Measure & Integration
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Lecture - 14 B
Measurable Functions

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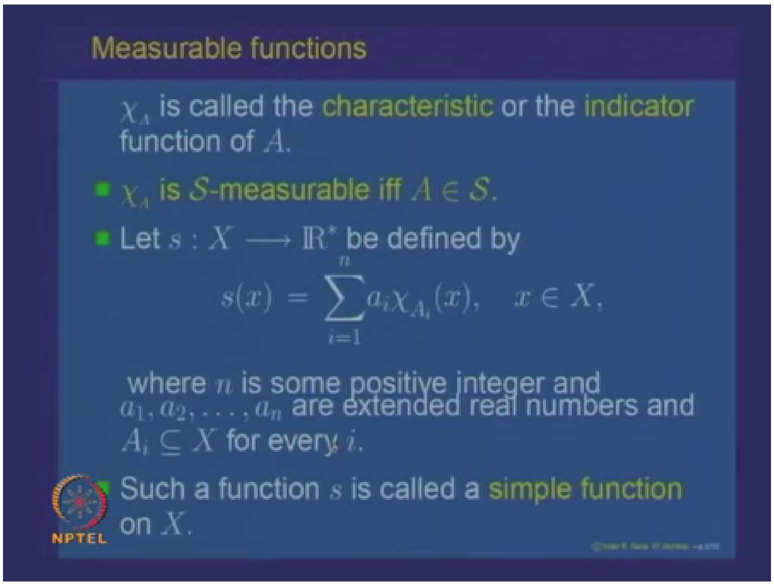

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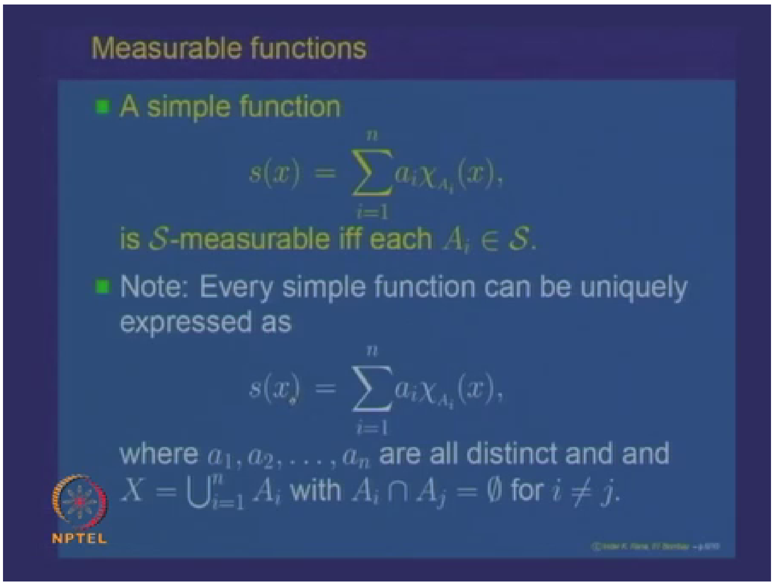

Our claim is that a simple function is measurable if and only if each one of the Ai is belong to S. So, we want to prove a simple function s which is sigma a i indicator function of A i, i equal to 1.

