Measure & Integration
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Lecture - 14 A
Measurable Functions

Welcome to lecture number 14 on measure and integration. Today will start looking at functions on measurable spaces, they are called measurable functions.

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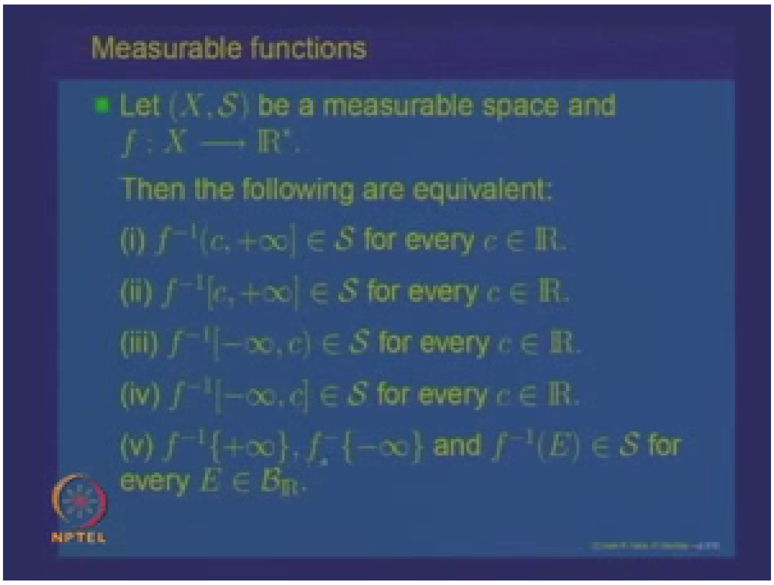

To start with will assume that we have a measurable space X S. So, is a set S is a sigma algebra of subsets of the set X and we have a function f defined on X taking extended real values. So, the R star denotes the set extended real line, that is a set of all real numbers together with plus infinity and minus infinity and with the possible operations that we have defined earlier.

So, will be looking at functions which are extended real valued defined on the set X. To start with you want to prove the following namely, for this function f the following statements are equivalent inverse image of the interval open interval c closed at infinity if you take the inverse image of any such interval then that belongs to the sigma algebra S. We will show that this is equivalent to say that the inverse image of the closed interval c to infinity belongs to S, for every c the real number. And also this is equivalent to saying that the inverse image the interval minus infinity to c minus infinity closed c open also belongs to the sigma algebra S. And then will show that this is also equivalent to saying that the inverse images of all the intervals of the type to minus infinity to c, c closed belongs to S for everything belonging to R.

So, will show that these 4 are equal and to each other and also these are all equivalent to the following namely the points f inverse of plus infinity and the set f inverse of minus infinity along with f inverse of every set E a borel set in R they belong to S. So, we will show that for a function f defined on X taking extended real values send it real number as the values these 5 conditions are equivalent. So, methodology is going to be will prove one is equivalent 2, 2 is equivalent 3, 3 is equivalent to 4 and any of them is equivalent 5. So, let us start proving this properties.

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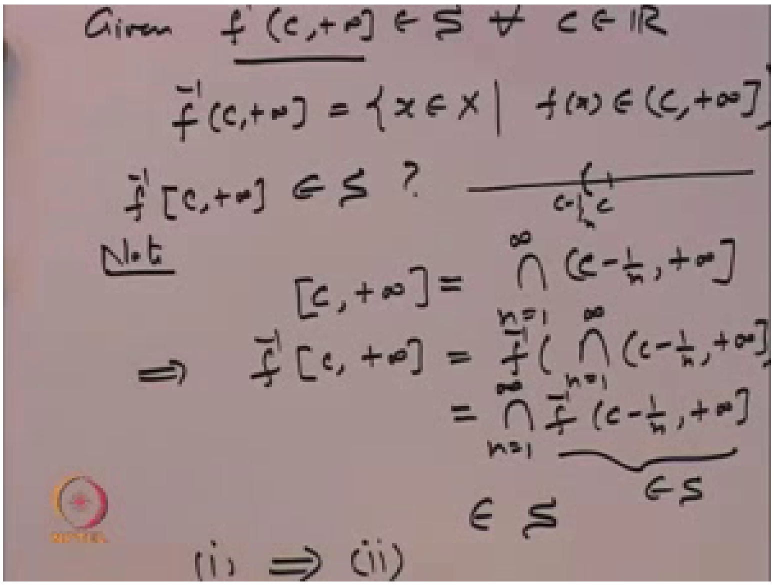

So, first property is we are given that f inverse of c to plus infinity belongs to S for every c belonging to R. And we want to prove the same property for f inverse of keep in mind what is f inverse? This is the f inverse of c plus infinity, is the set of all point X belonging to x.

