Measure and Integration
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Lecture - 13B
Characterization of Lebesgue Measurable Sets

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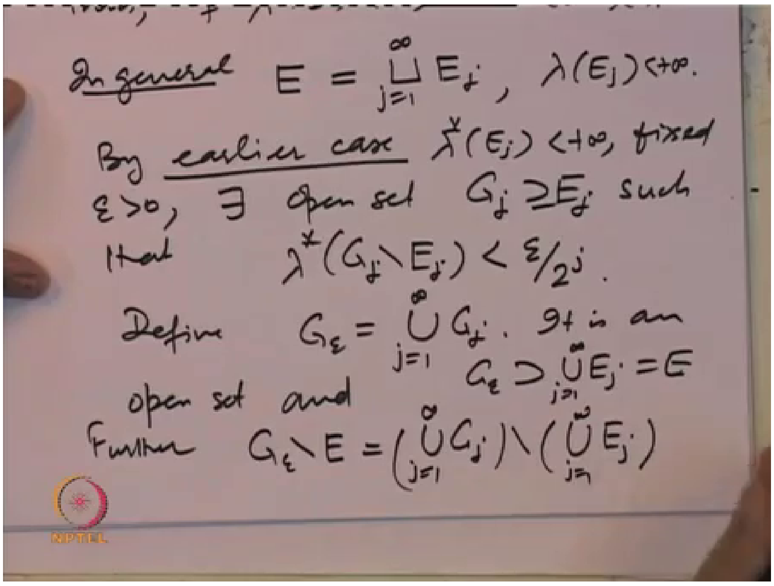

So, that is what we are going to do now, so let us in the general case, so in general the set E which is Lebesgue measurable may not have a finite Lebesgue measure, but the Lebesgue measure being sigma finite you can write E as a disjoint union of sets E j j equal to 1 to infinite such that lambda of E is E j is finite, it is a measurable set, and it is finite.

So, now by the earlier case because, lambdas star of E j is finite, for fixed epsilon bigger than 0 there exist open set call it G j which includes E j such that lambda star of G j minus the set E j is less than the small number epsilon, but we are going to write less than epsilon in to the power 2 to the power j, and will see soon why we are doing that because for each piece we are going to make it small, and getting going to add of these pieces. So, now, define the set G epsilon to be equal to union of G j j equal to 1 to infinity.

So, then G j is open, so it is an open set because it is a countable union of open sets G j's and G epsilon includes G j includes E j so, includes union of E js which is equal to E so, it is an open set which includes E. Now let us look at the difference further so, let us observe further that G epsilon minus E what is that equal to that is union of G j's minus union of E j's right, that is the refinish that is how we constructed, and now here it is a simple set theoretic property namely that, this is a subset of union j equal to 1 to infinity G j minus E j.

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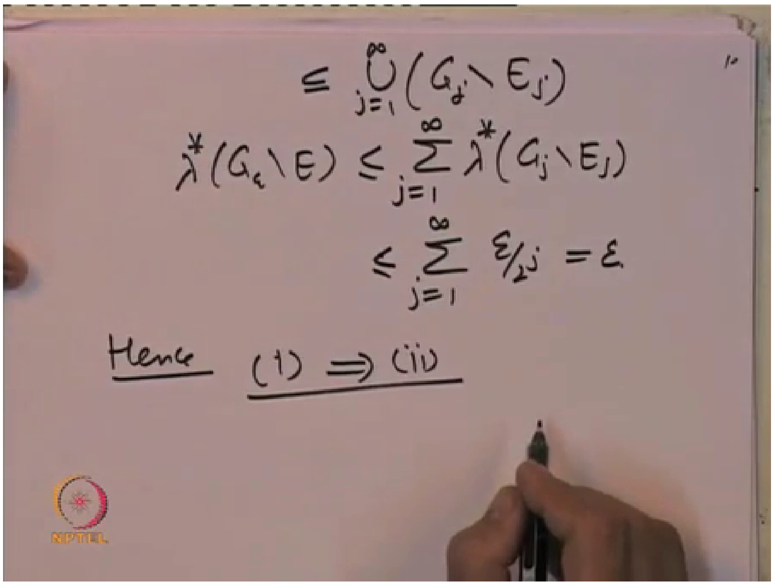

