Measure & Integration
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Lecture - 13 A
Characterization of Lebesgue measurable sets

Welcome to lecture number 13 on measure and integration. If you recall in the previous lecture we have been looking at Lebesgue measure, Lebesgue measurable sets and its properties. I will continue that study of Lebesgue measurable sets and its properties today itself. Will be (Refer Time: 00:40) looking at the translation invariance property of the Lebesgue measure and then Lebesgue measurable sets, we saw we topological in ice subsets of the real line.

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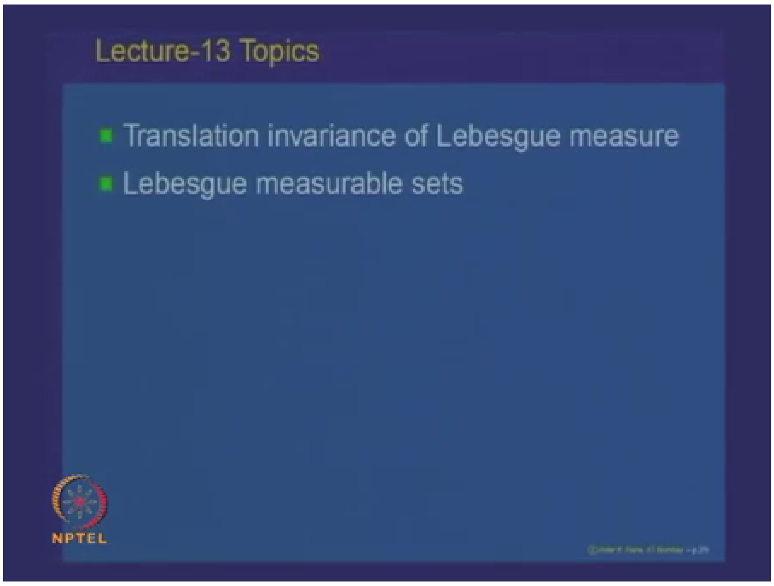

So, let us recall that we are defined what is called the Lebesgue measurable sets and that gaves give us the space the real line the Lebesgue measurable sets and the Lebesgue measure.

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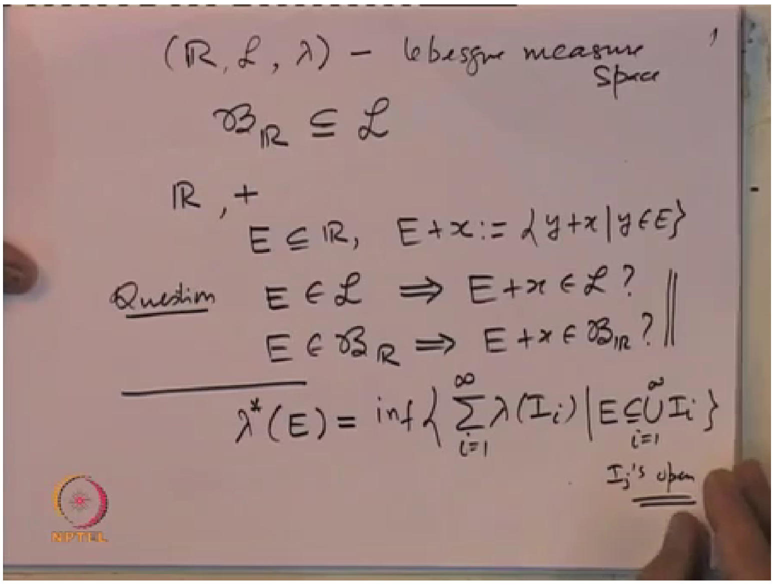

So, this was called the Lebesgue measure space and the Borel sigma algebra of real line the Borel subsets of the real line form a sub sigma algebra of Lebesgue measurable sets.

So, these properties we had seen and now today what we are going to look at is the following recall the on the real line there is a binary operation of addition you can add real numbers. So, this operation can be used to transform subsets of the real line. So, let us take a set E contained in real line and define what is called E plus x. So, E plus x is defined as all elements Y plus x such that Y belongs to E.

So, it is the said E which is translated by an element x. So, the question is E belonging to L, if E is Lebesgue measurable does this simply that E plus x is Lebesgue measurable and similarly we will also look at the second question namely if E belongs to B R if E is a Boral subset of real line does it imply E plus x belongs to B R .So, these are the two questions will start analyzing. So, the importance of these two questions is the class of Lebesgue measurable sets is it invariant under translations and is the Lebesgue is the class of Boral subsets invariant under the group operation of translation on the real line.

