying that is a same. So, what is the difference between sn, and this sn is the lower value f is something it between.

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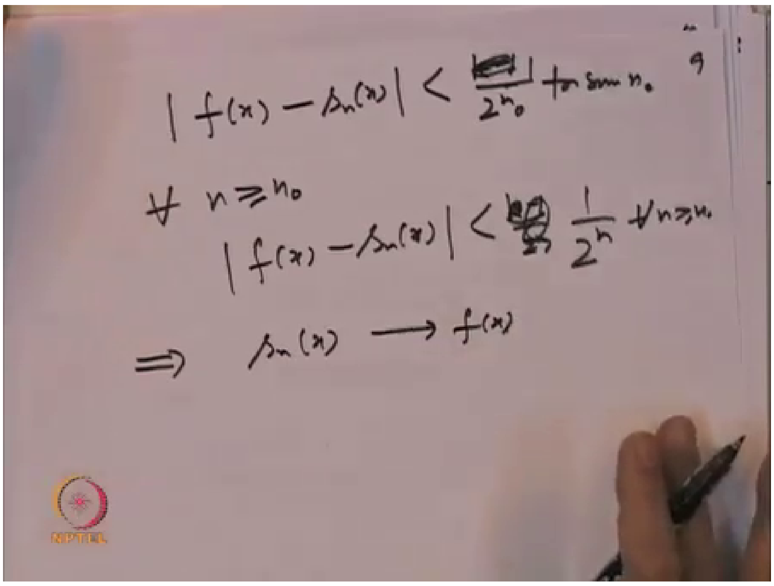

So, that implies that the absolute value of f of x or actually f of x is bigger. So, minus sn of x will be less than k minus 1 by 2 to the power n for some n and; that means, if this is happening for some n.

So, let us say for some n naught then for every n bigger than or equal to n naught f of x minus sn of x is less than right. it will be less than those k minus 1 by 2 to the power n for some k and some n.

So, it will be less than sorry not k minus I, 1 over 2 to the power the difference will be at the most both lie in the interval of length 2 to the power n naught. So, it will be less than 1 over 2 to the power n 2 to the power n for every n bigger than n naught and that implies that sn x converges to f of x. now so let me just go over to the construction once again to understand because this is a important construction.

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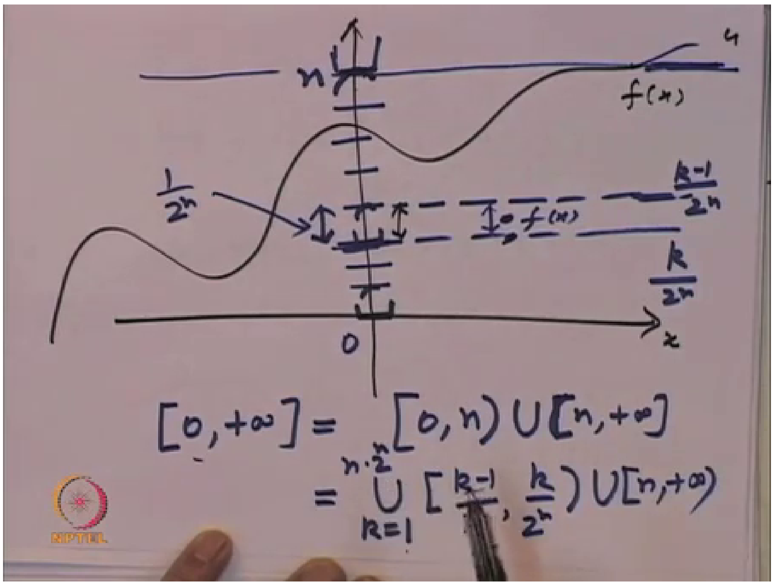

So, it says that I want to even a function f which is non negative measurable we want to construct a sequence of simple functions, which are non negative which are increasing and they converge to f of x. So, what we do we divide the range. So, this is the range of the function is a subset of it. So, it divided into a partition the range. So, partition into 0 to n. So, this is 0 and this is n union n to infinity upwards. So, this is and the portion 0 to n is divided into sub intervals each of length 1 over 2 to the power n. So, this will look like k minus 1 by 2 to the power n k by 2 to the power n k equal to 1 from 1 to how many such intervals will be there each of length 2 to the power n total length is n. So, n times 2 to the power n right.