Such that f X belongs to c to plus infinity. So, this is set of all points X in the domain which are mapped in 2 the intervals c to infinity, f inverse does not mean that the function is invertible or anything this is the symbol used it is a pull back off the points which go into this c to plus infinity. So, we want look at f inverse of closed interval c to plus infinity and you want to show that this belongs to S. So, to show that let us observe the simple set theoretical equality namely the closed interval c to plus infinity can be written as intersection of look at the open interval c to c minus 1 by n to plus infinity and look at the intersection of all this intervals.

So, keep in mind here is c and hear is c minus 1 over n if the take this open interval. So, this open interval c minus 1 by n 2 plus infinity includes this close interval c 2 infinity for every n. So, intersection also included and actually this is equal because given any point which is slightly bigger than c can be excluded by taking n sufficiently large. So, c minus 1 over n converges to c that is the basic idea. So, this is the simple identity about intervals which would be easy to prove.

So, and then this implies that the f inverse of c to plus infinity is equal to f inverse of intersection and equal to 1 to infinity of c minus 1 over n 2 plus infinity. And here is the another of simple observation that the inverse images of intersection are same as intersection of the inverse images. So, this is equal to f inverse of c minus 1 over n to plus infinity closed. And we are given that whenever the interval is the type c to plus infinity open inverse image is s, so, each one of this sets belongs to S is the sigma algebra. So, intersection belongs to S. So, this belongs to S. So, basically what we have done is the close interval c to plus infinity is written as a intersection of open intervals c minus 1 over n to plus infinity. And observing that the inverse images of intersections are interactions of inverse images we get that f inverse of the closed interval c to plus infinity belongs to s.

So, this implies. So, we approved one implies 2, let us show that 2 also implies one.

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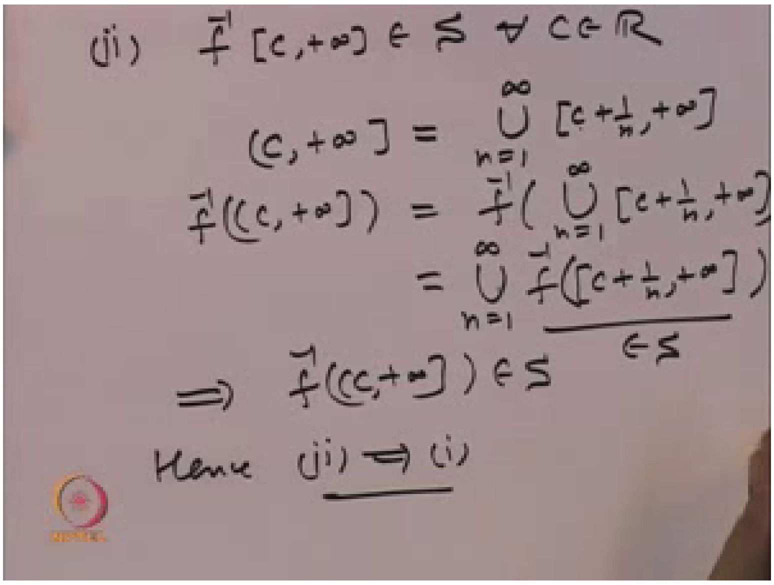

So, what is a statement 2? Statement 2 says f inverse of the closed interval c to plus infinity belongs to S for every c belonging to R. So, that is the statement 2. So, we want to now prove the same thing for open intervals. So, the idea is open interval c to plus infinity can be expressed as union of the closed intervals c plus 1 over n to plus infinity n equal to 1 to infinity. And that is quite easy to verify with the interval c plus 1 by n is to infinity is inside the interval c to plus infinity.

