Measure and Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 15A
Properties of Measurable Functions

Welcome to lecture 15 on measure and integration. In the previous lecture we had defined what is called a measurable function on a measurable space x s and then we had looked at some equivalent ways of looking at measurable functions. We looked at examples of what are called simple measurable functions, they are nothing but finite linear combinations of indicator functions of subsets of the set x. The simple measurable functions are sort of the core of a class of all measurable functions. We showed that sum of simple measurable functions product of measurable simple measurable functions and maximum and minimum of simple measurable functions are all simple measurable functions. And today we will start looking at some general properties of measurable functions f, and then we will characterize measurable functions in terms of simple measurable functions.

So, today's talk is going to be mainly on properties of measurable functions.

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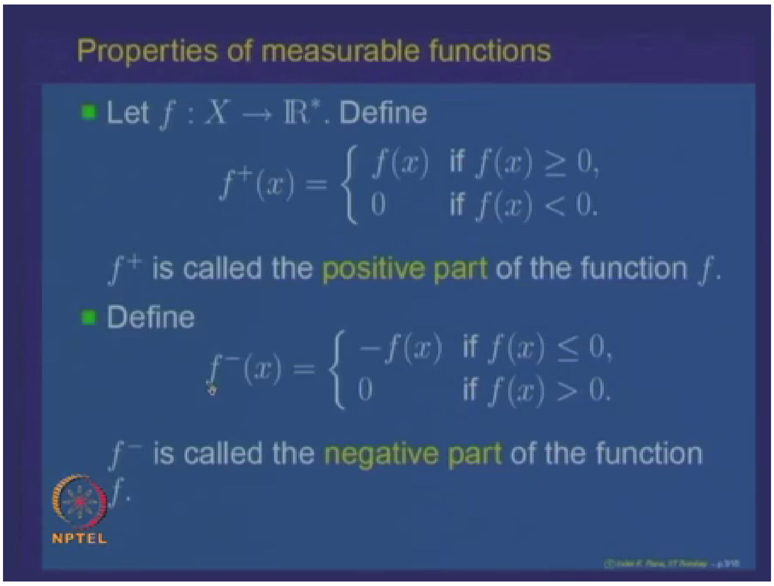

So, let us recall for simple function we define what is called the positive part and the negative part of a function. So, this can be defined for any function let f be a function defined on Xtaking extended a real valued functions extended real values R star. Then we defined what is called the positive part of the function. So, that is defined by f plus of x. This is again a function on the space x, and it is defined as f plus of x is equal to f of x if f of x is bigger than or equal to 0. And it is defined as 0 if f of x is less than 0. Essentially what we are saying is look at the graph of the function f of x, as long as the graph remains above the x axis keep the function as it is. So, f of x is kept as it is if f of x is bigger than or equal to 0, and as soon as the graph goes below the x axis we define it is value to be equal to 0. So, this is called the positive part of the function.

So, note that for any function f the positive part of the function is again a function on the space x, but it takes only non negative values. And similarly we can define the negative part of the function to be a function on x again such that. So, and it is denoted by f minus of x. So, f minus of x is equal to minus of f of x keep in mind we are putting a negative sign here if f of x is less than or equal to 0. So that means, as soon as the function goes is on x axis or bellow the x axis we reflect it against x axis and put it is value as minus of f of x. So, f of x with a negative sign. So, if f of x is negative. So, this will always be a non negative quantity. And it is function is defined as 0 if f of x is bigger than 0.

Essentially once again it is looking at the graph of the function as long as the graph of the function is above the x axis we put it value equal to 0, and it is minus of f of x, if f of x is less than or equal to 0. So, these are called the positive part and the negative part of the function and it is quite obvious from the definition that f can be written as f plus minus of f minus.