So, this is we have partitioned range now, given a point x f of x either it will be beyond n or given n either it will be beyond n, or it will be between 0 to n. if it is beyond then we define sn of x to be equal to n. So, if the value of f of x is bigger than n then we define it to be equal to n, n if it is not then it will be between the interval 0 to n. So, it will fall into one of the sub intervals is somewhere here in some k minus 1 by 2 to the power n k by 2 to the power n. So, we define this lower value of that interval that is k minus. So, this is k and this is k minus 1.

So, the lower value whether to be equal to the value of the function sn x. So, this sequence is increasing see for any point x f of x, and sn will be at the most very difference of 1 over 2 to the power n right for n large enough or if not then sn will go to infinity. So, that is the idea that it converges and increasing once again comes from the fact that we are taking the lower value at every stage. So, at any stage either sn plus 1 is bigger than n plus 1, in that case it will be bigger than n also. So, sn plus 1 will be bigger than sn. if not it will be in one of those sub intervals or le length 2 to the power n plus 1, but how did we get those. So, that is so, this total length is 1 over 2 to the power n.

So, when you want to divide the next stage from sn to sn plus 1 we divided it into 2 equal parts. So, if f of x is here then sn plus 1 is the lower value, here or if a sn plus 1 here it is a lower value. So, in either case sn is always going to be the lower value. So, that says it is increasing and convergent. So, that proves the theorem that.

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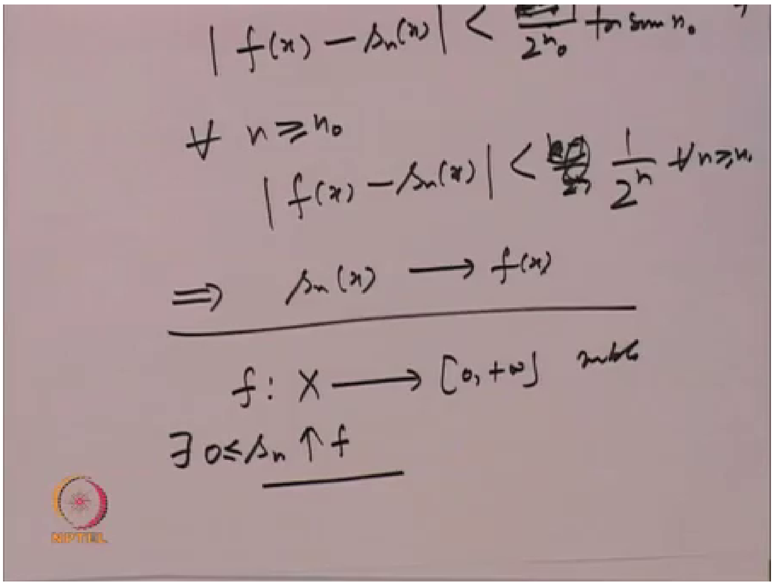

So, what we have shown is the following given a function f x to 0 to plus infinity measurable, there exists a sequence sn or non negative functions which are simple and measurable increasing to f. So, that is what we have proved.

So, let us come back to the theorem which said that.