So, this union is inside it and conversely is easy to check that because c plus 1 over and goes to c this is actually equal to c 2 infinity. So, now, once again observing that the inverse image of c 2 plus infinity is equal to f inverse of the union n equal to 1 to infinity c plus 1 over n to plus infinity. And once again a simple observation that the inverse images is of union is union of the inverse images is so that gives us that this is f inverse of c plus 1 over n to plus infinity. And we are given that each one of them belongs to S and this union of intra union of sets in S s is a sigma algebra. So, implies that this set also f inverse of c to plus infinity also belongs to s.

So, hence we have shown that 2 implies 1. So, 1 implies 2 and 2 implies 1. So, thus we have shown that the statement one, implies statement 2 and the statement 2 implies 1. And if you will see the proofs carefully in both of them we are just try to represent a closed interval as a intersection of open intervals. And also in the 2 implies one we have tried to use the fact that you can represent in open interval as a union of closed a intervals, similar facts are used in prooving the remaining statements. So, let us just prove the statements namely 2 implies 3.

So, let us prove 2 implies 3. So, the statement 2 is regarding closed intervals.

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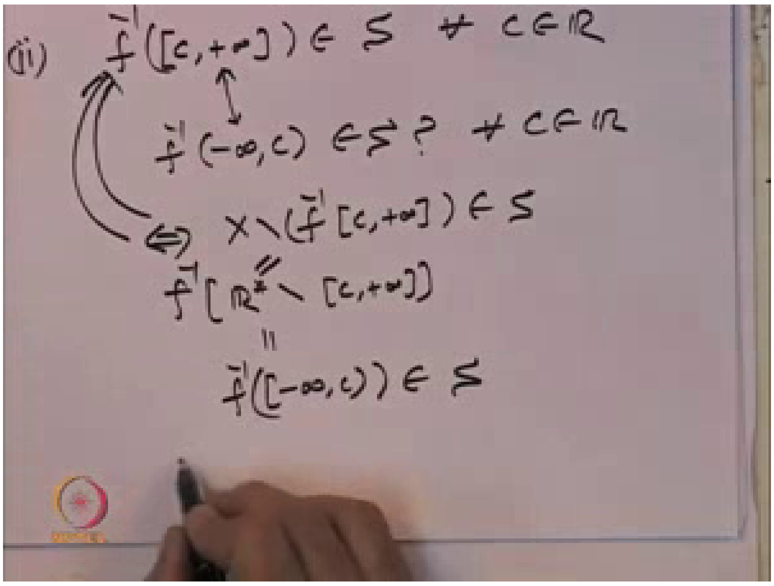

So, we are given that f inverse of this belongs to S for every c belonging to R. So, that is the statement 2 which is given and we want to show that f inverse of minus infinity to c open belongs to S for every c belonging to R. And if you look carefully this interval and this interval are related with each other they are namely complements of each other. So, the given statement implies that this because this belongs to S. So, it is complement. So, X minus f inverse of c to plus infinity also belongs to S. And this set the complement of this is nothing, but. So, here is a small observation that complement of the inverse image is nothing, but the inverse images compliment.

So, this set is equal to f inverse of R star minus c 2 plus infinity. And that is equal to f inverse of minus infinity to c open because here is c is closed. So, this also belongs to S because S is a sigma algebra. So, if a set belongs it is complement belongs. So, this belongs to S. And see all this statements are reversible if this belongs then it is complement belongs. So, these are if and only if statements. So, 2 implies 3 is obvious by taking complements. Let us prove 3 implies four. So, the statement 3 implies. So, what is given.

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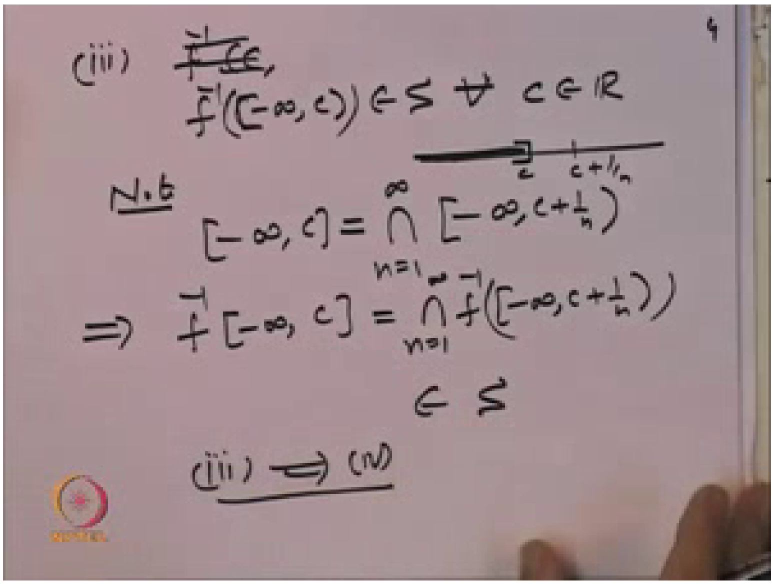