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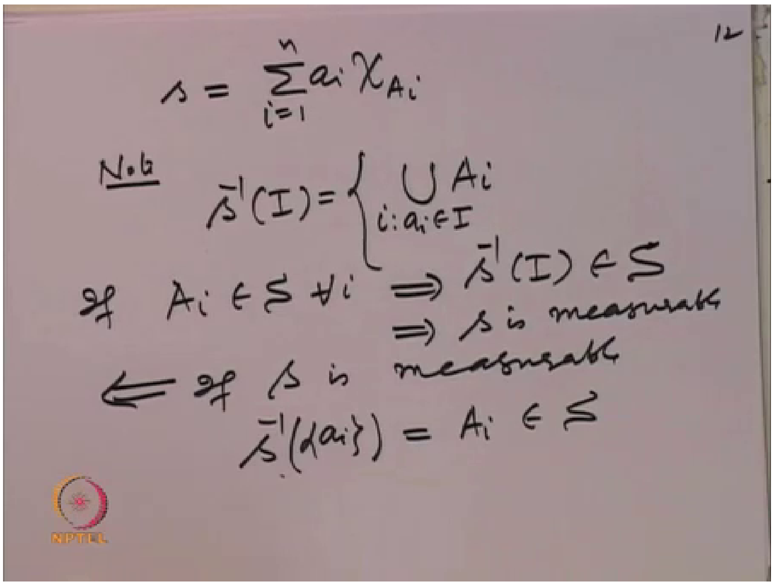

We want to show that this is measurable if and only if let us observe. So, note to check measurability we have to look at s inverse of n interval I. So, what is that going to be that is the function s takes values a is small a is on the set A i. So, this is the main thing to be observed that a infinite linear combination of the indicator functions is a function which takes only finite number of values namely a 1, a 2 a n and the value small a i is taken on the set capital A i. So, so what will be s inverse of I that will be union of those sets A i s union over i such that a i belongs to the interval I right. So, clear.

So, let us once again observe that s takes values a 1, a 2, a n. So, look at the inverse image of interval i. So, look at those is say that a i belongs to i the interval I. So, this will the pull back of this will be the set A i. So, look at the unions of this A is. So, s inverse of i is union of a is right.

So, if each A i belongs to s for every i this will imply that s inverse of intervals belong to s. So, implies that s is measurable because inverse image of every interval belongs to i and this interval is in the extended real numbers. So, plus infinity and minus infinity are included in this. So, is measurable and conversely if s is measurable, then look at s inverse of singleton a i. So, that will be equal to A i and hence measurability implies this belongs to s of course, here slight care one has to take we can assume that all the a i s are distinct because if they are not distinct we can put together those a is into one box.

So, that leads to our. So, anyway so that says at simple function is measurable if and only if all the sets involved in the representation as a i times chi of a i are all measurable. So, as observe just now they have given a simple function s we can also write it in the form we can represent as a i summation a i indicator function of a i where all the a is are distinct and this capital A is are disjoint. Because if sets are not disjoint we can put them together in the same value is taken on two distinct sets, then we can put them together in one box and call that set as nu a i. So, this is sometimes called a standard representation of a simple function where the a is are all distinct and this capital A i is form a partition of the whole space x.

So, a simple function is nothing, but a finite linear combination of indicator functions so that is example of measurable function.

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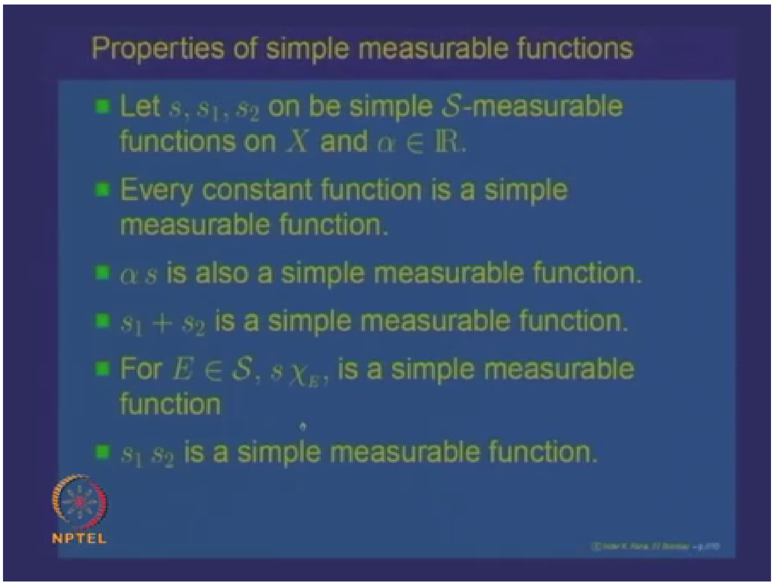