So, these two questions we will answer in the first to start with. So, to answer the first question let us recall that Lebesgue measure is nothing, but the restriction of the outer Lebesgue measure. So, and the Lebesgue outer measure for real line is defined as the infirmum of sigma lambda of intervals I i where these intervals form a covering of E is a subset of union I i's i equal to 1 to infinity and I i is are all intervals. And keep in mind the remark we said you can choose these intervals I i is I j is to be open if necessary. So, whether you take all possible coverings of E by intervals or all possible coverings of E by open intervals both will give you the same value namely the Lebesgue measure of the Lebesgue outer measure of the set E.

So, let us start.

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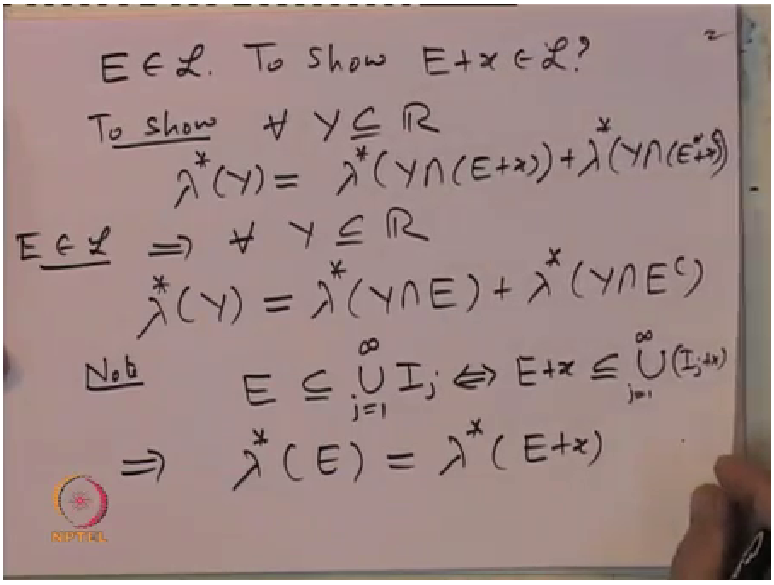

So, let us start with a set E which is Lebesgue measurable to show that E plus x is also Lebesgue measurable. So, this is the question. So, to show that recall what is definition of a measurable set. So, to show that for every subset Y of real line we should have Lebesgue outer measure of Y is equal to Lebesgue outer measure of Y intersection E plus x plus Lebesgue outer measure of Y intersection E complement plus x E plus x compliment sorry. So, that is what we have to show.

Now, let us start observing that we are given that E is Lebesgue measurable. So, E Lebesgue measurable implies for every subset Y of real line, the Lebesgue measure of Y is equal to Lebesgue outer Lebesgue measure of Y intersection E plus outer Lebesgue measure of Y intersection E complement.

Now, our aim is to transform this E to E plus x; that means, we should be looking at the properties of the outer Lebesgue measure of a set in terms of translation. So, let us observe. So, let us note that if a set E is covered by a union of intervals I j one to infinity, that is if and only if E plus x is covered by union of the translated intervals that is I j plus x j equal to one to infinity; that means, every covering of the set E gives a corresponding covering of the set E plus x by the intervals I j plus x note that if I j is the interval I j plus x also is an interval.

So, if E is covered by intervals I j we get a corresponding covering of E plus x by the intervals I j plus x, and conversely given a covering of E plus x we can construct back a covering of E by looking at by translating by minus x. So, this property obviously, implies that Lebesgue measure of a set E is same as the Lebesgue measure of the set E plus x. So, this observation implies that the Lebesgue measure of a set remains invariant under translations. So, this is a property. So, this is a important property of the Lebesgue outer measure we are going to use to conclude that if E is Lebesgue measurable than E plus x also is Lebesgue measurable.