So, this is a simple set theoretic that union of G j's minus union of E j's is a subset of union of G j minus E j, and once that is verified which is easy to verify we get that lambda star of G epsilon minus E is less then are equal to sigma j equal to 1 to infinity lambda star of G j minus E j is which by our choice is less than epsilon to this power j, so this is less than are equal to summation j equal to 1 to infinity epsilon to 2 the power j which is equal to epsilon. So, that proves the second property completely in the general case also. So, hence what we are shown is that 1 implies 2, namely if E is Lebesgue measurable, then I can find given epsilon we can find a open set G epsilon such that the outer measure of this is small less than epsilon.

So, now let us go to the second step of the verification. So, we have verified the first step namely 1 implies 2.

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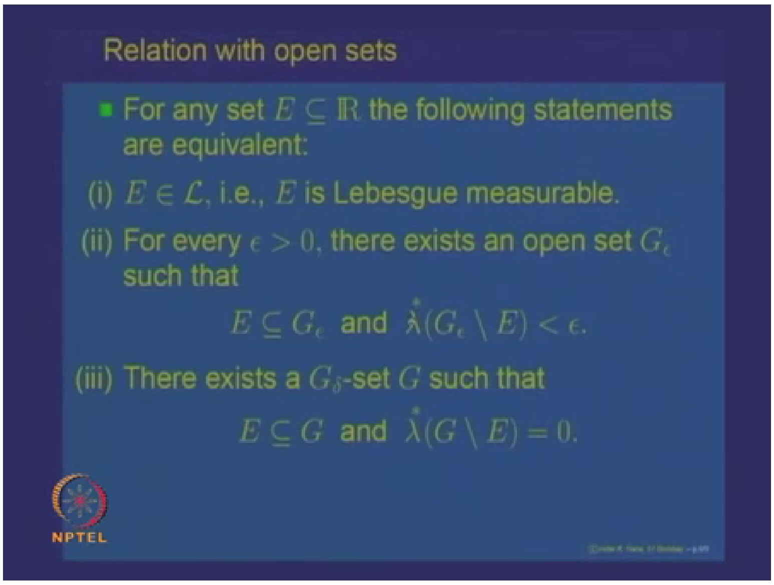

Now, let us verified that 2 implies 3, so let us assume 2 holds.

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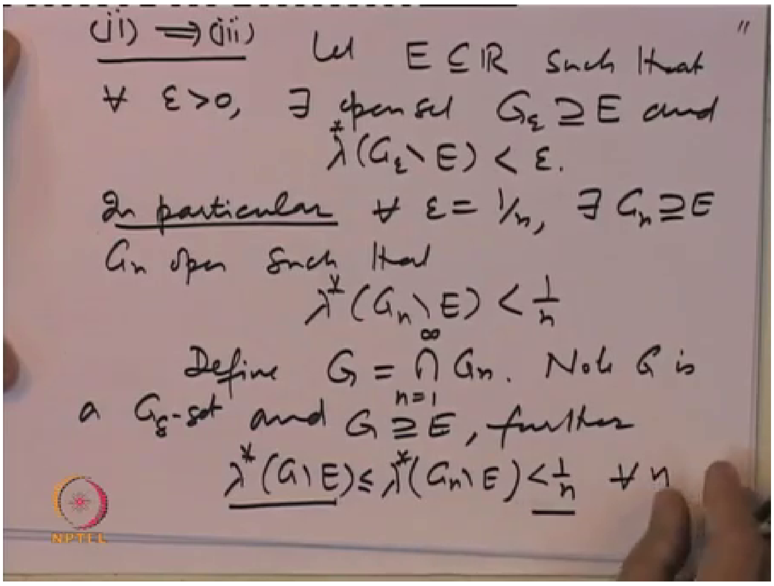

So, 2 implies 3, we are given a set E. So, given 2 means let E be a subset of real line, such that for every epsilon bigger than 0 there exist a open set G epsilon which includes E and lambda star of E is a just a set so, we can write a lambda, now it will lambda star of G epsilon minus E is less than epsilon. So, that is what is given to us, and where to construct a set such that the difference has got measure Lebesgue measure zero. So, the obvious ways make this epsilon small and small, so, in particular that says for every epsilon equal to 1 by n there exist G n including E G n open such that lambda r of G n minus E is less than 1 by n and so, we are specialised this a given condition for each epsilon equal to 1 by n, and we got an open set G n which includes E.