We will studies some more properties of the simple functions or this class of simple measurable functions. So, let us start with s, s 1, and s 2 be simple measurable functions and alpha is a real number. So, first of all we want to observe that every constant function is a simple measurable function what is a constant function? A constant function is nothing, but a function which takes a single value everywhere on the set. So, we can think it as if the constant value taken is c then it is c times a indicator function of the whole space x. So, every constant function is simple measurable because it is a constant multiple of indicator function say a alpha times a simple function is also a simple measurable function.

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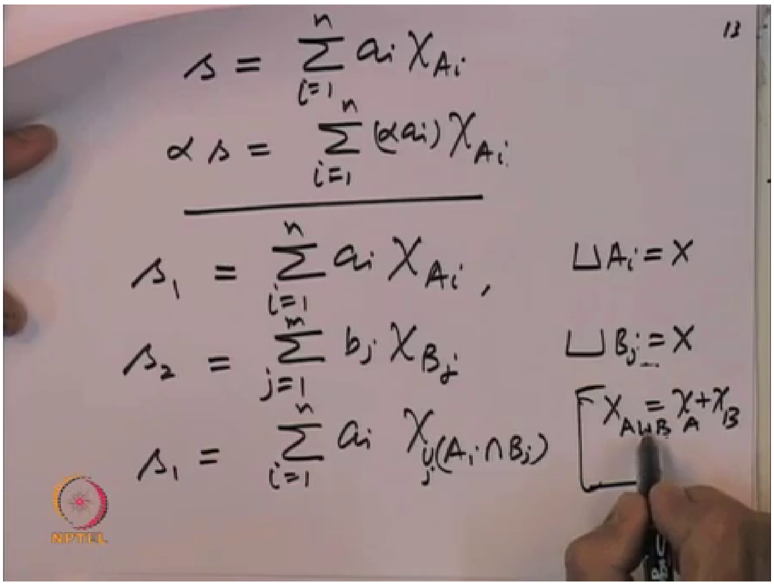

Because of the fact that if a simple function s is equal to summation a i chi A i, i equal to 1 to n then alpha times s is equal to sigma alpha a i times chi of A i, i equal to 1 to n.

So, alpha s is again a simple function and only its values have changed, but the sets on which these values are taken remain the same. So, clearly it indicates that if s is measurable then alpha s is also a measurable set. The next property we want to check is that if s 1 and s 2 are two simple measurable functions then s 1 plus s 2 is also a simple measurable function.

So, let us take a function s 1 which is sigma a i chi A i 1 to n. So, let us say s 1 has the representations sigma A i chi a i and s 2 has the representation j equal to 1 to m, b j chi of B j. Now whenever one is dealing with more than one simple function the idea is try to bring the sets involved in the representation to be same. So, here we are assuming that s 1 is a let us say we have ascended representation then union A i is equal to x and here union B js is also equal to x then i we can write s 1 as each a i can be decomposed into union of the b js.

So, we can write i equal to 1 to n, a i chi of A i intersection B j and union over js. So, each A i can be intersected with union of B js and now here is a simple fact a simple observation that the indicator function of disjoint sets. So, here is a observation if you have two sets A and B and they are disjoint then a union b is equal to chi of A plus chi of B. So, this we leave it for you to verify that the indicator function of the union of two sets is equal to sum of the indicator functions whenever the sets are disjoint. So, using that we can write s 1.

