Measure & Integration
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Lecture - 02A
Algebra and Sigma Algebra of a Subsets of a Set

Welcome to today's lecture on measure and integration. This is the second lecture and in this lecture, we are going to cover the following topics.

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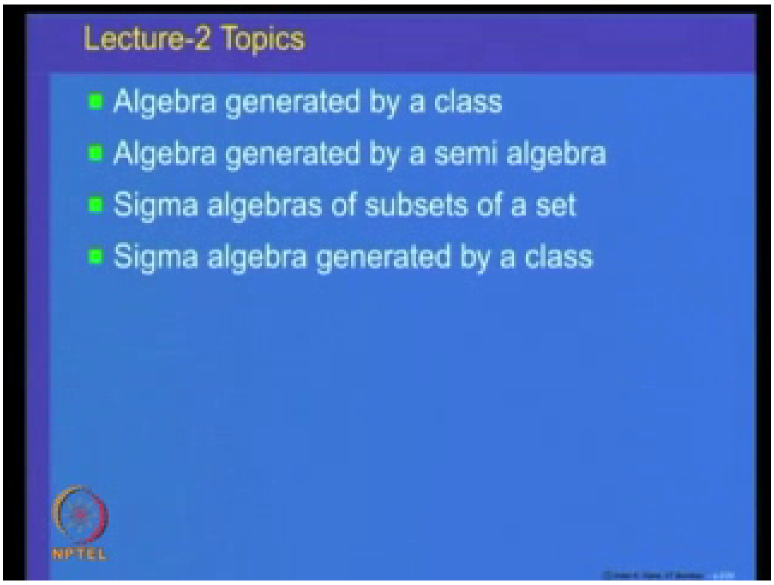

Algebra generated by a class, then we will look at algebra generated by a semi algebra. We will also look at what is called the sigma algebra of sets of X and sigma algebra generated by a class of subsets of a set X.

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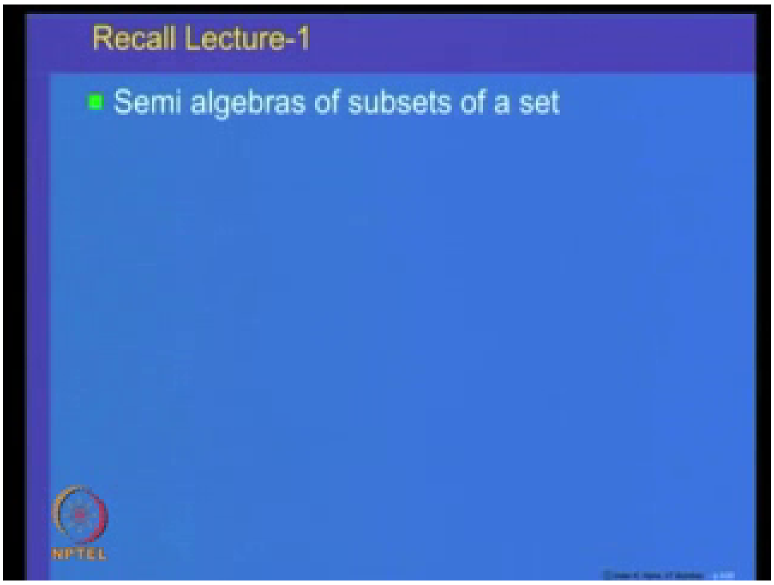

Let us just recall what we called as the semi algebra of subsets of a set X. It was the collection.

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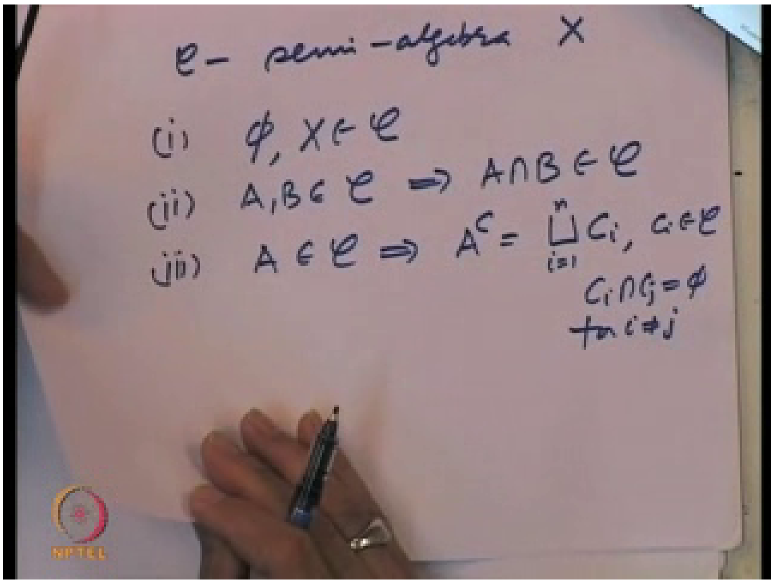

So, let me just recall a semi algebra C is a semi algebra of subsets of a set X, if at the following properties empty set and the whole space belong to it. And secondly, for A and B belonging to C, implies that the intersection of these 2 sets is also an element of C. That is the class C is closed under intersections and third property was that if A belongs to C then this implies that A compliment can be represented as if it is finite disjoint union of elements of the class C.