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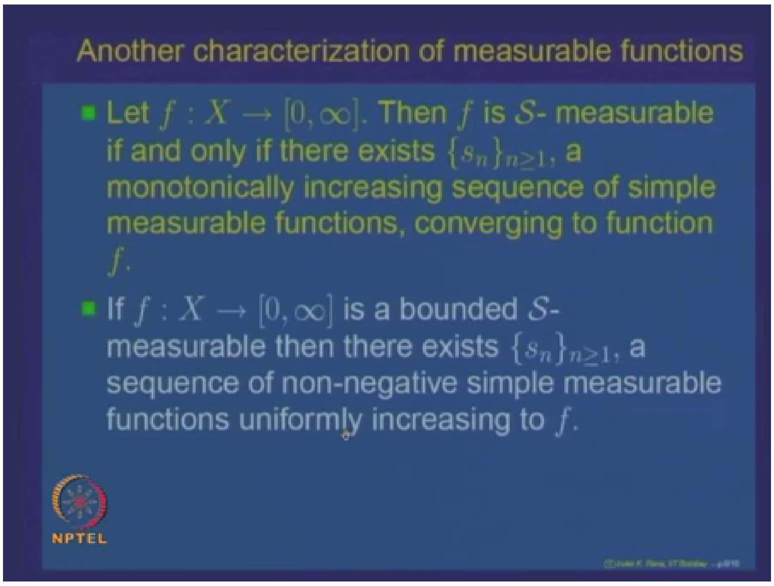

Because this is one of the key theorems in the notion of the for the concept of measurable functions. that every non negative measurable function can be approximated can be obtained as a limit of non negative simple measurable functions and these non negative simple functions can be selected to be a increasing sequence. So, you can approximate a non negative function as a limit of increasing sequence of non negative simple measurable functions. So, this immediately gives us a corollary for functions which are not non negative, but let us before that let me just observe that this sequence this in the proof if the function f is bounded, then the sequence can be chosen to be sn to be uniformly increasing to f not only it converges point wise to f you can actually claim that it converges to f uniformly. So, to prove that it converges to f uniformly if we just observe because the function is bounded. So, let us just observe that if the function is bounded; that means, what; that means, f is a bounded function.

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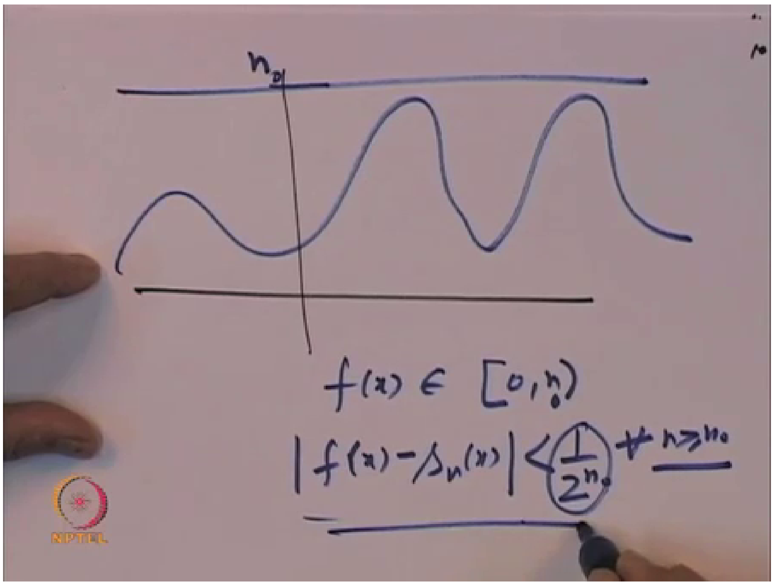

So, it is graph. So, there is going to be n. So, that the graph of the function always stays bellow this right. So, once n is fixed; that means, f of x is always going to belong to 0 to n, for some n and for that fn f of x right.

So, let us say n naught. So, n naught is the bound for the function. So, then fn is minus sn of x is going to be less than 1 over 2 to the power n naught for every n bigger than n naught. So, this works for all. So, given epsilon bigger than 0, I can select n naught such that this is true; that means, the same epsilon works for every x; that means, the sequence sn converges uniformly to f of x. So, this is a observation which we do not may not be needing it, but is good to observe that if f is a bounded measurable function that. So, this theorem says that if f is a bounded measurable function, which is non negative then there is a increasing sequence of non negative simple functions uniformly converging uniformly increasing to f. So, this is the case for when the function is non negative. in the general case for a general measurable function, we can look at the positive part and the negative part of the function approximate the positive part by a sequence approximate the negative part by a sep sequence of simple functions. and then look at the difference of the 2, and that will give us a sequence of simple measurable functions converging to f, they will not have any longer be monotonic.