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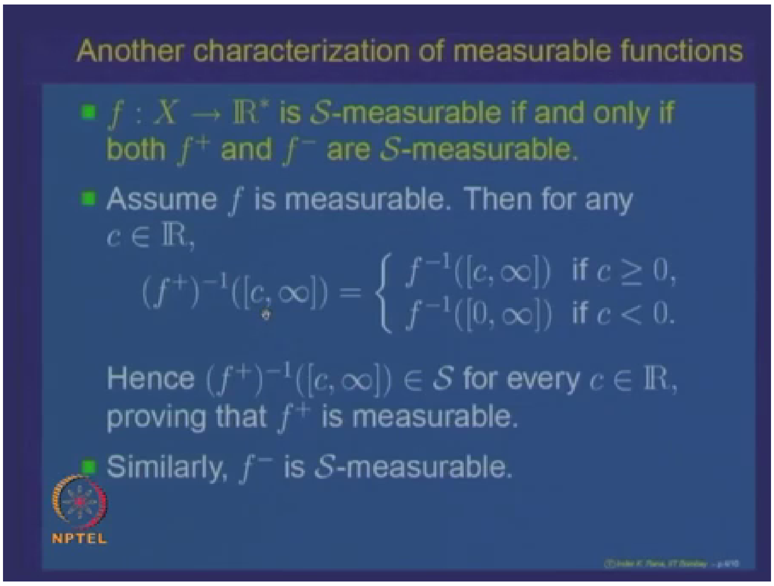

So, and we want to prove that if f extended real value is a measurable function. We want to show that in that case the positive part and negative part are both measurable functions and conversely a positive part and negative part are measurable then the function f is measurable. So, namely saying a function f on X taking extended real valued a is measurable if and only if both f plus and f minus are measurable functions and the proof is rather simple. So, let us assume first f is measurable. Then for any point c in R look at the inverse image of the end closed interval c to plus infinity. So, f plus inverse image a closed interval c to infinity. So, that we know is because f of x f plus of x is equal to f of x if x is f of x is bigger than 0. So, if this value c is bigger than 0 then f plus of x will be equal to f of x. So, this inverse image of the closed interval c to infinity under f plus is nothing but the inverse image of f, of the interval c to infinity if c is bigger than or equal to 0.

Because in that case f of x is always going to be positive, and it is equal to the inverse image of the interval 0 to infinity if c is negative because then we do not want to look at the remaining part. So, in either case because f is measurable both these sets are in the sigma algebra. So, f inverse f plus inverse of c to infinity belongs to the sigma algebra if f is measurable. So, that proves that f plus is measurable. A similar argument will prove that f minus is also measurable. So, essentially what we are saying is the f plus inverse that is the inverse image of the interval c to infinity under f plus can be represented as inverse image of a interval of f under f of some interval and both are f measurable implies that whenever that f inverse of interval is a set in the sigma algebra.

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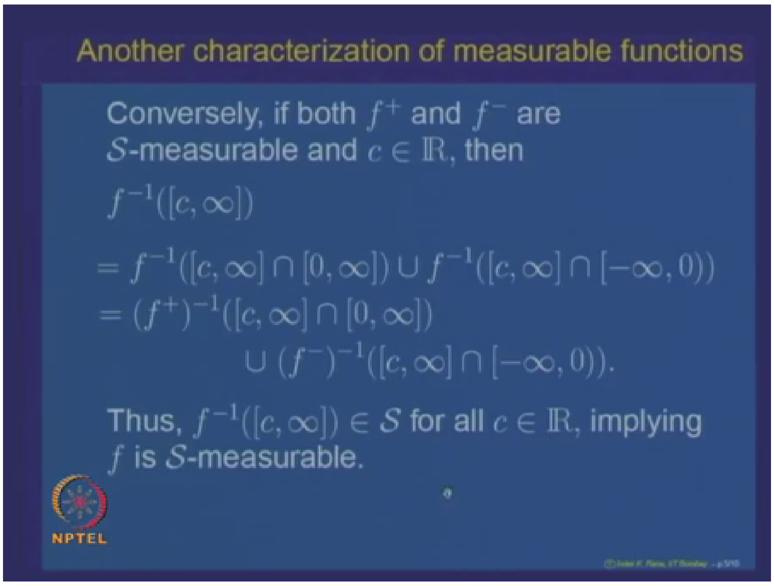

So, we have shown that if f is measurable, then f plus and f inverse are f minus are both measurable. Let us prove the converse part suppose both f plus and f minus are measurable, and let us take a point c belonging to R, then we should show that the inverse image f inverse of c to infinity belongs to the sigma algebra s for every c in R.