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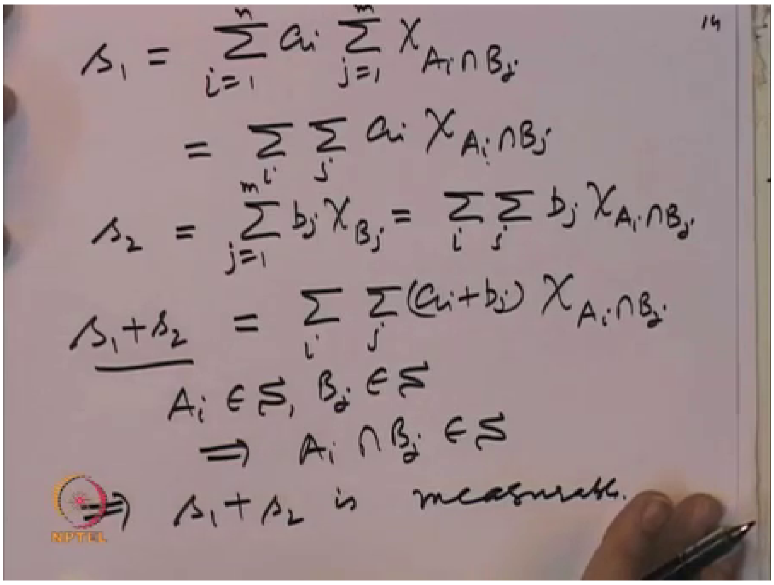

So, this s 1 can be written as summation i equal to 1 to n a i summation over j equal to 1 to m chi of A i intersection B j.

So, this is same as summation over i summation over j a i chi of A i intersection B j and similarly for the second simple function s 2 which had the representation b j chi of B j, j equal to 1 to m we can write this as summation over i summation over j b j chi of A i intersection B j. So, what we are saying is that whenever you are given two or finite number of simple functions, we can assume without lose of a generality that the indicator functions involved are of same sets.

So, s 1 is equal to summation over i summation over j, A i times indicator function of A i times B j indicator function of A i intersection B j and similarly s 2 can be written as summation over i summation over j b j indicator function of A i. So, then what is s 1 plus s 2? S 1 as plus s 2 is nothing, but summation over i summation over j of a i plus b j indicator function of A i intersection B j and that should be clear because if I take a point x then if x belongs to A i intersection B j then s 1 gives the value a i and s 2 gives the value b j. So, sum will give the value a i plus b j outside the value is 0. So, one doesnt have to bother

So, s 1 plus s 2 can be given the representation sigma summation over I, summation over j a i plus b j of this and now since A i belong to the sigma algebra, B js belong to the sigma algebra. So, that implies A i intersection B j is also belong to the sigma algebra. So, s 1 plus s 2 is written as a linear combination of indicator function of sets which are in the sigma algebra. So, that implies s 1 plus s 2 is measurable right.

So, this proves the property that the class of simple measurable functions is close under addition, first property said it is close under scalar multiplication this says is closed under addition.

Next let us take a fix a set any set e in the sigma algebra and multiply s with the set indicator function of e then claim is this is also a simple measurable function and that comes on very simple observation.

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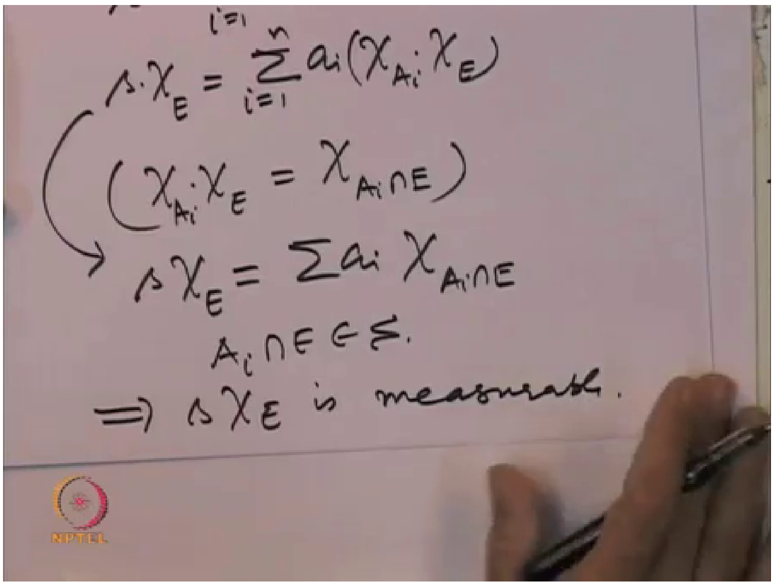

So, let us take a set E belonging to S and S is a simple function which is sigma a i indicator function of A i then s times the indicator function of E is nothing, but summation i equal to one to n a i indicator function of A i times indicator function of E.