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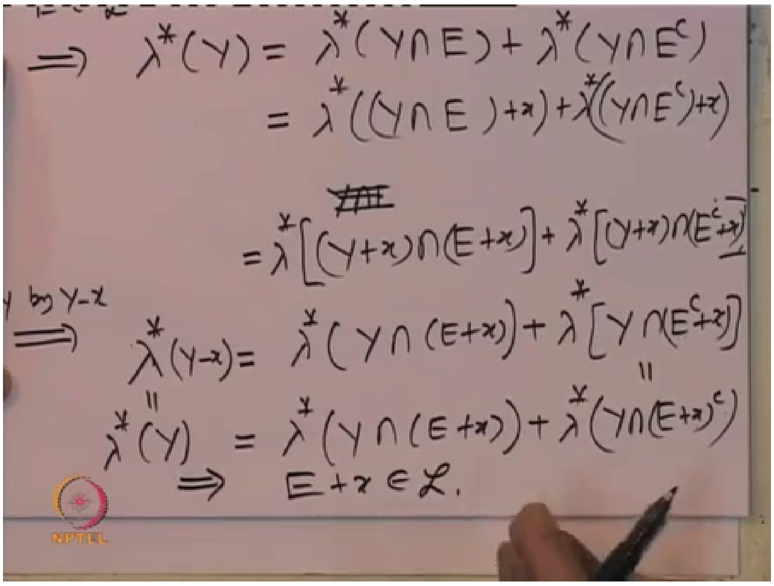

So, as we said E Lebesgue measurable implies for every subset Y we have got lambda star of Y is equal to lambda star of Y intersection E plus lambda star of Y intersection E complement. And now observing that Lebesgue measure is invariant outer Lebesgue measurable is invariant and translations we can write these as lambda star of Y intersection E plus x plus lambda star of Y intersection E complement plus x ok.

So, here we are using the fact that lambda star is translation invariants and now a simple observation they will tell you that Y intersection E plus x is same as why intersection it is it is same as Y intersection Y plus x intersection E plus x. So, meaning that if you take intersection and translate that is same as translating and intersections. So, very simple set theoretic property.

So, the first term here is equal to lambda star of Y plus x intersection E plus x and similarly the second one will give you it is lambda star of Y plus x intersection E complement plus x. So, that is using the fact that lambda star is translation invariant and intersection translation is same as translation intersection they commute with each other and now this property is true for every subset of Y.

So, i can the replace Y by Y minus x. So, implies replace Y by Y minus x we get this is also equal to. So, lambda this property is true for every. So, let us replace. So, lambda star of Y minus x is equal to lambda star of, Y plus x minus x. So, that is Y intersection E plus x plus lambda star of Y plus x minus x. So, that is Y intersection E complement plus x. So, in the our equation where replace Y by the set Y by Y minus x. So, this is true and now observe lambda star of Y minus x is same as lambda star of Y we translating Y or by x or minus x does not defect.

So, we get for every subset Y lambda star of Y is equal to lambda star of Y intersection E plus x plus lambda star of Y intersection E complement plus x. Now a simple observations tells you that this set E compliment plus x is same as E plus x compliment.

So, first take the compliment and then translate that is same as saying first translate and then take compliment. So, is a purely a simple set theoretic exercise which you should be able to verify easily. So, we get lambda star of Y is equal to lambda star of the first Y intersection E plus x plus lambda star of Y intersection E plus x complement and hence this implies that E plus x is Lebesgue measurable.

So, we have proved the first property namely if you take a Lebesgue measurable set E and translate then the translated set also is Lebesgue measurable. So, another way of saying the same thing is that the Lebesgue measurable sets are translation in variant they remain the class of Lebesgue measurable sets is translation invariant. And we already seen the Lebesgue outer measure is translation invariant so; that means, that the length function is translation invariant on the class of all and Lebesgue measurable sets. So, this proves the first property. So, we have answered the question that if is Lebesgue measurable then E plus x is also measurable.

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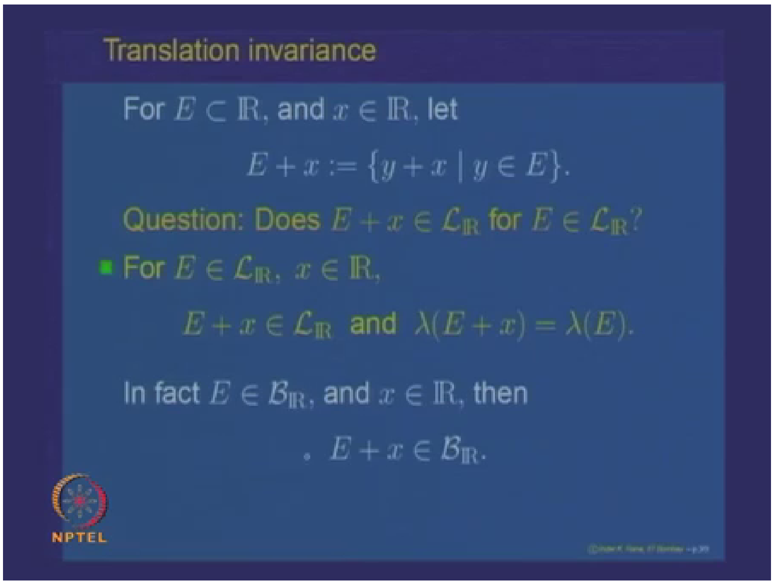

Let us look at question that if E is a Borel set can we say that E plus x also is a Borel set. So, to answer this question we need some a topological properties of the real line. So, let us look at the topological property is what we are going to look at. So, the question is if E is a Borel subset of real line, does that imply E plus x is also a Borel subset of real line.