That is a compliment is a union of elements C i where each C i belongs to C and C i intersection C j is empty for i naught equal to j. So, such a class was called a semi algebra of sub sets of X.

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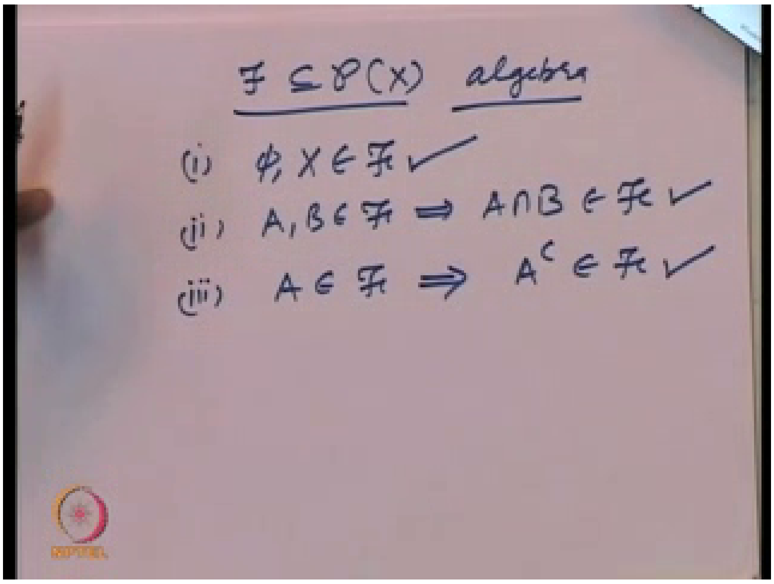

Then we looked at what we called as the algebra of sub sets of a set X. So, a class F of subsets of a set X is called an algebra, if it had the following properties namely empty set and the whole space belong to F as in the case of semi algebra. Secondly, A and B belonging to F imply that A intersection B also belongs to F as was the case for the semi algebra. And third property which is different from the semi algebra which is a bit stronger than the semi algebra is whenever I said A belongs to F this should imply that a compliment also belongs to F.

So, an algebra of subsets of a set X is a collection of subsets of x, which is which includes the empty set and the whole space it is closed under intersections and it is also closed under compliments. And the last time we looked at some examples of algebras and semi algebras.

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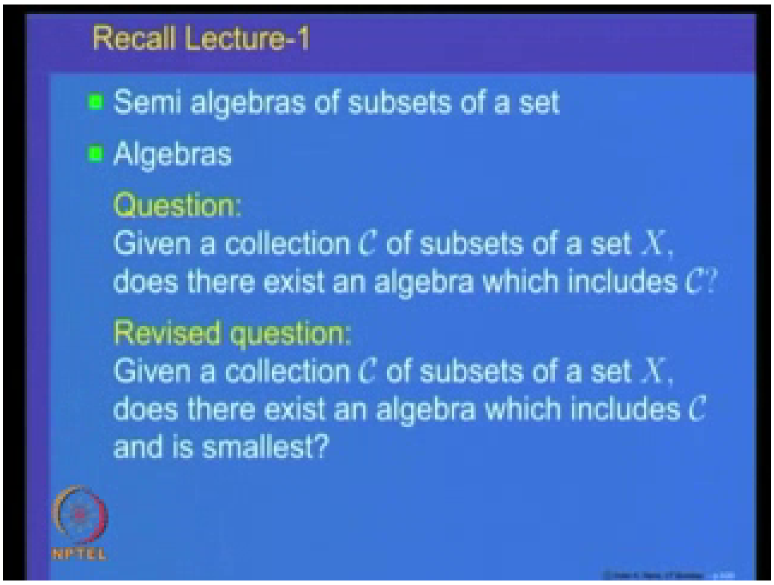

So, today what we are going to look at is the question is given a collection C of subsets of a set X does there exist an algebra which include C. So, the given collection C may not be an algebra of sub sets of X it is an arbitrary collection. And we would like to know can we find a collection of subsets of X which includes this collection and is an algebra. Of course, there is one obvious answer namely the power set. For example, the power set of X is always an algebra because it is collection of all subsets of X.