So, here is a observation that the product of indicator functions is nothing, but the indicator function of the intersection is nothing, but the product of indicator functions is equal to indicator function of the intersected set.

So, if we use this then the set the function s times indicator function of E can be written as sigma a i indicator function of A i intersection E. So, it is again a linear combination of indicator functions of sets A i is intersection E and since A i belong to the sigma algebra, E belong to the sigma algebra. So, this belongs to the sigma algebra. So, the function s multiplied by the indicator function is a linear combination of characteristics function of sets which are in the sigma algebra. So, implies s indicator function of E is measurable. So, that proves our next property that and using this it is easy to check that the product of simple functions is also simple measurable function is also a simple measurable function.

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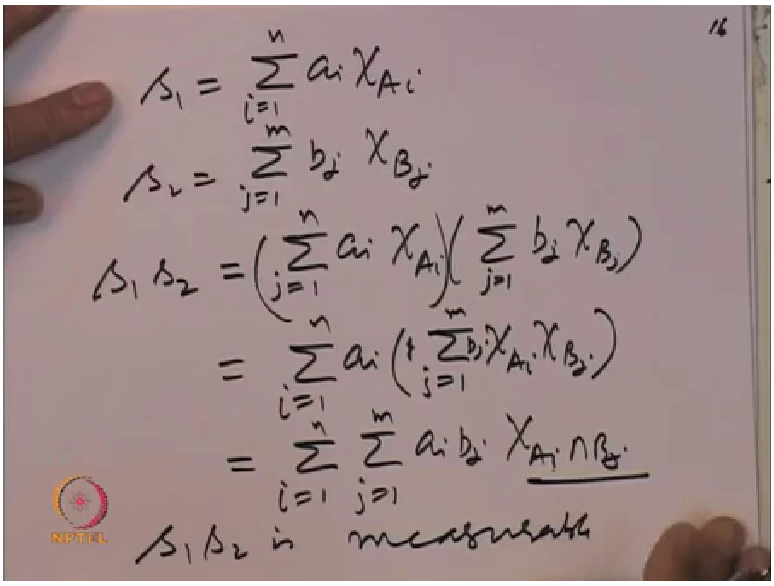

So, for that let us take s 1 is sigma a i indicator function of A i and s 2 is sigma j equal to 1 to m, b j chi of B j then s 1 multiplied with s 2 is nothing, but we can do distributive law. So, 1 to n a i of chi A i summation b j 1 to m chi of B j. So, we can write this as summation over i 1 to n a i summation over j 1 to m chi of A i chi of B j into that constant B j. So, let us write that the B j to be here right.

So, anyway we did not have done that much we could have just said that is chi indicator function of a i times s 2 each one of them is a simple. So, anyway this we can be written as summation over i 1 to n summation over j equal to 1 to m a i b j chi of A i intersection B j and. So, once again s 1 into s 2 is a linear combination of indicator function of sets where A i belong to s because s 1 is measurable b j s belong to the sigma algebra as s because s 2 is measurable. So, intersection is measurable. So, s 1 s 2 is measurable.

So, the product of a simple measurable functions is again measurable.