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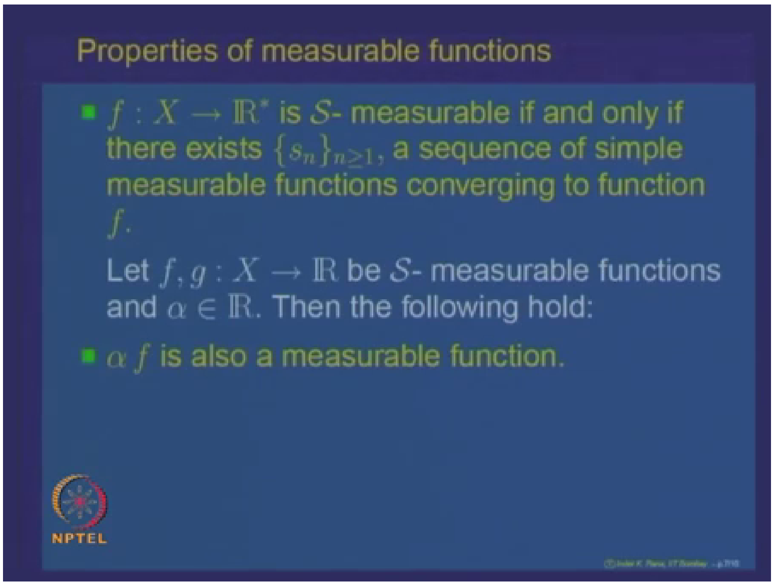

So, as a consequence we had that if f is non necessarily a non negative function, if f is a measurable function right. Then there exist a sequence of simple functions converging to it. So, what we are saying is if f is measurable.

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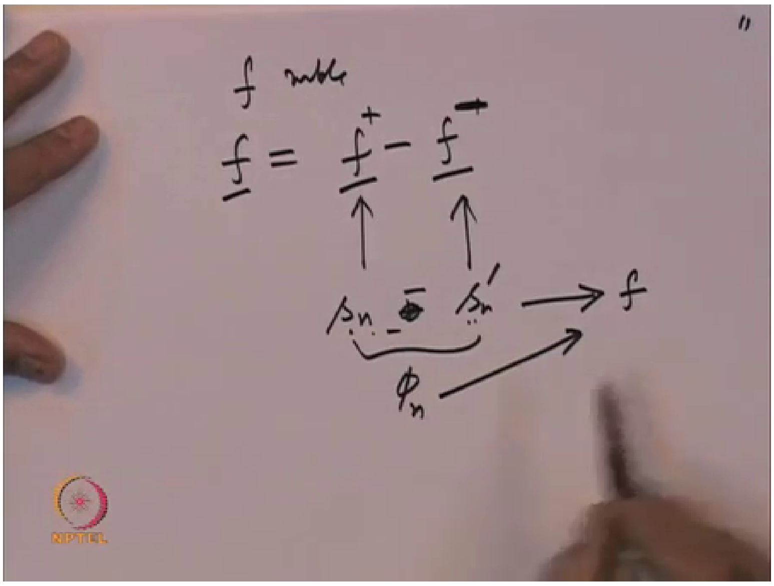

Then I can write f is equal to f plus minus f minus, f minus and we just now observed f measurable implies both of them are measurable. and for this there is a sequence sn which is increasing to of simple measurable functions non negative increasing to this there is a sequence another sequence call it as say sn dash which is again non negative simple measurable functions increasing to f dash. So, if I look at this plus this then that sorry this minus this minus then this will converge to f. So, call this as your new sequence. So, this is call that as phi n. So, phi n is a sequence of because difference of simple measurable functions is measurable.