So, let us fix a c and look at this. And we have to interpret this in terms of inverse image is of some intervals in terms of f plus and f minus. So now, let us observe that f the inverse image if the interval c to infinity can be decomposed into 2 parts, let me f inverse of c infinity intersection 0 to infinity. So, look at the intersection of c to infinity with 0 to infinity. And the intersection of this interval c to infinity with m infinity to 0. So, thus interval c to infinity is decomposed into 2 parts, it is intersection with minus infinity to open interval 0, and it is intersection with closed interval from 0 to infinity. So, f inverse of c infinity c to infinity is nothing but f inverse the inverse image of the interval c infinity intersection with 0 to the part of the interval which lies in the positive part. And inverse image of the part of the interval which lies in the negative part, but; that means, in the first part we are looking at whenever the function is in 0 to infinity; that means, function is non negative.

So, the first inverse image is nothing but the inverse image of this in same interval under f plus. And similarly the second one is a inverse image of the interval c infinity into intersection with minus infinity to 0 with respect to f inverse. So, the part of the function which lies in 0 to infinity is written as an inverse image under f plus of an interval and the other part is written as inverse image under f minus. Since both f and f minus f plus and f minus are measurable functions. So, these 2 sets f plus inverse of that interval and f minus inverse image under that interval, these both are sets in the sigma algebra s. So, their union is also in this sigma algebra s So that means, f inverse of c to infinity is in the sigma algebra for every c belonging to R, and that proves that this is f is a measurable function. So, we have shown that if function f is measurable if and only if it is positive part and negative part both are measurable functions.

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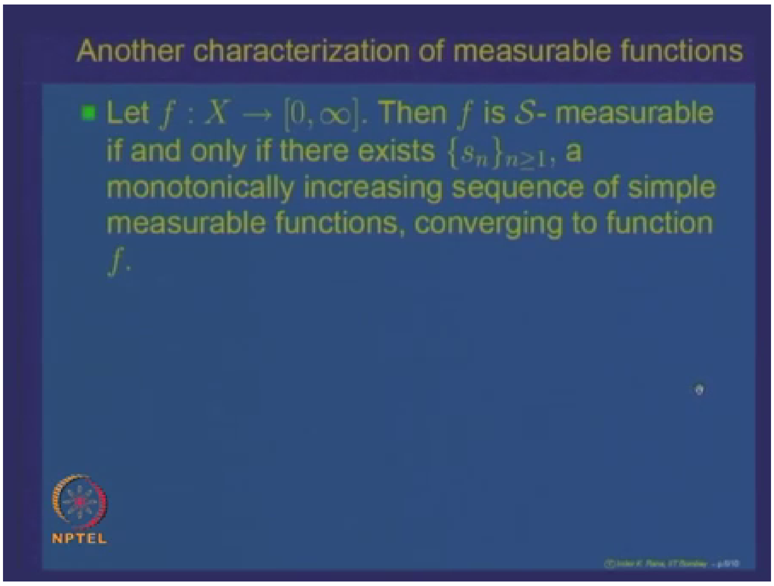

Next let us look at some more properties of measurable functions. We want to give a characterization of measurable functions in terms of simple functions. So, we start with a non negative function let f be a non negative function on defined on x. So, taking values on 0 to infinity, we want to show that this f function is measurable if and only if there exist a sequence S n of simple monotonically increasing functions again I n. In fact, non negative we can also say they are non negative simple measurable non negative sequence of functions which are monotonically increasing to the function f; that means, if a function f is non negative and measurable this can happen if and only if s can be written as a limit of simple functions which are non negative and the sequence S n is increasing.

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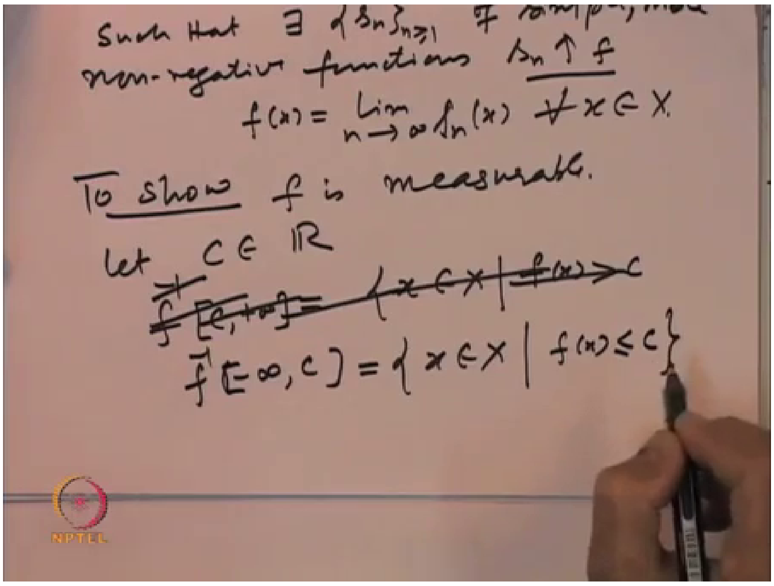