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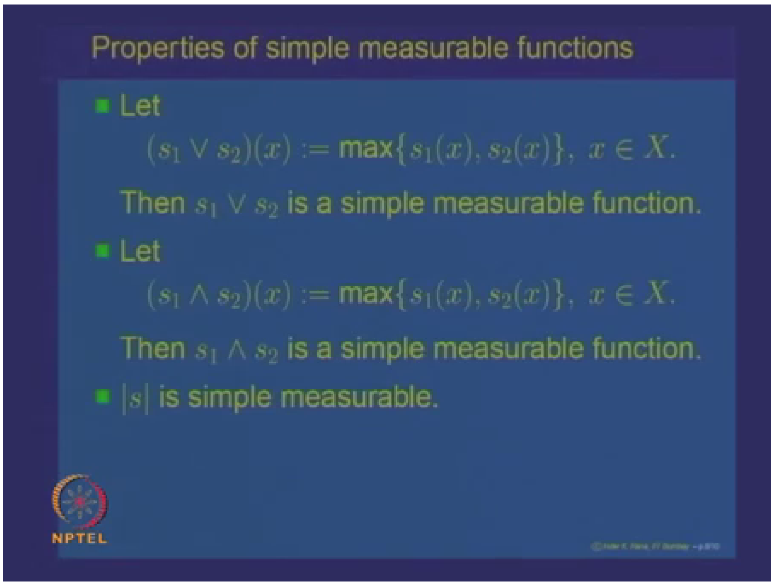

Let us go a step further and let us define for given two simple functions s 1 and s 2 let us define what is called s 1 maximum of these two function s 1 V s 2. So, what is s 1 V s 2? This is a function whose value at a point x is defined as the maximum of the numbers s 1 x and s 2 x. So, at every point x compare the values of s 1 and s 2 whichever is higher define the value to be that number. So, the claim is s 1 V s 2 is also a simple measurable function; once again the technique is same as for the sum.

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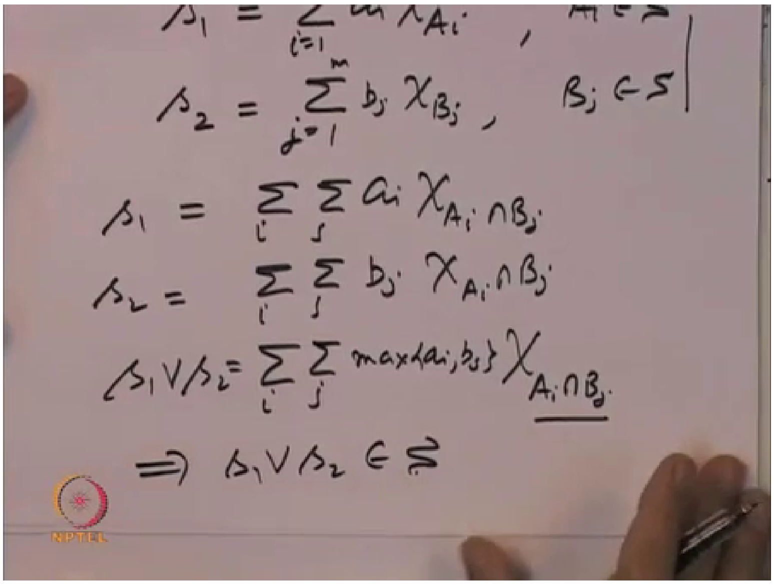

So, let us write s 1 equal to sigma a i chi of A i, i equal to 1 to n and let us assume s 1 is simple; that means, all the a is are in the sigma algebra s.

Similarly, s 2 is measurable. So, let us write s 2 as j 1 to m and b j chi of B j where b js belong to s and now we bring them to common sets as before. So, let us write s 1 as sigma over I, sigma over j a i chi of A i intersection B j and s 2 equal to sigma over i sigma over j, b j chi of A i intersection B j then s 1 maximum s 2 at any point x. So, we want to define what will be at any point x the value of this. So, look at a point x either it will be in one of the sets A i intersection B j then s 1 will give the value a I, s 2 will give the value b j and maximum of that has to be put.