So, it is it has all the properties namely it is closed under intersection and complement and includes the empty set in the whole space. And; obviously, C is a subset of p X. So, in some sense p X is the largest algebra of subsets of X which includes any collection C. So, we should modify our question namely given a collection C of subsets of a set x, does there exist and algebra which includes C and is the smallest.

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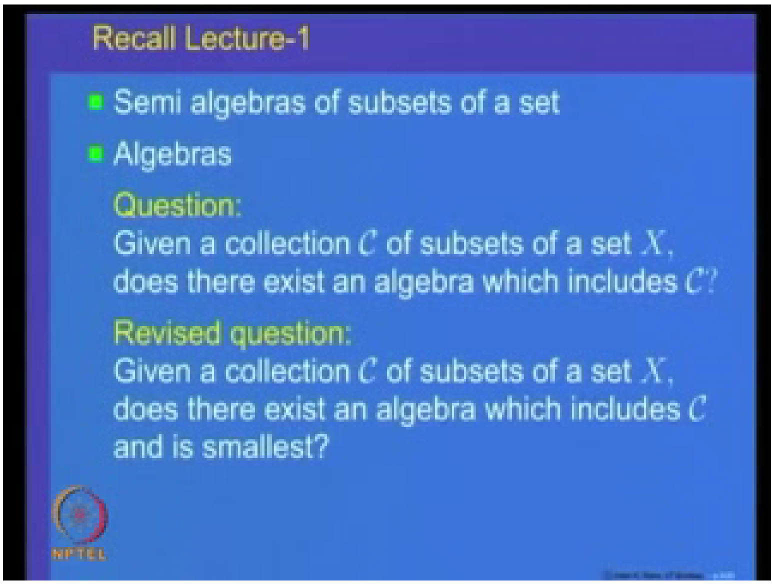

So, to answer that question let us look at the following. So, let us collect together.

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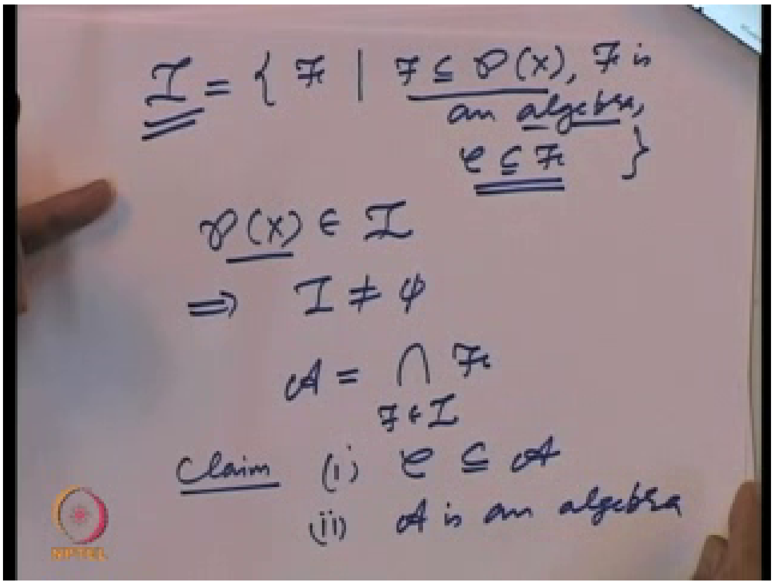

So, let us look at the collection all F such that F is a subset of p X it is a collection of subsets of x, and such that F is an algebra and C is contained in f, C is the collection which is given to us which may not be algebra.

So, let us collect all those collections of subsets of F X. So, call them as F such that. So, F is a collection of subsets of X such that F is an algebra and includes C. So, it has 2 properties one, C is a subset of X this collection C is inside the collection F and F is an algebra. So, first of all let us observe that p X is an element of this collection. So, let us call this collection as say i. So, this collection of all algebras which includes C is a non empty collection because the power set of X the collection of all subsets of X is an algebra and include C. So, this is non empty. So, implies that this collection is not empty. And now what we do let us define a to be equal to intersection of all this F such that F belongs to i. So, let us take the all the collections in this collection all the algebras which are members of this collection I and take there intersections.

So, keep in mind each F is a collection of subjects and it is an algebra and we are taking the intersection of these collections. So, the claim is that one C is a subset of this collection a which is obvious because the collection i of all the algebras F has that property C is a subset of each member. So, this property is obvious. Secondly, we claim that a is an algebra. So, to prove that a is an algebra, let us observe what we have to do?