So, let us prove this fact. So, let us start with. So, let f is from x to R and we are given a sequence such that there exists a sequence S n of simple non negative functions S n increasing to f; that means, what; that means, f of x is limit of n going to infinity of S n of x for every x belonging to x. And this S n's are a monotonically increasing. So, this is mon this is written as x. So, to show non active simple functions and each has simple of course, measurable right. So, to show that f is measurable.

Now So, let us let c belong to R, and let us look at the inverse image of the interval c to plus infinity. So, what is that? That is all x belonging to x such that f of x is bigger than c. So, or it will be easier if you look at the other sets. So, let me instead of this let me look at the set, which is f inverse of minus infinity to c. You will soon see why I am taking this instead of the earlier set why I am taking this. Because this proof becomes slightly simpler. So, what is this? This is all x belonging to x such that f of x is less than or equal to c. And what is f of x?

Recall So just now we said f of x is limit n going to infinity of S n of x, and S n x is increasing. So, if the limit of S n x which is f of x is less than or equal to c; that means, each S n has to be less than or equal to c, because even if one goes above c then the limit has to be bigger than c.

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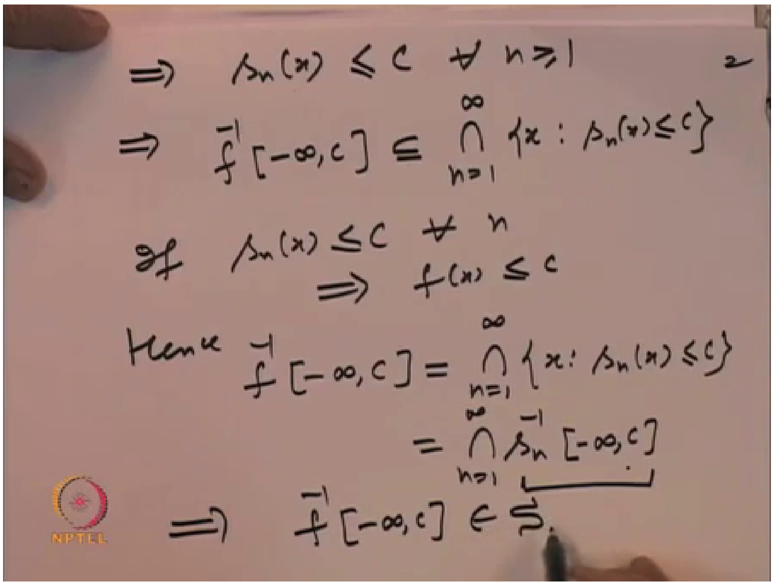

So, this condition So, implies that S n x is less than or equal to c for every n bigger than or equal to 1. So, implies that the set f inverse minus infinity to c. So, if x is such that f of x is less than or equal to c, that implies S n of x is less than or equal to c. So that means, this is contained in this is for every n. So, this contained intersection n equal to 1 to infinity of all x such that S n of x is less than or equal to c. And if conversely if So, note if S n of x is less than or equal to c for every n then at this automatically implies that f of x because f of x is a limit that is also less than or equal to c.

So, hence So, what we are saying is in this is actually an equality. So, hence f inverse of minus infinity to c can be written as intersection n equal to 1 to infinity of x such that S n of x is less than or equal to c. And that is same as n equal to 1 to infinity. So, this is S n inverse of the interval minus infinity to c. And S n's being simple measurable functions, each one of them is an element each one of this sets is an element in the sigma algebra s. So, implies that f inverse of minus infinity to c belongs to the sigma algebra s.

So that means, hence f is measurable. So, what we have shown is if there exist a. So, what we have shown is if f can be written as a limit of increasing sequence of simple measurable non negative simple measurable functions, then f is measurable.