And now, because we want it small s we want to let this becomes smaller and smaller, so that it says the following, so define G equal to intersection of G n n equal to 1 to infinity. So, what is G g is intersection of open sets, so note G is what is called a G delta set, so it G delta is by definition in intra section of open sets, and this set G includes E because each G n includes E, so includes E. so G also includes E further lambda star of G minus E note G minus E is a subset of G minus E because G is a introspection.

So, G minus E is a subset of G n minus e so, by countable by monotone property lambda star of G minus E is less than lambda star of G n minus E which is less than 1 by n. So, lambda star of G minus E is less than 1 by n for every n.

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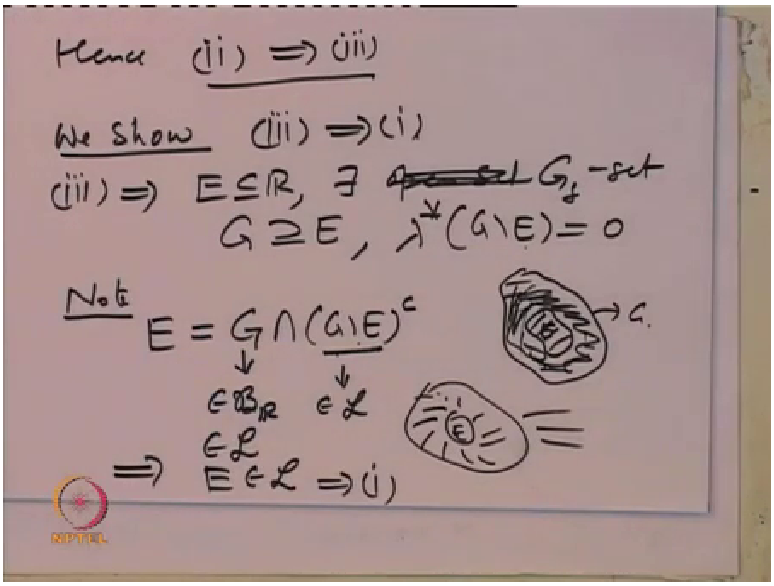

So, that implies the fact lambda star of G minus E is equal 2 0. So hence we are shown 2 implies 3 and now let us conclude the proof by showing, so let us show we show that 3 implies 1.

So, what is 3? So, three says that by 3. So, three implies that for E a subset of R there exist open set G sorry there are exist sorry not an open set there exist a G delta set G including E and lambda star of G minus E equal to 0. So, here is the set E that is the set E inside and this is a set G which recover set. So, that the remaining part has got measure

0, but note what is E. E is same as you take the set G this is a full set, and introspected with the compliment of the outer portion.

So, this is the compliment look at the complement of this. So, intersection G minus E compliment. So, is simple observation because what is G minus E compliment. So, this inside portion is right let me just draw a picture again. So, this is inside is E and outside is G. So, this shaded portion is E minus. So, what is complement? Complement is the outside potion here and E. So, we interested with G you get E. So, is nothing, but G intersection G minus E complement and this is a G delta set, and a G delta set is a intersection of open sets. So, this set belongs to B R and hence this that also belongs to L. So, it is Lebesgue measurable and this set G minus E has got outer measure 0.

So, G minus E belongs to is a Lebesgue measurable set because all sets of outer measure 0 are Lebesgue measurable. So, this set also belongs to L and so intersection of 2 Lebesgue measurable sets is Lebesgue measurable. So, this implies that E belongs to L. So, three implies. So, this implies one. So, we this proves completely the fact that the three properties that E is Lebesgue measurable is equivalent to saying for every epsilon bigger than 0.