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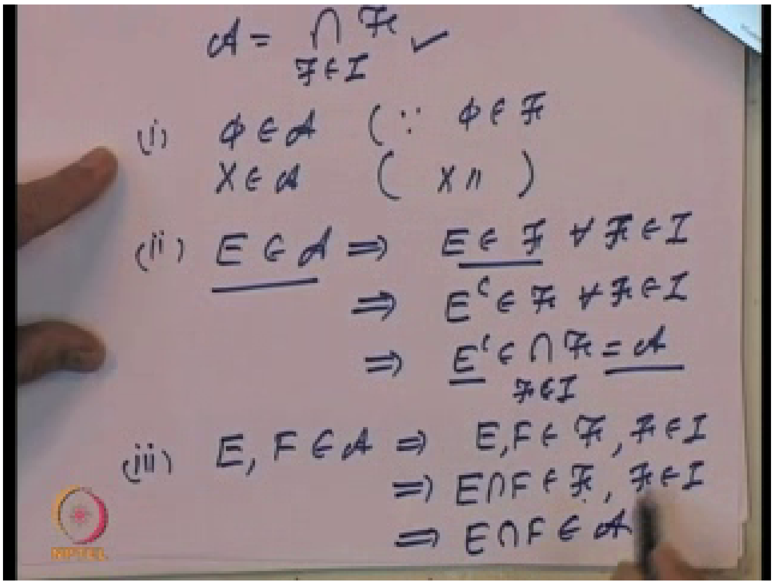

So, what is a is the intersection of all algebras F which are inside i. So, first we want to show empty set belongs to a. Why? Because we know empty set belongs to each F because each F is an algebra. Similarly X belongs to A because of the same reason because X belongs to F and F is in algebra. And secondly, let us take a set E belonging to A then that will imply by the very definition that E belongs to F for every F inside the collection i.

But that implies because E belongs to F and F is an algebra that implies that E compliment also belongs to F or every F belonging to i, but then that implies it is equal and to saying this implies that E compliment belongs to the intersection of all these f, F in i and that is precisely our a. So, we have shown if E belongs to A then E compliment also belongs to a. So, the class A is closed under compliments, and let us finally, show that it is closed under intersections also. So, let us take 2 sets E and F belonging to A then that implies that E and F both belong to each F F belonging to i, but that implies that E intersection F belongs to the collection F because F is an algebra. So, that is crucial. So, that is being used again and again E and F belong to the collection F which is an algebra. So, the intersection also belongs and hence E intersection F belongs to A.

So, what we are showing is this collection A, which is intersection of all the algebras which include C is an algebra. So, we are shown the second property namely that this collection.

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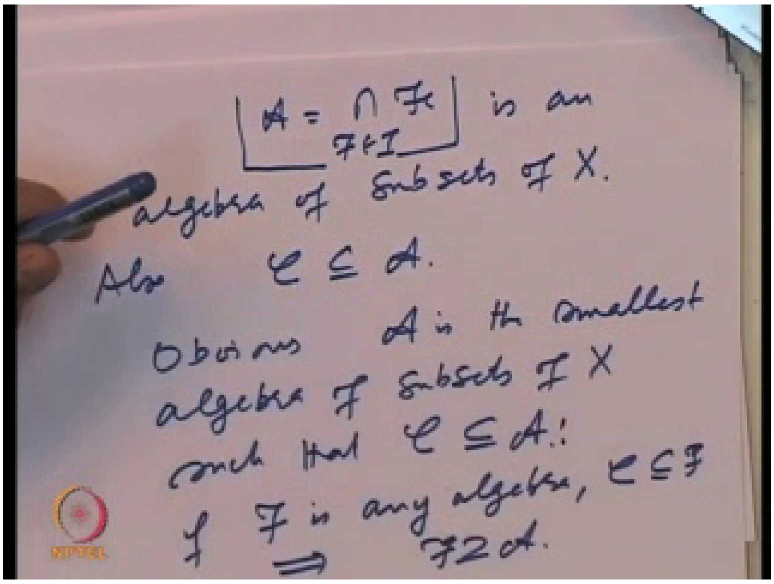

A of intersection of all the algebras which include C is an algebra of subsets of the set X also C is inside A. And by the very nature because it is the intersection of all the algebras which include A it should be obvious. So, obvious property namely A is the smallest algebra of subsets of X such that C is inside A. What do you mean by that; that means, that if. So, this is same as saying F F is any algebra and C is inside F that implies that this F have to include A.

So, what we have shown is that given any collection of subsets of a set X, if we define a by this a is the intersection of all the algebras including i, then this such an algebra then this collection exist because there is at least one algebra which includes C namely the power set and it is the smallest s. So, what we are show is given any collection of subsets of a set X there is an algebra of subsets of X which is small s and includes A. So, we have answer this question that given a collection C of subsets of A set X.