To us a f inverse of c to sorry f inverse of minus infinity to c open that belongs to S for every c belonging to R. And from here we want to conclude this closed interval.

So, note. So, once again note that the closed interval to c is equal to. So, let us you want to include the point c inside. So, it is nothing, but look at minus infinity to c plus 1 over n the open interval. So, here is c and here is c plus 1 over n. So, this interval the closed interval is already inside c plus 1 over n for every n. So, if I take the intersection of all these. So, that will give us the closed interval minus infinity to c. So, similar to the earlier argument this implies that f inverse of minus infinity to c which is equal to. So, f inverse of the intersection that is the intersection of the inverse image is f inverse of minus infinity to c plus 1 over n, open and each one of them belongs to this S belongs to S. So, 3 implies 4. So, if f inverse of minus infinity to c open belongs then f inverse of minus infinity 2 c close also belongs. So, 3 implies 4 l h proof the converse statement namely 4 implies 1. So, we are given.

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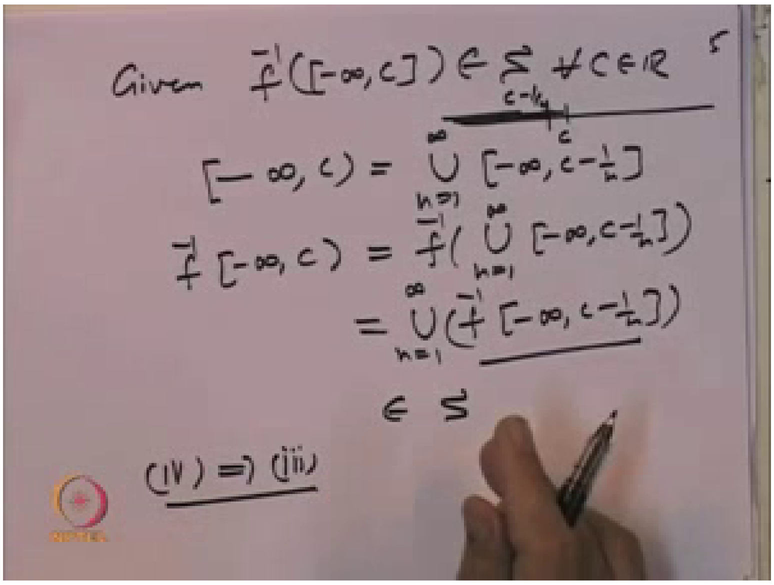

F inverse of minus infinity to c close belong to S for every c belonging to R. And we want to look at f inverse of the open interval c.

So, that. So, once again the similar situation that is minus infinity to c it is here. And we want to look at the open interval. So, let us look the union of intervals minus infinity to c minus 1 over n equal to 1 to infinity. So, there all here is c minus 1 over n. So, these are all inside it closed interval. So, the unions will give us this open interval. So, once again taking the inverse images is minus infinity to c is equal to f inverse of the union n equal to 1 to infinity. And f inverse is union is union of the inverse images. So, that gives us minus infinity to c minus 1 over n. And all of them each one of them is given to be inside s.

So, that implies that this belongs to S. So, 4 implies 3 also 2. So, what we are shown till now is that all the first 4 statements are equivalent to each other. So, first statement was about intervals of the type minus infinity to c to infinity, open the next one was c close and then next was minus infinity to c. So, all inverse images is of all these types of intervals are inside R inside S. They all the statements are equivalent to each other. Now let us prove that this implies that f inverse of plus infinity and f inverse of minus infinity and f inverse of every borel set is inside S.