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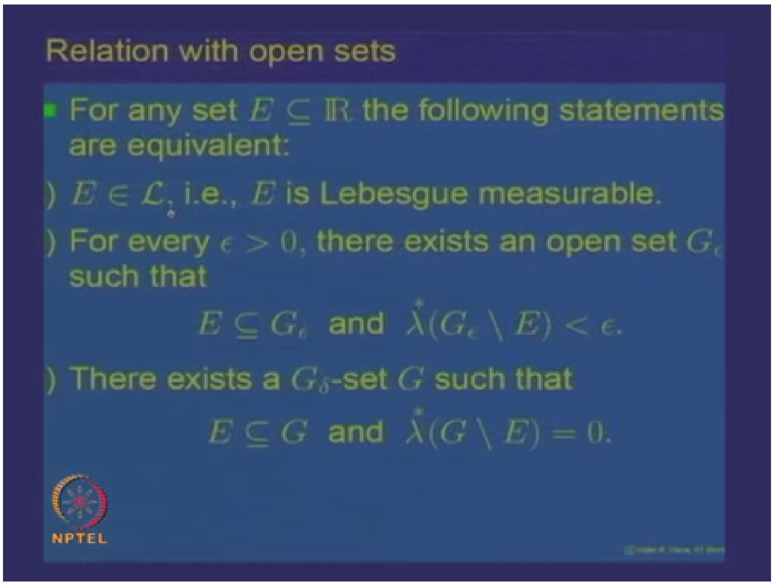

There is an open set G epsilon such that E is a subset of G epsilon and lambda star of G epsilon minus E, the difference is outer measure small and that is equivalent to saying that for the set E there exist a G delta set covering it so that the differences got measure 0.

So, this is. So, this gives us a characterization of Lebesgue measurable sets in terms of open subsets of a real line. A corresponding characterization of Lebesgue measurable sets is obtained in terms of closed sets, let us state that also and prove it. So, let us look at the next that for any set E in R the following is true, namely E is Lebesgue measurable is equivalent to saying for every epsilon reaction bigger than 0.

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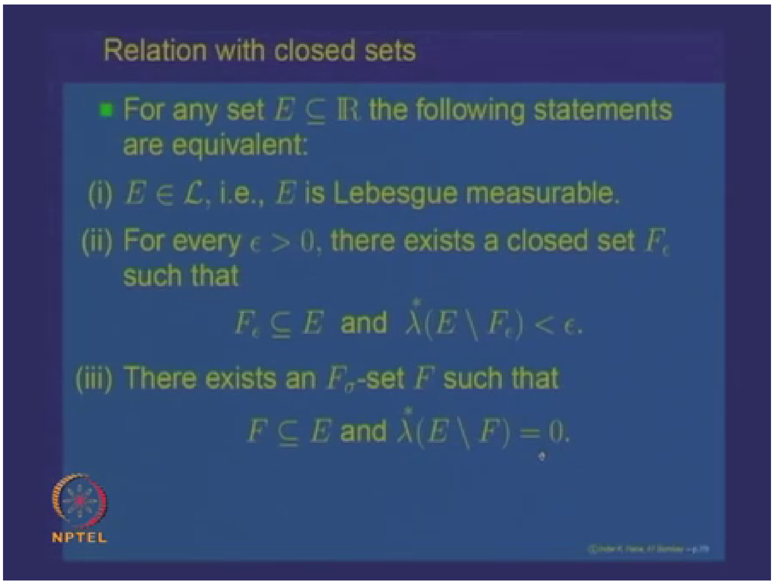

Now there is a closed set inside E said that Lebesgue measurable of the difference is small and that is equal and to saying that there is a F sigma set. So, what is F sigma set? A F sigma set is nothing, but a set which can be expressed as a countable union of closed sets such that.

So, there is an F sigma set F inside E such that lambda star of E minus F is equal to 0. So, let us quickly prove this and this will use the earlier characterization. So, suppose. So, let us suppose one holds E belongs to Lebesgue measurable sets.