So, this is a sn is non is a simple measurable function sn dash is a simple measurable function. So, phi n is a simple measurable function sn converges to f plus sn dash converges to f minus. So, the difference will converge to the difference which is f only thing is so, this is a phi n converge to f, but we cannot say phi ns are increasing, anymore this each one of them is increasing, but the difference may not be increasing. So, that proves that for a general measurable function is measurable if and only if there is a sequence of simple measurable functions converging to f.

So, now let us look at some more general properties of measurable functions. let us take, we are going to look at the various properties given 2 functions f and g which are measurable given a scalar whether some of the measurable functions is measurable or not, whether the product or measurable functions is measurable or not whether scalar multiple of a measurable function is a measurable or not. So, let us list all the properties which are true. So, first say if f is measurable and alpha is a scalar then alpha times f is also a measurable function.

So, to for that this alpha could actually be any extended real number also depending upon because we are taking only. So, keep in mind I am taking only real valued functions for the time being f and g are both real valued functions which are measurable and alpha is a real number. So, the claim is alpha f which is again a real valued function is measurable. now we can for this we can apply our sequential criteria because f is measurable. So, there is a sequence of simple measurable functions converging to it and So, look at the sequence.

So, let us look at the proof of this.

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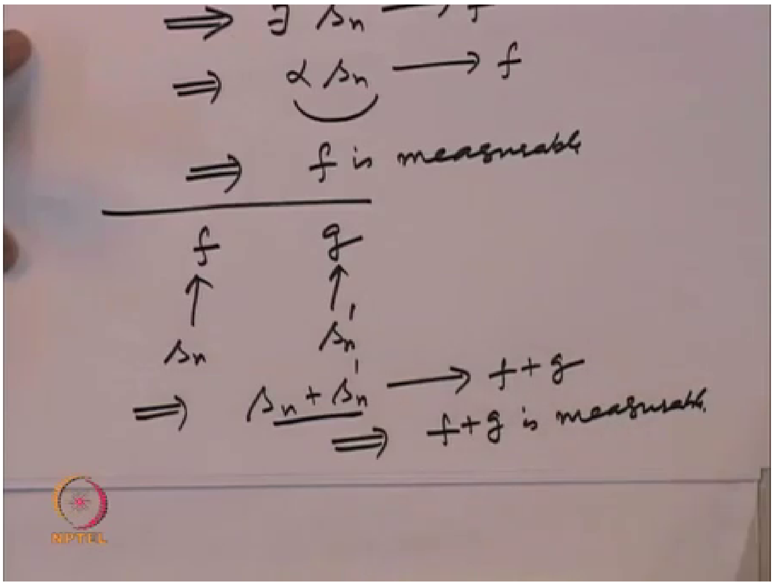

So, that says f measurable implies there exist a sequence sn of simple measurable functions converging to f, but that implies by the properties of sequences alpha sn converges to f. because sn is simple measurable a constant times a simple measurable functions are again a measurable. So, this is a sequence of simple measurable functions converging to f. So, implies by the previous theorem f is measurable. and the same proof works for some of 2 functions, let us say f and g are 2 measurable functions. we want to prove that f plus g is measurable. So, f measurable implies there is a sequence of sn of simple measurable functions converging to it g is measurable. So, that is the sequence sn dash of simple measurable functions converging to it. So, that implies that sn plus sn dash convergence to f plus g and this is once.

Again, this is a sum of simple for every n this is a sum of simple for measurable functions. So, this is again a simple measurable function. So, we got a sequence of simple measurable functions which converges to f plus g. So, implies that f plus g is measurable. So, that implies f plus. So, we approved the next step namely if f and g are measurable then f plus g is also measurable, let us look at the next property if f is measurable then mod f is also measurable. why is mod f measurable you can look at 2 different ways.