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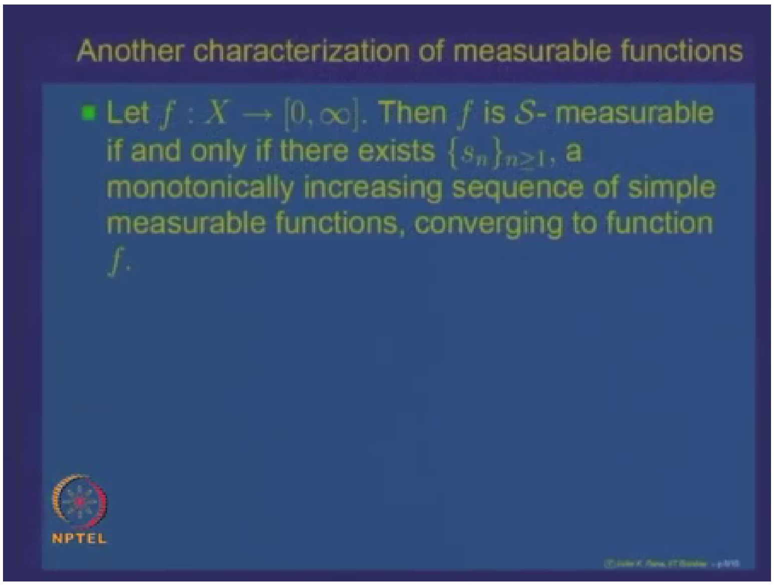

Let us look at the converse part of it which is going to be slightly no not so obvious.

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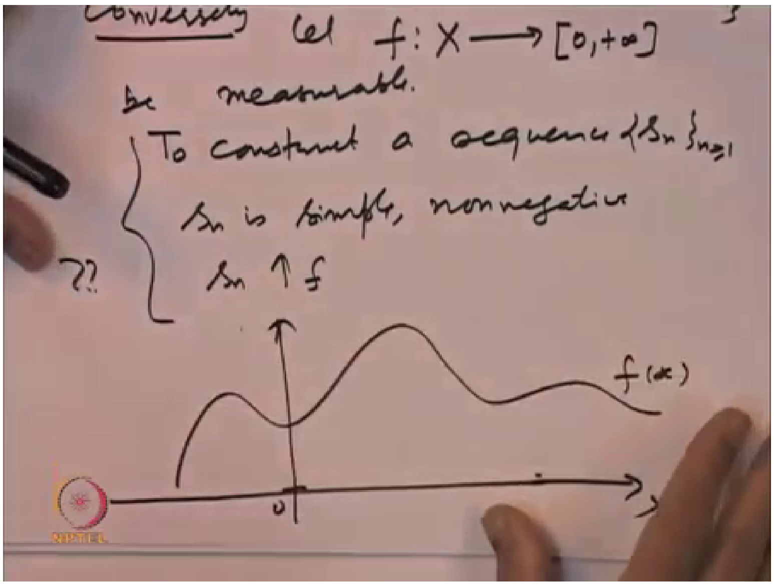

So, conversely let f from x to is a non negative function. So, 0 to plus infinity be measurable. So, we want to show to construct a sequence S n of functions such that S n each S n is simple, non negative S n is increasing to f. So, this is all we want to do. So, this is this construction is intuitively very obvious, but needs to be explained.

So, let us look at in the picture. So, a let us draw a picture of the function. So, let us draw. So, this is x axis, and this is the values the real number they are all values are non negative. So, the graph is going to be above the x axis. So, this is going to be the graph of the function f of x. And the range of the function is a subset of 0 to infinity. So, what we are going to do is we are going to partition the range first into smaller intervals. So, to do that let me draw a picture on a slightly bigger piece.

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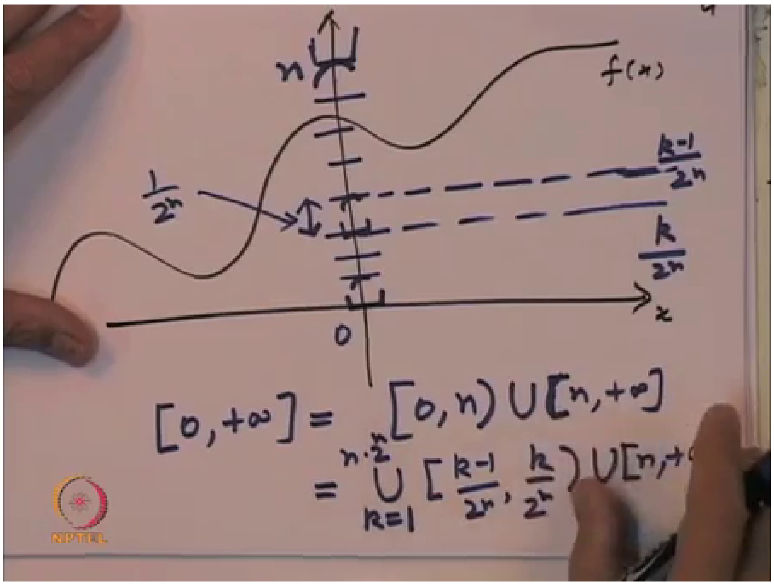