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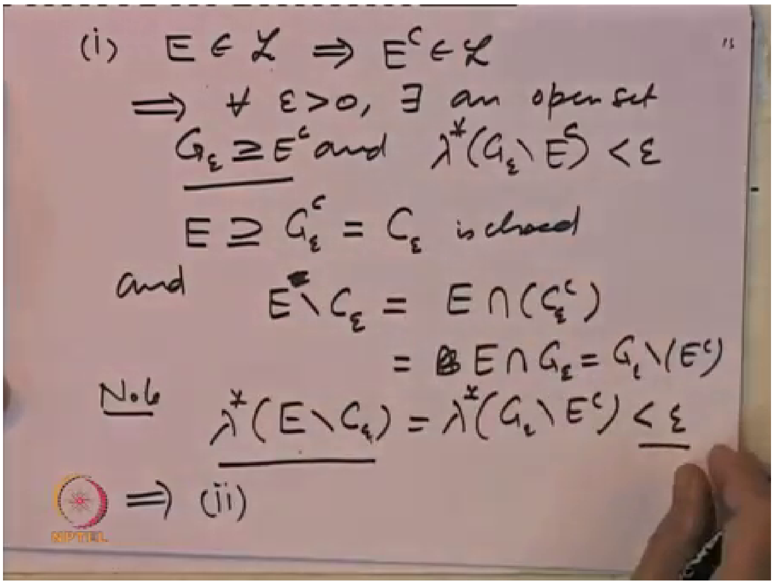

Now, E belongs Lebesgue measurable sets implies there is a open set which covers E with difference of measure small, but we want close sets. So, let us observe E belongs to L also implies that E compliment belongs to L right because its Lebesgue measurable sets is sigma algebra. So, if E is measureable complement also measurable. So, this implies by just now what we proves, E complement is Lebesgue measurable. So, for every epsilon bigger than 0 there exist an open set G epsilon such that this includes E, and outer Lebesgue measure of G epsilon minus E is less than epsilon right this is just now, we proved this fact, but.

So, this is E compliment, we are applying the previous just now proved result for E compliment and now. So, if E complement is inside G that means, E will be including G epsilon complement and note G epsilon is an open sets. So, its complement is a closed set. So, let us call it has C epsilon. So, C epsilon is closed, it includes E and we want to find what is the Lebesgue outer measure of a E minus C epsilon. So, what is that? That is E intersection what is this is the C epsilon compliment, by set theory and that is same as E intersection this is G epsilon.

So, sorry do we want a E compliment, you want to find out what is the difference between the closed set inside a E compliment. So, that is sorry no no no no to compliment E. So, E minus C epsilon E intersection C epsilon complement and C epsilon complement is nothing, but G epsilon G epsilon. So, this is G epsilon intersection E. So, we want to control. So, now, note that Lebesgue outer measure of E compliment. So, Lebesgue measure of we had E minus.

So, what is this? So, this we can also write it as G epsilon minus E complement right because E compliment will be E. So, minus C epsilon is same as outer measure of G epsilon minus E complement and that is less than epsilon. So, what we are shown if E is Lebesgue measurable, then there is a closed set inside it such that the measure outer measure is less than epsilon. So, implies 2 holds. So, one implies 2 we have proved and here we are just use the fact that set is open if and only if its complement is closed, and applied the previous criteria. So, one implies 2 we have proved.

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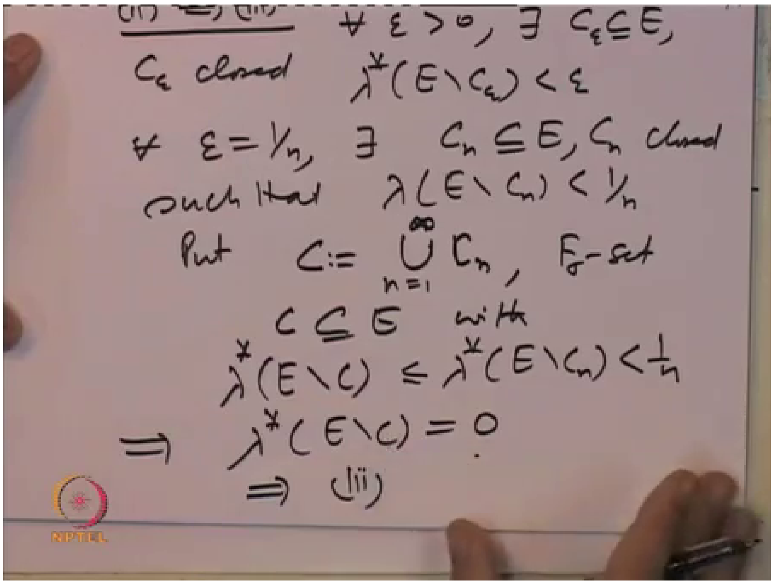

Let us look at 2 implies 3, and that is once again for every epsilon what is given that there exists a closed set C epsilon closed inside E, C epsilon closed with the property that outer measure of E minus C epsilon is less than epsilon.