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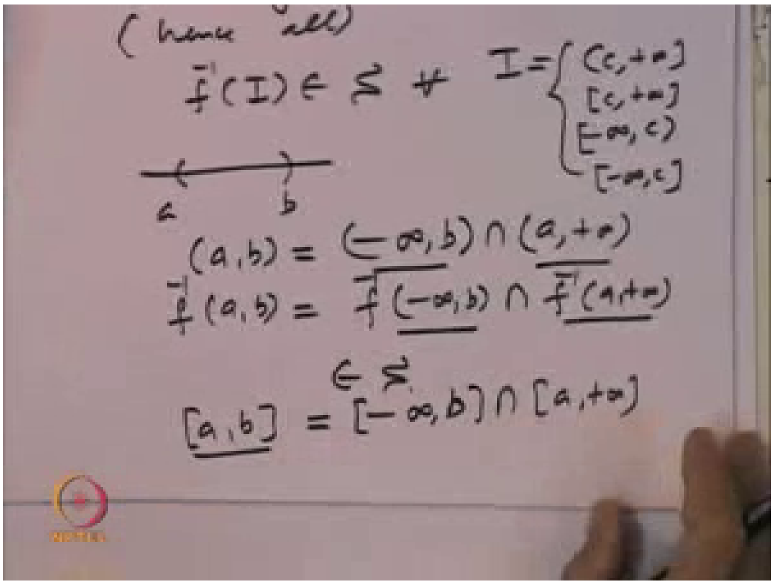

So, let us assume any one of 1 to 4. So, assume any one of statement 1 to 4. hence all because they are all equivalent. So, we know that f inverse of a interval belongs to S whenever for every interval i which looks like either c to plus infinity or looks like closed c to plus infinity or it looks like minus infinity to c open or minus infinity c closed. So, for all this 4 type of intervals the first 4 state any one of the first 4 statements implies they belong to S. Now look at any other interval, supposing i is a open interval a to b open interval a to b, then we can write this open interval a to b as minus infinity to b open interval intersection with minus infinity to b intersection with open interval a to plus infinity. And we know that inverse image of this interval belongs to the sigma algebra inverse image of this belongs to the sigma algebra.

So, that will give us that the inverse image of a b is equal to f inverse of minus infinity to b intersection f inverse of a to plus infinity and both belongs to the sigma algebra. So, this is belong to the algebra S. So, what you am trying to say is that any one of the statements 1 to 4 imply that inverse image of every open interval also belongs. And similarly we can take actually closed interval also for example, a close interval a b can be written as minus infinity to b, intersection a to plus infinity and similarly argument with imply that in f inverse of this interval also belongs to S.

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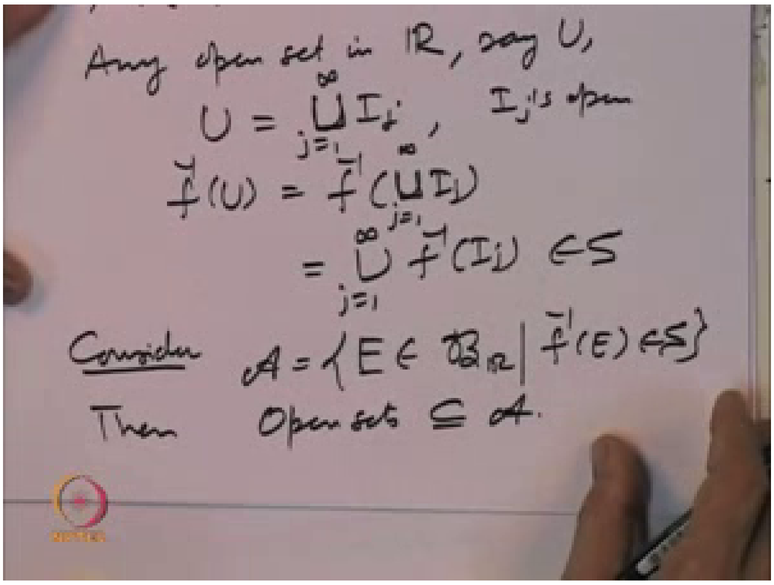

So, if you assume any one of this statements 1 to 4 then that implies imply that f inverse of every interval belong to S for every interval. Now recall that in the real line any open set say open set in R is a countable union of open intervals. So, in R say open set is u then u can be written as union of i js j equal to 1 to infinity i js open. Actually you can be write it as a disjoint union of open intervals also accountably disjoint union of open intervals. So, f inverse of u will be equal to f inverse of union disjoint union of i js and which is same as union of f inverse of i js j equal to 1 to infinity. I think disjoint is not needed, but any way this is. So, and f inverses of every open interval belongs to S. So, this belongs to s.