So, that we are able to look at So, this is. So, this is the graph of the function f of x and this is x. So, let us mark of let us put a point n here. So, this is a point 0. So, from 0 to n and then from n to onwards. So, we have a dividing the range of the function the range is a subset of 0 to infinity.

So, we are divided the range. So, I am writing 0 to plus infinity as equal to 0 to n open and union with the closed n to n to plus infinity. So, I have divided the range into 2 parts, and now like what we do is the portion 0 to n. So, this portion from 0 to n we I am going to divide for every n into smaller pieces of length 2 to the power n. So, cut it into pieces right such that the length of each piece. So, length of each piece is nothing but 1 over 2 to the power n. So, this is a length of each piece.

So, let us call the interval. So, this is my general interval. So, the upper point here will denoted by say k minus 1 by 2 to the power n, and the lower part as k by 2 to the power n. So, I am going to write this is equal to So, and we are going to look at open at the bottom close at the bottom open at the top. So, I am going to write as union of intervals of the type k minus 1 by 2 to the power n comma k by 2 to the power n union that other part will leave it as it is n to plus infinity. And this union how many such small pieces will be there total length from 0 to n each has got sub interval has got length.

So, 2 to the power n. So, this starts with k equal to 1, and goes up to n times 2 to the power n. So, this is what we do. Now for every n I want a function S n of x. So, for any x look at the value of f of x. So, let us look at those values.

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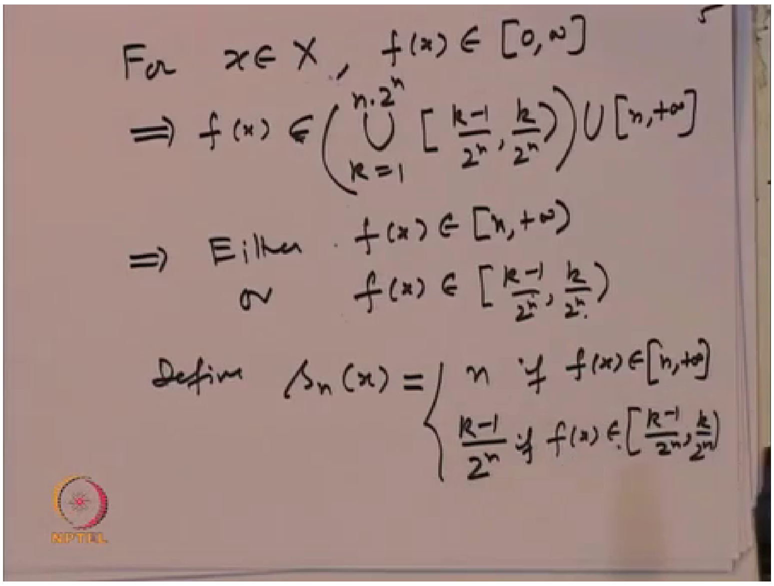

So, for every for x belonging to x f of x belongs to 0 to infinity right. So that means And so, this union is a disjoint union. So, whichever let us write. So, f of x belongs to that partition that we have designed. So, k equal to 1 to n times 2 to the power n of the interval k minus 1 by 2 to the power n to k by 2 to the power n to k by 2 to the power n open and this union n to plus infinity.

So now if f of x belongs to this and this is a disjoint union. So, f of x will belong to only one of them. So, implies either f of x belongs to n to plus infinity, or f of x will belong to one of the sub intervals let us call it as k minus 1 2 to the power n over k by 2 to the power n right. So, these are the 2 possibilities. And now we want to define a function S n. So, define S n of x you want to define. So, what should be the value of it? Say if the value is bigger than n if f of x is bigger than n then let us keep the value to be equal to n if this happens. If f of x belongs to n to plus infinity. And let us define it. So, if f of x is inside this k minus 1 by 2 to the power n to k by 2 to the power n, let us take the lower value So, let us k my. So, define S n x to be equal to 2 to the power n if f of x belongs to k minus 1 by 2 to the power n to k by 2 to the power n.