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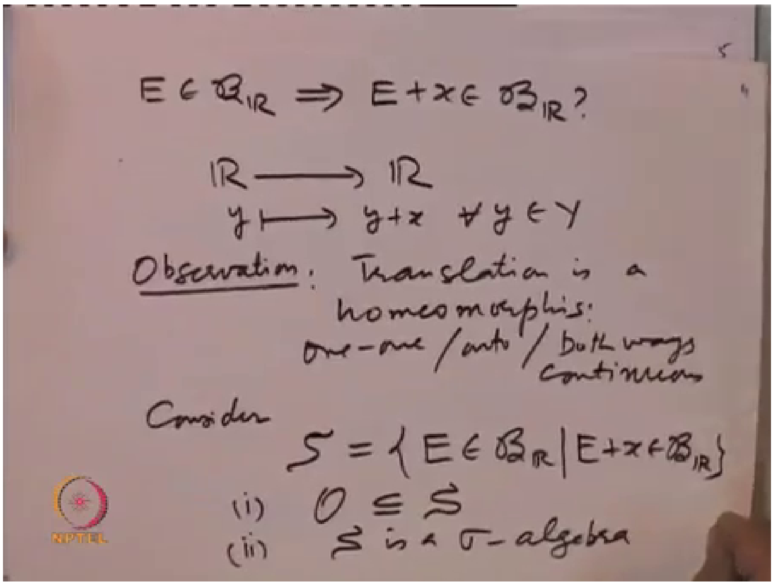

So, for that let us observe consider the map. So, consider the map from real line to real line where Y goes to Y plus x for every Y belonging to y. So, x is fix. So, this is translation. So, this is translation map from real line to real line, and the observation is that this is this map translation is a homeomorphism.

So, what does homeomorphism means? That is it is 11 on two and both ways continuous continuous; that means, this is continuous and the universe map because is one one on two that also is continuous. So, this is an important property very basic, but yet important property that we are going to use to prove that if E is Lebesgue measure if E is a Borel set then E plus x is a Borel set.

So, let us look at. So, consider. So, to prove our requirement let us consider the collection say S. So, s is a collection of all subsets which are boreal and which have that required property namely E plus x belongs to B R .So, look at all Borel subsets of real line which sets that the translation translated set is also a Borel set.

So, we are going to prove two things about this one that the class of all open sets are subsets of S and second thing will prove that s is a sigma algebra. So, once these two are proved all open sets are inside S and S is a sigma algebra. So, this these two properties will imply that the sigma algebra generated by the class of all open sets is inside S and that is nothing, but the Borel sigma algebra. So, the Borel sigma algebra will come inside as and that will prove the required property.

So, we have to only prove these two facts namely that if you take an open set then it belongs to S; that means, if a taken open set and translate that should be a Borel set, but that is again obvious by the fact that translation is a homeomorphism.

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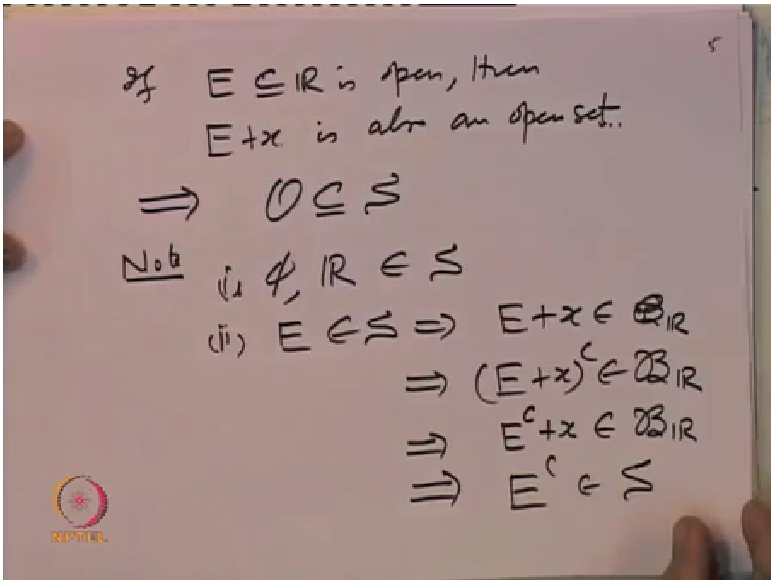