So, if we assume any one of the 4 conditions then that implies that f inverse of every open set is in the sigma algebra. So, now, this here is the sigma algebra technique. So, consider the class a of all sets belonging to S all sets belonging all subsets in b R such that f inverse of E belongs to S. Then just now what we showed the open sets are inside A. And it is easy to check that a is a sigma algebra and A is a sigma algebra. So, let us check that that a is a sigma algebra. So, why a is sigma algebra. So, clearly empty set.

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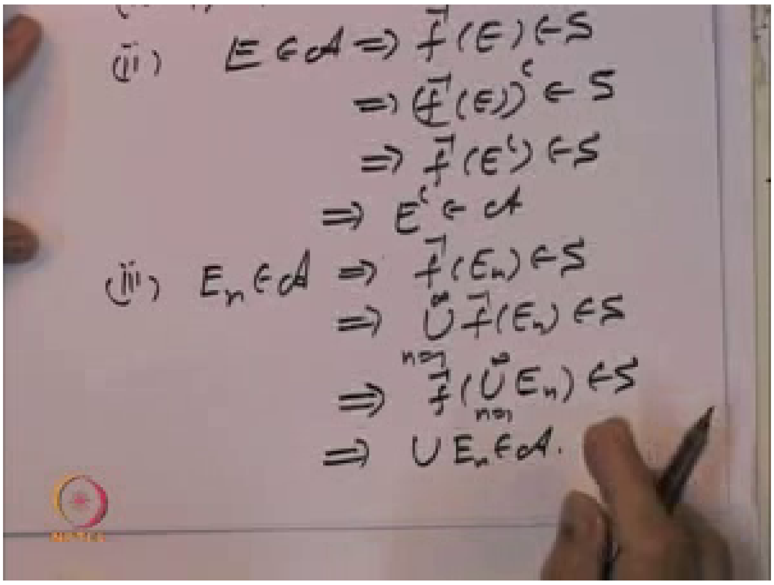

And the whole space and the whole space R belong to a right because X, and the empty set belong to S. Secondly, let us observe that if a set E belongs to A; that means, that f inverse of E belongs to S and that implies that f inverse of E compliment belongs to S because S is the sigma algebra and that is same as f inverse of E complement belong 2 S.

So, implies E compliment belongs to A. So, A is closed under complements and finally, if E ns belongs to A that implies and f inverse of E n belong to S. So, implies union of f inverse of E ns also belong to S. Because S is the sigma algebra and hence that implies that union of inverse images is inverse image of the union. So, union E n also belong to S. So, implies union E ns belong to A. So, we approved verified that a is a sigma algebra. So, this is the sigma algebra including open sets. So, it must include the borel sigma algebra inside, but is already a sub class of borel sets. So, A is equal to the class of borel sets; that means, if you assume any one of this first 4 conditions in the statement that we just now stated, then that implies the statement that f inverse image of every borel set is in the sigma algebra S.

Let us verify that the inverse image is of the points plus infinity and minus infinity are also.

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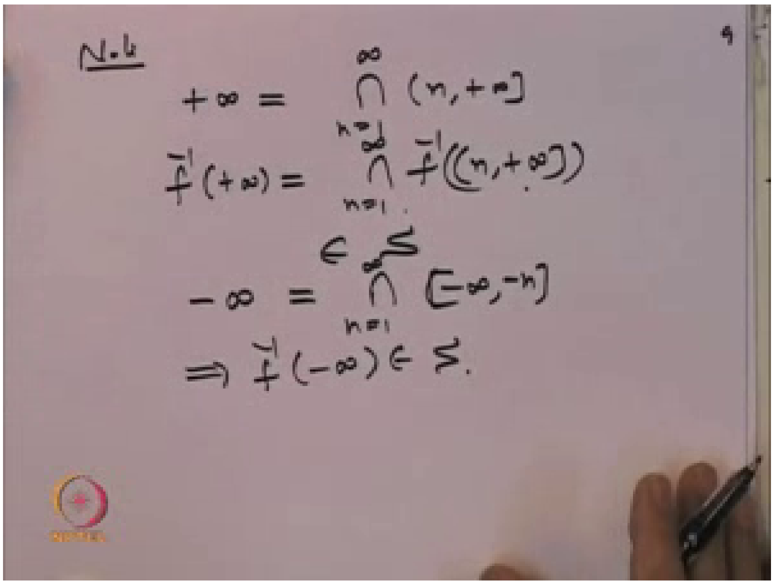