So, in particular for every epsilon equal to 1 by n, there exist a set C and which is inside E, C n closed such that Lebesgue measure of E minus C n is less than 1 by n. So, put. So, technique is same. So, put C equal to union of these sets C n. So, this one to infinity. So, this is a F sigma set because it is a union of closed countable union of closed sets and.

Each C n is inside E. So, this set E is also inside E with the property that lambda star of E minus C, C is the union of all the C ns. So, if I take only one of them C will be a subset of E minus C will be a subset of that. So, it is less than or equal to lambda star of E minus C n which is less than 1 by n. So, implying lambda star of E minus C is equal to 0 and that proves. So, hence three holds. So, for given for every epsilon there is a closed set inside it, we have shown there is a closed set inside E with difference are there is a not a closed set E sigma set with the measure 0 and finally, let us put that a prove that three implies one.

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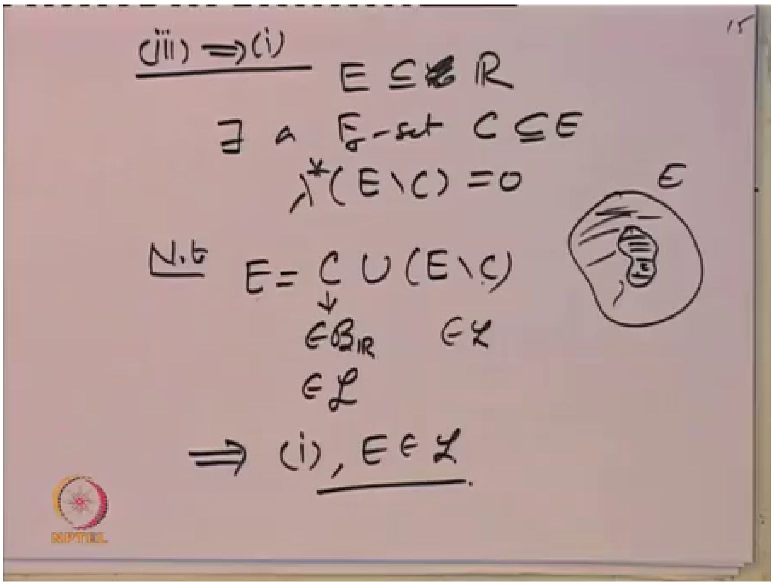

So, E is a subset of real line with the property that there exists a F sigma set C contained in E with the property, that lambda star of E minus C is equal to 0. So, here is the set E and I want a set C inside it. So, this is the set C inside it say that the differences got measure 0, but now note the set E can be written as C union E minus C right.

So, this is this portion outside and union the inside portion and this is E as sigma set. So, it is a Borel set and hence it belongs to is a Lebesgue measurable set. The set E minus C is a set of measure 0. So, that is a Lebesgue measurable set. So, E is a union of 2 Lebesgue measurable set. So, implies 1 that is E is Lebesgue measurable set.

So, we have approved third property namely 2 implies 3 and now we have just now proved three implies 1. So, the Lebesgue measurable set are very nicely connected with topological nice sets namely open sets and close sets. So, for every Lebesgue measurable set can be covered by are open set, set is the differences got Lebesgue measure small and which is equivalent to saying a set is Lebesgue measurable that is also a characterization and similarly a set E is Lebesgue measurable if and only if you can find a close set inside it sides at the differences got measure small.

So, Lebesgue measure is a very a nice measure on real line and it is defined all Lebesgue measurable sets. So, in particular it is defined for all Borel sets, and it is translation invariant it gives nice is compatible with the group structure and it also has nice properties with respect to the topological structure namely with respect to open sets and close sets.

So, with that we conclude our analysis of Lebesgue measure and Lebesgue measurable sets. In the next lecture will start looking at functions on measurable spaces and their properties.

Thank you very much.