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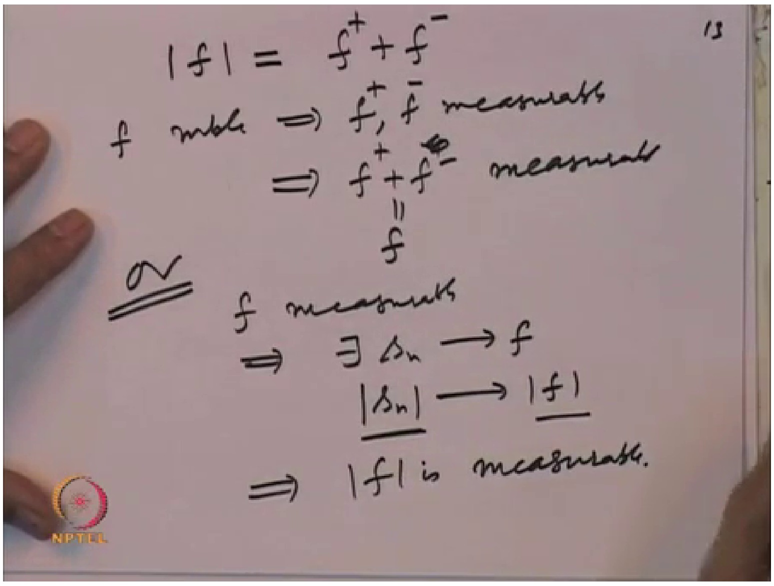

Now, we have got enough techniques to conclude this see either we can write mod f is equal to f plus f minus. this is a observation which will play a role later on also. this is the positive part of the function this is a negative part of the function f measurable implies both f plus f minus measurable. implies f plus f minus f minus measurable and this is precisely my f. So, that is one way of looking at it or you can also look at from sequence point of view f measurable, implies there is a sequence sn of simple measurable functions converging to f, but then a simple argument which works for sequences which what are have already seen that mod of sn convergence to mod f and observation. if sn is simple then mod f also is mod sn is also simple for every n.

So, this is a sequence of simple measurable functions converging to f mod f; that means, mod f is measurable. So, either you can look at sequences or you can look at the positive part and negative part either one will be help you to conclude that if f is the measurable. then mod f is also measurable. let me look at another property of a measurable functions namely that if we are seen this property for simple functions that if e is a set in the sigma algebra s and you f is measurable then product of f times indicator function of e is also a measurable function.

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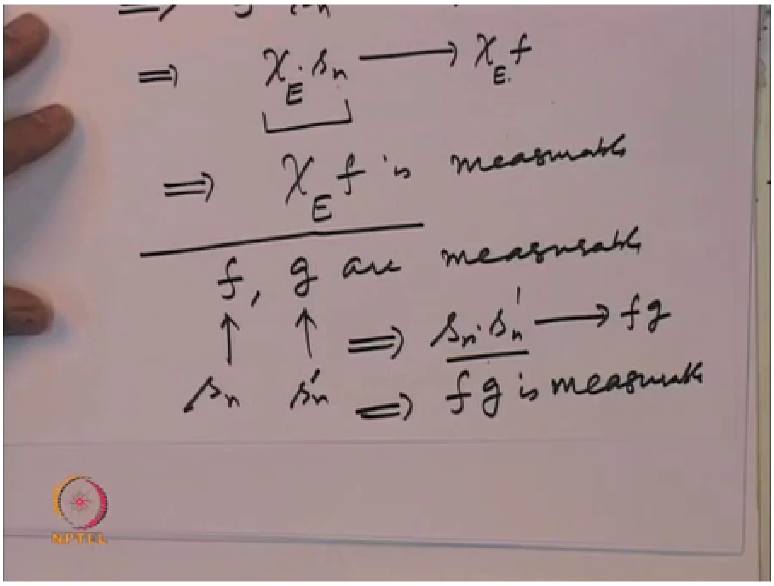

So, once again we can take the help of the criteria. just now approved, f measurable implies that there exist a sequence of simple measurable functions converging to f at every point, but once that is true if sn converges to f then that implies look at chi e times sn that will converge to chi e times, f right. because this remains multiplying by a function. So, this converges to the simple properties of sequences and now observe that, this is if sn is simple measurable function then the indicator function of e times sn is also a simple measurable function.