So, note. That plus infinity can be written as intersection of n to plus infinity n equal to 1 to infinity right. So, f inverse of plus infinity is equal to intersection n equal to 1 to infinity, f inverse of n to plus infinity f inverse of this is equal to f inverse of right hand side right inside is a intersection. So, it is intersection of the inverse images and each one is the interval. So, that f inverse image inverse image of each one of the intervals belong to S. So, intersection belongs to S. So, this belongs to S and a similar argument for minus infinity will implies was minus infinity can be written as intersection of n equal to n equal to minus 1 to infinity of minus infinity to minus n.

So, inverse image of this will be intersection of inverse images and will imply that f inverse of minus infinity belongs to S. So, we have shown that if you assume any one of the 4 conditions stated above then that implies that the inverse image of the point plus infinity and inverse images of very borel set belong to the sigma algebra S. The converge statement is obvious because every interval is a borel set.

So, saying that statement 5 implies any one of the statements for above is obvious because every interval is a borel set that is a special case. So, we are proved this theorem namely for a function f define on a set X taking non extended real valued functions all these 5 conditions are equivalent to each other. And if assume any one of them then other will also hold. So, a function which satisfies any one of these conditions is called a measurable function.

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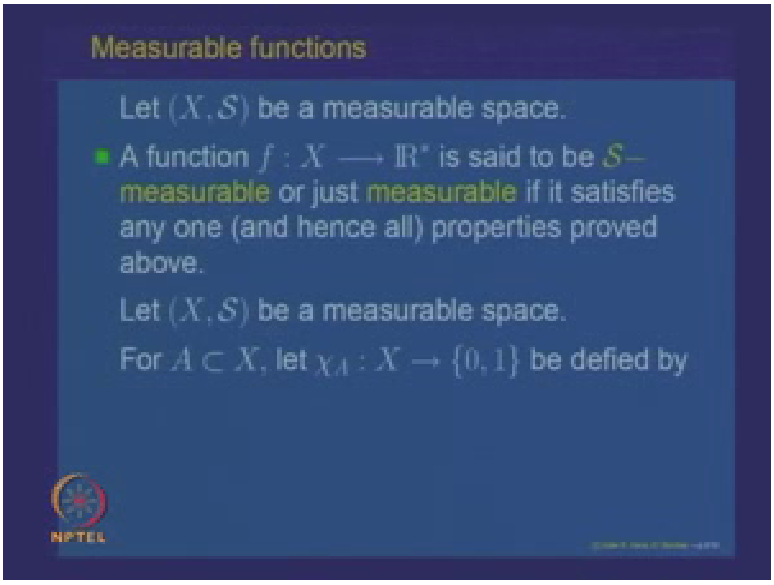

So, a measurable function on X taking extended real valued is a function which satisfies any one of those 5 conditions as stated here. So, these are going to be important class of functions for us to deal with.

Let us look at some examples. The first example is that of what is called the indicator function of a set. So, let us look at what is called the indicator function of a set x. So, let us take X any set

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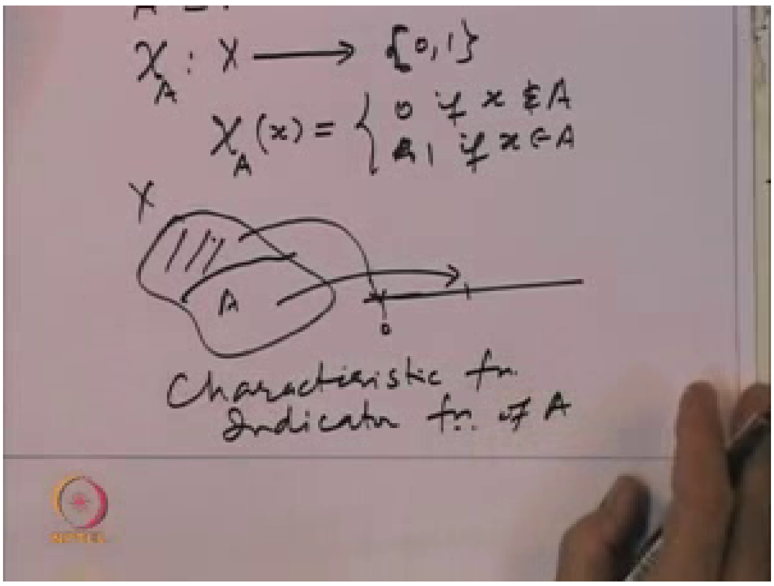