So, it is maximum of a i b j if x belongs to Ai intersection B j. So, s 1 V s 2 is nothing, but summation over i summation over j of this. And this is once again a finite linear combination of characteristic function where the sets involved are in the sigma algebra. So, this will imply s 1 wedge s 2 belong to s. A similar argument will imply that the corresponding minimum of the two simple function measurable functions is also a measurable function.

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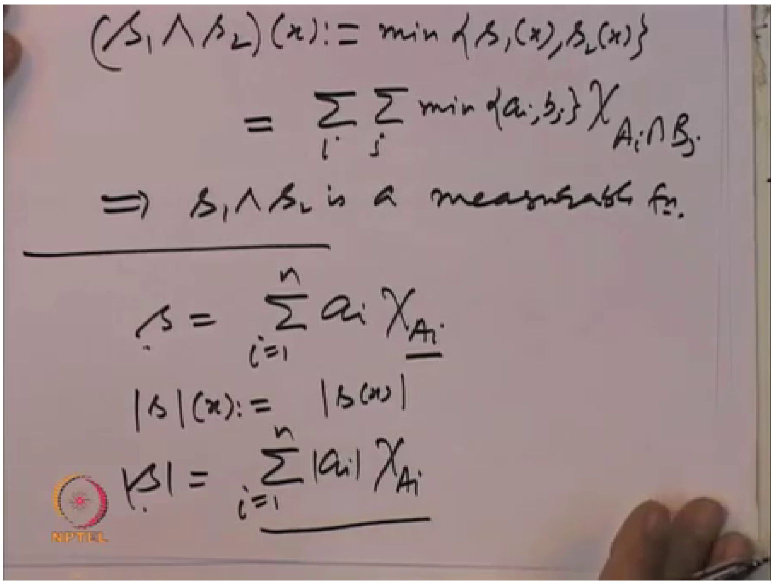

So, what is the minimum function? So, s 1 wedge s 2 at a point x is defined as the minimum of s 1 x, s 2 x that is called the minimum of the two functions and we want to show that also belongs to is a simple measurable functions. So, once again if s 1 is defined as this and s 2 is defined as this then what is s 1 wedge s 2. So, this can be written as simply sigma over I, sigma over j minimum of a i b j indicator function of A i intersection B j. So, once we write it that way it becomes clear that the minimum also is a. So, implies that s 1 wedge s 2 is a measurable function whenever s 1 and s 2 are measurable function.

So, not only the maximum the minimum also is a simple measurable function, whenever s 1 and s 2 are measurable functions and let us finally, prove that if s is simple measurable than mod s is also a simple measurable function. Well that is there are many ways of looking at this one can also look at if s is equal to sigma a i chi of A i 1 to n then what is mod s? Mod s is a function defined at x to be equal to it is mod of s of x right.

So, mod s is nothing, but sigma mod of a i, i equal to one to n indicator function of a i. So, this also is a measurable because it is if s is measurable each A i is a measurable set and mod s is a linear combination of indicator function of sets which are measurable. At this point it is worthwhile noting few things about mod of a function.

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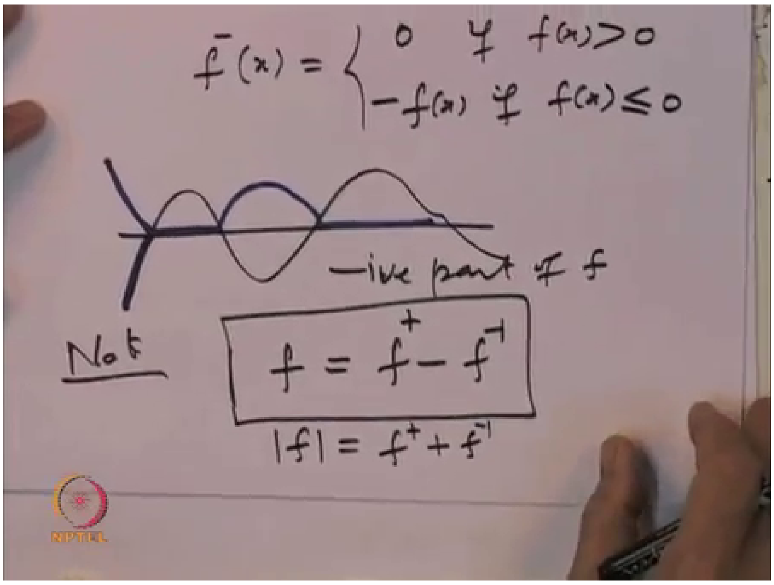