So, that converges to indicator function of e times. So, that implies that indicator function of e times f is measurable right. In fact, we can go a step further and prove that you can multiply by the same argument espousing f and g are measurable, for f we have got a sequence sn of simple measurable functions converging to f we have got a sequence sn of simple measurable functions converging to sn dash converging to g. So, that implies if I multiply sn dash that converges to f times g and product of simple measurable functions.

We have already seen is again a simple measurable function. So, a sequence of simple measurable functions converging to f of g; that means, implies that fg is measurable. So, product of measurable functions is also measurable.

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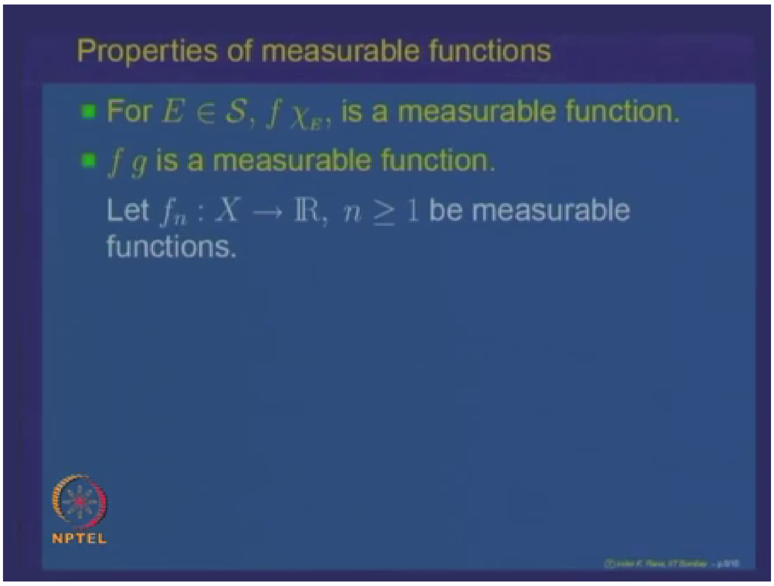

I think will close here today and look at the sequences of measurable functions next time. So, today what we have proved we have looked at the an important criteria characterization of measurable functions a function f defined, on a set x taking extended real valued functions is measurable if and only if keep in mind it is a characterization.

So, f measurable f is a function defined on x is measurable if and only if we can find a sequence of simple functions converging to it. and if f is non negative we can find the sequence of simple functions sn which is increasing and converging to f. if in addition we know that f is a bounded measurable function then you can have the sequence of simple functions sn which converges uniformly to f. So, that have the important criteria we have seen some applications today, we will see more application later on also and then we looked at the algebra of measurable functions, we proved that if f is measurable then scalar times f is also a measurable function.

If f and g are measurable then f plus g is also measurable, f into g is measurable the mod f is also measurable. this is for the real valued functions in case the functions are extended real valued you while defining f plus g and f into g you have to be slightly careful, because f may take the value plus infinity at a point and g may take the value minus infinity. then how will you define f plus g. So, for such kind of problematic sets we can separate them out right. So, separate out a set a on which f of x is plus infinity or g of x is equal to minus infinity or f of x is minus infinity and g of x is the plus infinity. So, on this set a we may not be able to define what is f plus g, but outside that we can define f plus g and this set, where f is plus infinity and g is equal to minus infinity or other way around is a measurable set is in the sigma algebra.

So, we can change the values we can define f plus g to be equal to anything we like and still that f plus g will be a measurable function. So, modifications of the algebra of measurable functions properties still remained true when the functions are extended real valued. we will continue the study of sequences of measurable functions next lecture.

Thank you.