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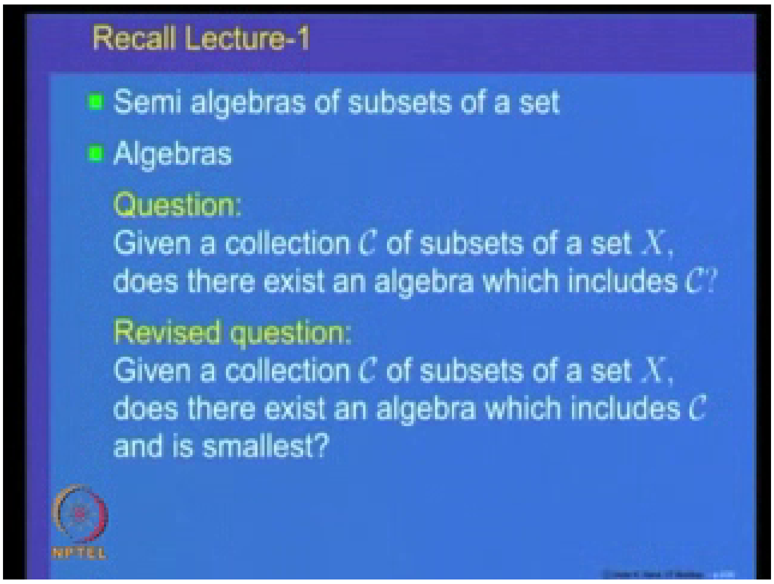

Does there exist and algebra which includes C and a small is s the answer is yes. So, let us state this as a theorem namely let X be any set.

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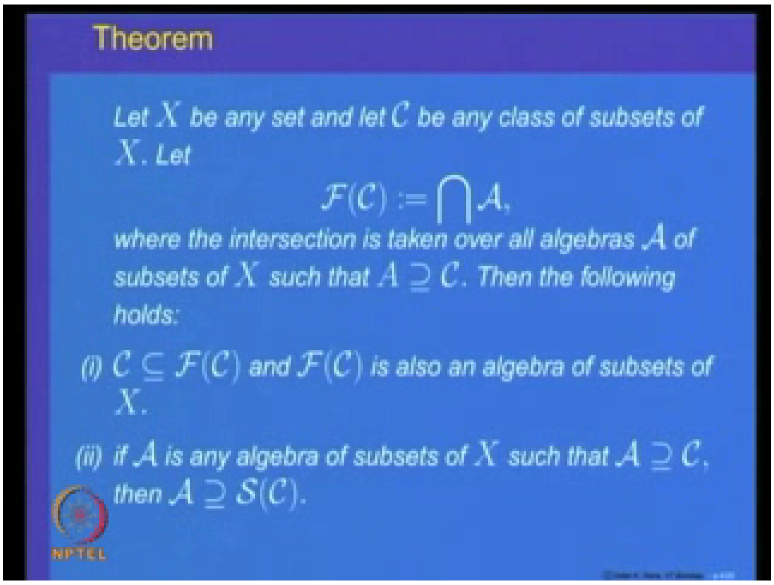

And let C be any class of subsets of set X let F of X denote the intersection of the algebras A which include the collection C.

So, F of C is the intersection of all the algebras that include the collection C then what we have guess known shown is that C is a subset of x; that means, F of X (Refer

Time: 11:37) include C and F of C is an algebra of subsets of X. So, F of C is an algebra which includes X and it has additional property, it is a smallest with that property that is if a is any other algebra of subsets of X such that A include C, then a must include F of C. So, this is not S of C it is F of C.

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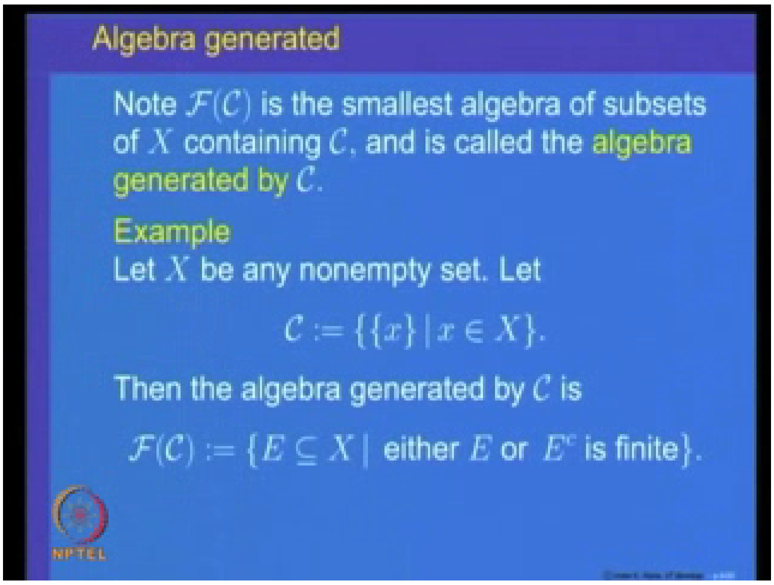

So, F of C is called the smallest sigma algebra of subsets of X containing C and is called the algebra generated by the class C. So, what we have shown is that given any collection of subsets of a set X, we can always fine and algebra of subsets of X which is smallest and includes it. Let us look at some examples. So, let us look at the collection X is any non empty set let us look at the collection of all singleton subsets of this collection X.