So, let us take any function f from x to R or R star let us look at function let us define f plus of x to be a function on x as follows it is equal to f of x if f of x is greater than are equal to 0 and it is 0, if x f of x is less than 0. So, what we are saying is look at the value are the function f of x, if it is bigger than or equal to 0 then you keep the value of the function as it is as soon as it goes below you cut it off by the value zero. So, if this is the function then f of x then what is f plus. So, f plus when it goes below you keep the value to be zero because is going below, because it is up you keep it as it is now it is going below. So, keep the value to be 0 and now it is going up right.

So, this is the value of this is the function f plus. So, this is called the positive part of the function. So, this called the positive part of the function similarly we can define what is called the negative part of the function to be as follows. So, given a function f from x to r R star the negative part of the function f of x is defined as 0, if f of x is bigger than 0 as soon as becomes positive we make it zero and we make it equal to minus of f of x if f of x is less than or equal to zero keep in mind the negative.

So, what it says, look at if this is the graph of the function then what do we do? We look at the graph as soon as it it is below we keep it as it is when it is going above when it is it is zero if f is positive. So, on the positive part we keep it is below we reflect it up. So, it is this this this and this and so on. So, note that. So, this is called the negative part part of f.

So, let us observe the function f is written as the positive part minus the negative part. Every function can be represented as the positive part and the negative part and both these functions are non negative functions and mod of f can be written as f plus f minus. So, that is the mod f, you can also think of the positive part f plus can be also thought of as the maximum of f and the function constant function 0 and f minus can be thought as maximum of minus f and 0 so, other ways of looking at it.

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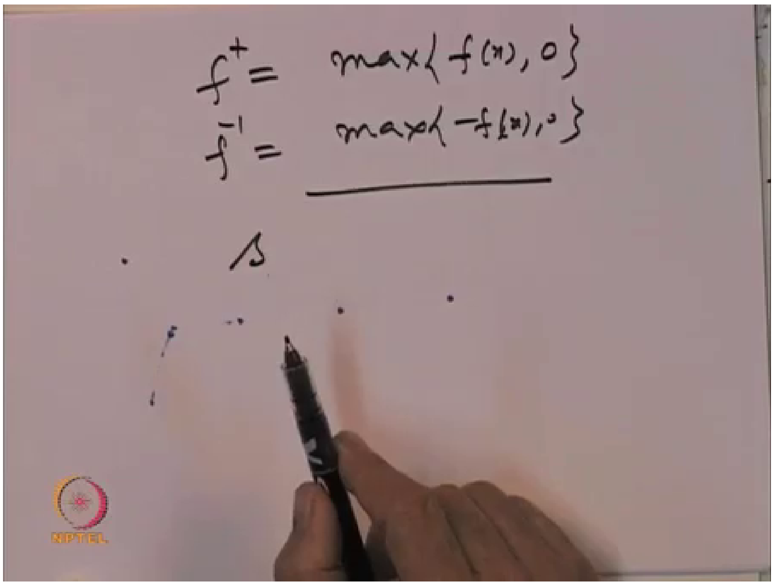

So, so if for a simple function saying that mod f is measurable can also be looked at because if s is measurable simple function is measurable, the maximum of simple function and 0 is measurable, the positive part is measurable the negative part is measurable and hence mod f will be also measurable. So, we will continue properties of measurable functions in our next lecture, and in the next lecture we will prove an important theorem namely we look at how sequence is a measurable functions behave where the limits of sequences is a measurable functions are measurable or not.

Thank you.