And A subset of x. So, we define a function called the chi of a this is a greek letter chi. So, lower case of X A function on X taking 2 values 0 or 1. So, this is called the characteristic function or the indicator function. So, this function takes the value at the adder a point X the value is 0, if X does not belong to A and at A, at the point The value is one if X belongs to A. So, here is the set X here is the set A. So, on a it gives the value one and on outside a it gives the value 0.

So, it is a 2 valued function it indicates. So, the points where it takes a value one is exactly the points in the set a. So, this is called the characteristic function. So, characteristic function or the indicator function of the set A. So, this is called the indicator function of the set A and the claim is. So, let we want to look at. So, X is set S is sigma algebra.

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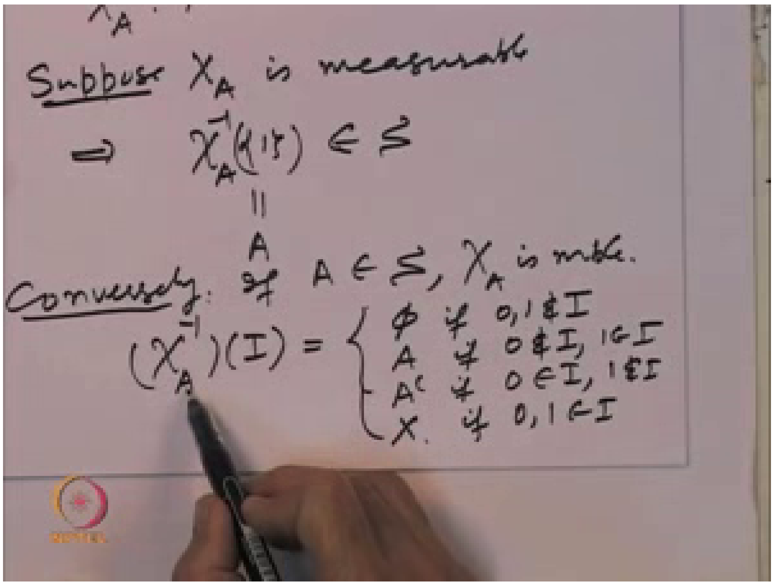

and we are got the indicator function a of the set A on X taking values. Of course, only 2 values. So, we consider it as a function in taking extended real values. So, we want to know is it measurable. So, suppose the indicator function of A is measurable

So, that implies, if I look at chi A inverse of the singleton point one that belongs to s, but what is that value. So, what are the points where it takes the value one that is precisely A. So, that is the set A. So, A belongs to S. So, if the indicator function is measurable then we get a belongs to S conversely if a belongs to S we claim that chi of a is measurable. So, for that look at chi a of chi a inverse of any interval i. So, what is that going to be the inverse image of a interval is going to be equal to empty set if 0 or one does not belong to the interval i because then there is no point which goes to the interval and it is equal to it is equal to A if 0 does not belong to i and 1 belongs to i and similarly it is a compliment if 0 belongs to i and if 0 belongs to i and 1 does not belong to i and is equal to X if both 0 and 1 belong to i.

So, in either case either it is empty set or it is a set a or a compliment or X. And all of these are elements of the sigma algebra S. So, inverse images is of every interval is in S. So, hence the indicator function is a measurable function. So, what we have shown is that the indicator function is measurable. So, this indicator function which is defined as one if X belongs to a and 0 if X does not belong to a. So, the characteristic function is measurable if and only if the set a belongs to s.

So, these simplest example of measurable function, let us consider a linear combination of the indicator functions. Suppose S is a function defined on X size and S of X is equal to a i times the indicator function of a set a I at value added X i equal to 1 to 1.

So, look at sets a 1 a 2 a n subsets of X look at their indicator functions and take a linear combination of them, a i times the indicator function of a i such a function is called a simple function on X such a function is called a simple function on X.