So, C is the collection of all singleton is X, where x belongs to X. We want to know what is the algebra generated by it and the claim is that the algebra generated by this C is nothing, but all those sets E in X say that either E or E compliment is finite. So, let us prove this how do you prove such kind of a session.

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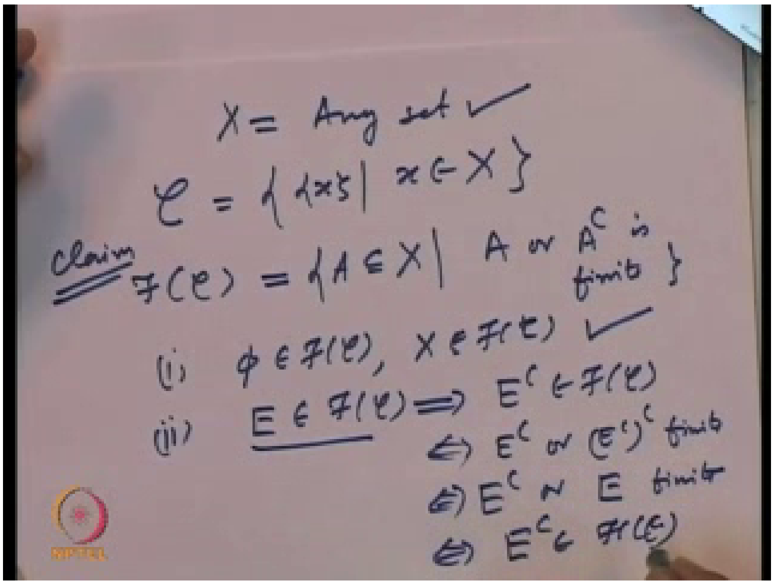

So, we have got x any set and C is the collection of all singleton sets belonging to which are subsets of X. And then we are looking at we claim. So, this is our claim that F of C is nothing, but all sets a contained in X say that a or a compliment is finite. So, let us observe.

Does empty set belong to F of C yes because by definition empty set is a finite set does X belong to F of C. So, for X to be an element of F of C either X should be finite, which we do not know because X an any set right it may or may not be finite, but we know it is complement which is empty set is finite and in F of C. So, by the second criteria a or a compliment. So, X may not be finite, but it is complement is empty set which is finite. So, this property is true let us check the second property namely, if E belongs to F of C does this imply E compliment belong to F of C is this true. Well for E compliment to belong to F of C either E should be finite or E compliment should be finite.

So, this will be true if and only if E compliment or E compliment, compliment finite. Which is same as E compliment or E compliment is E finite which is same as saying. So, this is if and only if this is if and only if this is if and only if E compliment belongs to F of C. So, E belonging to F of C is true if and only if E compliment because our definition of F of C is symmetric will respect to A and A compliment. So, the collection F of C of all those sets for which a or a compliment is finite includes empty set the whole space that is closed under compliments.

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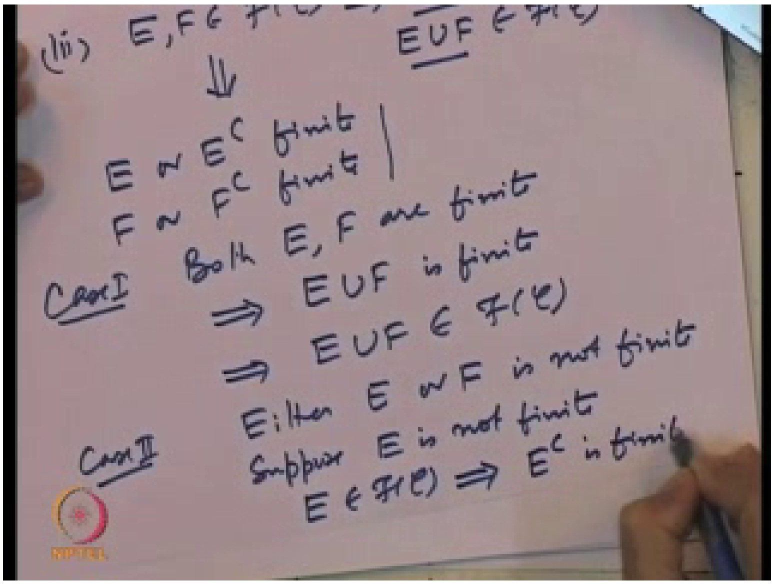

And let us show the third property namely that if E and F belong to F of C. Then that implies there intersection also belongs to F of C. Or equivalently will be this is equivalent to saying that a union F belong to F of C. Because we have observed that from union you can go to intersection by complements because the class is already closed into compliments.