So, this is how we are going to define the function S n. So, let us look at in the from the picture point of view. So, what I am going to do is. So, because this is a range of the function. So, f of x is going to be somewhere. So, if f of x is above n above this line above this line n say for example, that is happening here, in the this is the function then for all these points my S n is going to be S n is going to be. So, it is going to be S n is going to be this constant. As soon as the function crosses n the value of S n is going to be that part. And all the other possibility that f of x lies in one of this interval.

So, let us say f of x is here. So, this is my f of x. Then what do I want my value of S n should be such that the difference between s and f should not be it should be small. And this is the smallness you have created. So, I will define my S n to be the lower value. So, this is going to be my S n. So, as soon as the function is inside this strip, wherever the function is inside the strip the value is this lower value, if it crosses n then that is the constant value n. So, the function s n x is defined to be equal to this. So, this is S n of x is n if f of x is bigger than or equal to n, and if it is strictly less than n then it will belong to one of those small pieces and that is defined as a lower value of that interval as a value or the function S n of x. So, our claim is that this is the required sequence.

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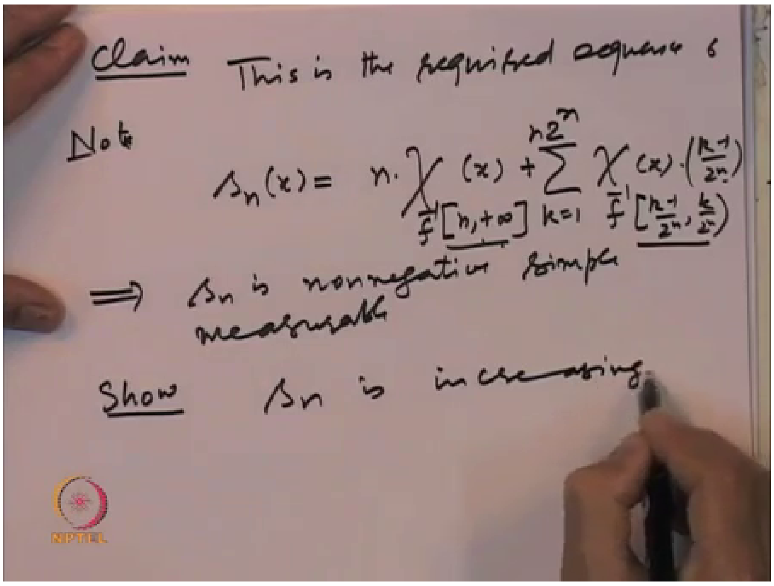

So, claim this is the, this I the required sequence. So, let us first observe what is S n. S n is defined as the I can write S n of x to be equal to n times on this interval. So, it is n times the indicator function of the interval n to plus infinity of x. If he point belongs here this number will be 1. So, the value will be n plus if it lies in that interval k minus 1 by 2 to the power n to k by 2 to the power n the value is this. So, it is this value times the indicator function of that set.

So, it is summation k equal to 1 to 2 to the power n times 2 to the power n of the indicator function of So, what is that set, that is nothing but the f inverse image of k minus 1 by 2 to the power of n and k by 2 to the power n open of x. So, from the picture from the earlier formula we get that my function is this function. So, it is clear from this it is. So, implies S n is non negative simple measurable. Why it is non negative? Because n is the value taken are either n or the sorry in multiplied this by k minus 1 by 2 to the power n that value we forgot to multiply. So, it either takes the value k minus 1 by 2 to the power n or n all are non negative numbers.

So, this is a non negative function, and it is a linear combination finite linear combination of characteristics function of sorry this is not n to infinity. So, this is f inverse right. So, let me just you should be careful. So, it is n if f of x belongs to this; that means, x belongs to f inverse of n 2 plus infinity. And f of x belongs to this interval means, x belongs to f inverse of this interval. So, that proves that S n is a. Now why are this sets measurable it is the linear combinations of indicator functions of sets and this set is measurable, because f is measurable this set is measurable, because f is measurable. So, f is measurability of f implies that inverse image is of intervals are elements in the sigma algebra. So, these are elements in the sigma algebra, and hence S n of x is a linear combination of indicator functions of sets in the sigma algebra s. So, it is a simple measurable function. And now let us prove that this is s is, so claim you want to show s n is increasing.