So, let us, but E and F belonging to F of C means E or E compliment finite and F or F compliment finite. So, various possibilities arise. So, let us take case one both E and F are finite. In that case that will imply that E union F is finite. Because union of finite sets is a finite set and that will imply that E union F belongs to F of C cover collection. So, whenever E and F are finite that is so what is a case to what is other possibility either E or F is not finite. So, let us for the sake of definiteness is suppose E is not finite, but E belongs to the class F of C. So, that implies E belongs to F of C, means E compliment is finite.

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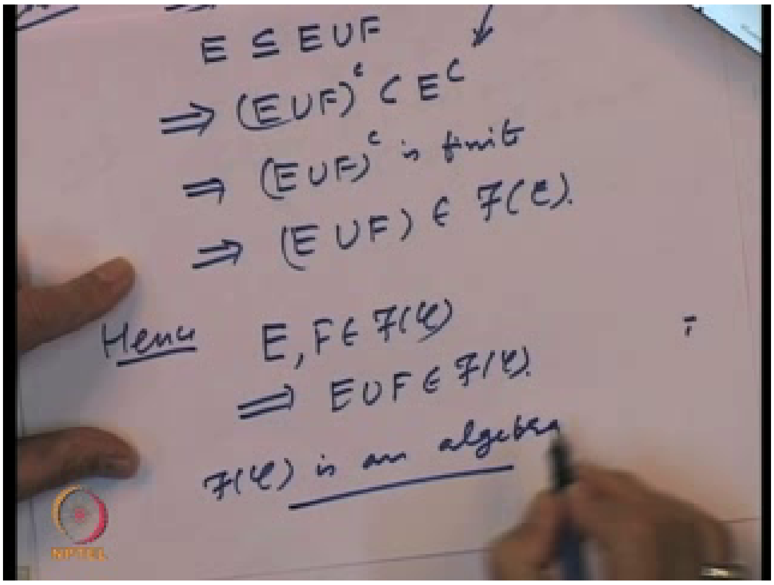

Now E compliment is finite let us look at now and E is contained in E union f, but that implies that E union F complement is contained in E compliment and E compliment is finite.

So, that implies. So, is finite. So, that implies E union F complement is finite and hence by definition this implies a union F belongs to F of C. So, we have shown in either case hence whenever 2 sets E and F belong to F of C that implies a union F belongs to F of C. So, thus we have shown that the collection. So, F of C is an algebra. So, F of C is an algebra had it.

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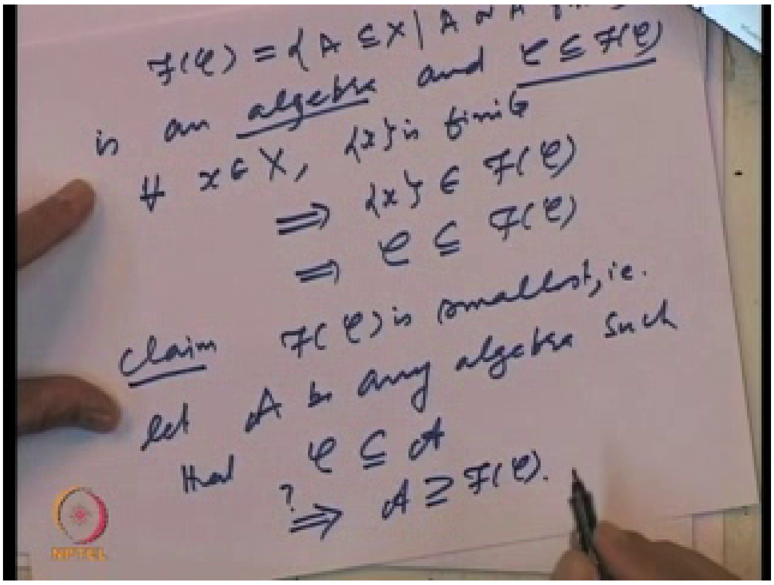

So, F of C which was defined as all sets a contained in X such that a or a compliment finite is an algebra. And C is contained in F of C. Is it clear why C is contained in F of C because for every X belonging to X the singleton X is finite. So, implying the singleton X belongs to F of C.

So, implying that C is a subset of F of C. So, thus F of C is an algebra and includes F of C claim finally, that F of C is smallest. So, that is let a be any algebra such that C is inside a then that should imply. So, that is the question that should imply that A includes F of C. So, let us show how is that true.

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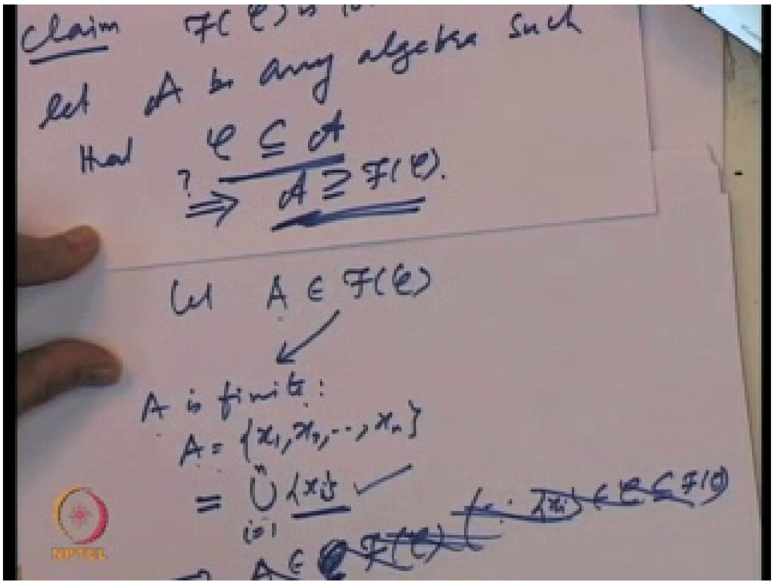

So, to prove this let A belong to F of C. So, 2 possibilities one A is finite. So, let us write a as X 1 X 2 sum x; that means, this is equal to union of singletons x 1 x i, i equal to 1 to n and each singleton x i is an element of class C and C is inside a F of C is inside A, C is given to be inside A. So, this implies that implies that A belongs C, A belongs to F of C is.

So, this so let me just go through again because a is finite A is written as X 1X 2 X n. So, I can write as a finite union each x i belongs to C. So, each x i belongs to F of C right. So, this implies. So, here is because each x i belongs to C which is subset of F of C. So, each one is in F of C, F of C in an algebra. So, A belongs to F of C, I am sorry this is not what you wanted to prove. So, this is this is not this is of course, I am improving what he was required. So, let a b finite. So, this we want to show that A is inside the class A, but. So, each x i belong to C, C is contained in a; that means, each x i belongs to A and A is an algebra. So, that implies that the union. So, A belongs to A same argument basically using that C is inside A and A is an algebra. So, that will prove that A is if A belongs to F then this implies that A belongs to A. So, that proves that A is a subset A always includes F of C. So, hence what we have shown is the following that if C.

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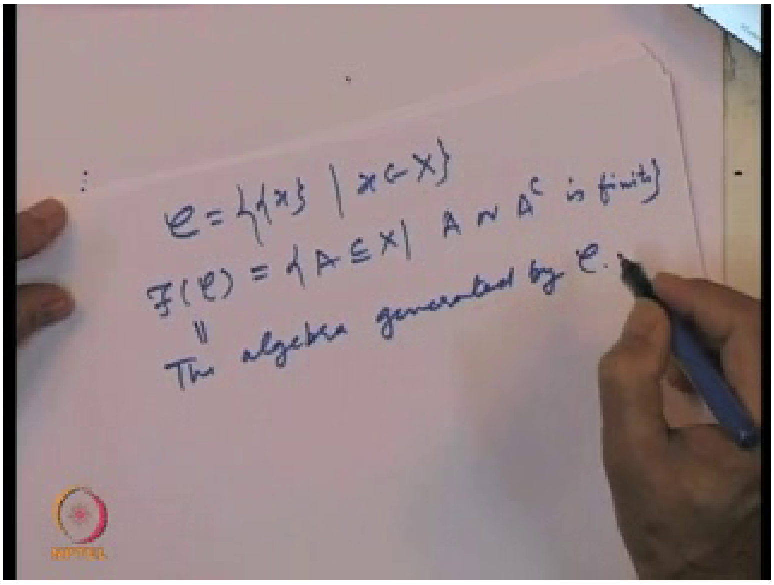

Is the collection of all singleton sets X belonging to X then the algebra generated by it is nothing, but all subsets a in X size that a or a compliment is finite a or a compliment is finite? So, this is the algebra generated by C. So, we have computed we have described the algebra generated by a collection C of singletons of a set X.