So, if E contained in R is an open set then the set E plus x is also an open set. So, that is also an open set. In fact, that is very simple property to an prove because if you take a if E is open that rebury point there is a open interval inside E and the translated one will be inside E plus x. So, that is easy to verify or one can simply verify by saying that E plus x is an open map or translation is a homeomorphism.

So, this an open set. So, this is implies the first properties that all open sets are subsets of S and the second thing that it is a sigma algebra. So, for that note empty set the whole space are both open. So, they belong to s and second if I set E belongs to s that implies E plus x belongs to s is a Borel set, but that implies this is sigma algebra. So, E plus x complement is also a Borel set right because is a sigma algebra. So, must be close and re compliments and now a simple observation that this set E plus x is nothing, but E plus x compliment is same as E complement plus x. So, this is same as this. So, that belongs to B R and that implies that E compliment belongs to E compliment belongs to s that is same as saying E compliment.

So, the class s of subsets of Borel subsets of real line says that it translates are Borel sets includes empty set the whole space it is closed under compliments and let us prove that this is also closed under countable unions.

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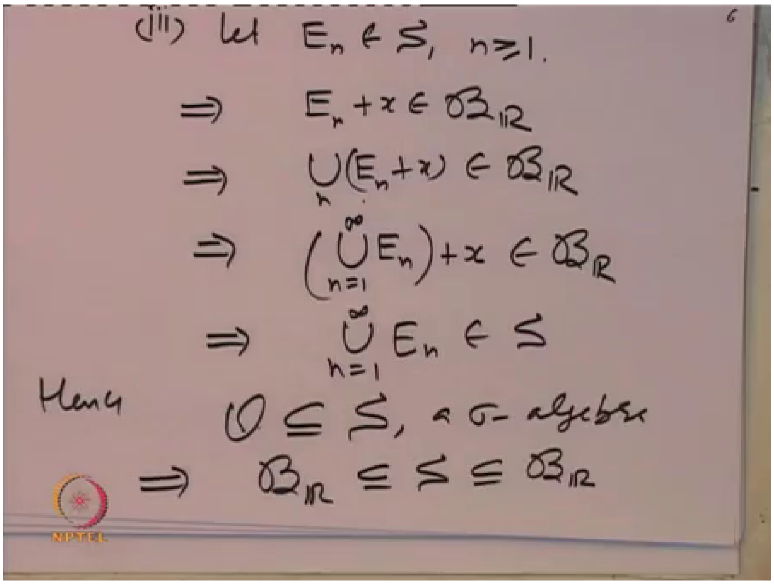

So, third property namely. So, let E n be a sequence of sets in S implies that E n plus x is a Borel set in R and Borel sets being a sigma algebra that implies union of E n plus x also belongs to B R, and hence now here is a simple observation that this set is same as you first take the union and then take the translation It is same as first translating and then taking the unions so, that belongs to B R .So, basically translation commutes with all set theoretic operations that is observation we have been using again and again.

So, this belongs to B R .So, that implies union n equal to one to infinity E n is a setting s. So, they approved. So, hence open sets are inside s a sigma algebra. So, implies that the Borel sigma algebra generated by open sets which is the Borel sigma algebra is inside the smallest sigma algebra generated by open sets namely the Borel sigma algebra must also come inside s and that is the subset of B R .So, hence all are equal.

So, this proves the second fact namely if E is a Borel set then its translated translation is also a Borel set and once again it emphasize the use of the technique that we had called as the sigma algebra technique namely you we wanted to prove that for every Borel set E the translation is a Borel set. So, note. So, we have collected all the sets which have this property and we proved two facts namely the open sets are inside this collection s and s is the sigma algebra. So, that imply that the smallest sigma algebra generated by open sets which is nothing, but the Borel sigma algebra also comes inside s. So, this technique we will be using we have been using and will be using quite often it is called the sigma algebra technique. So, we want to prove sum property about subsets of a set x collect them together and try to show that those sets form a that collection form the sigma algebra and includes the generators of the required sigma algebra.

So, we have proved that under translation the measurable sets the collection of Borel sets are very well behaved and we get a translation invariant measure on real line that is the length function.

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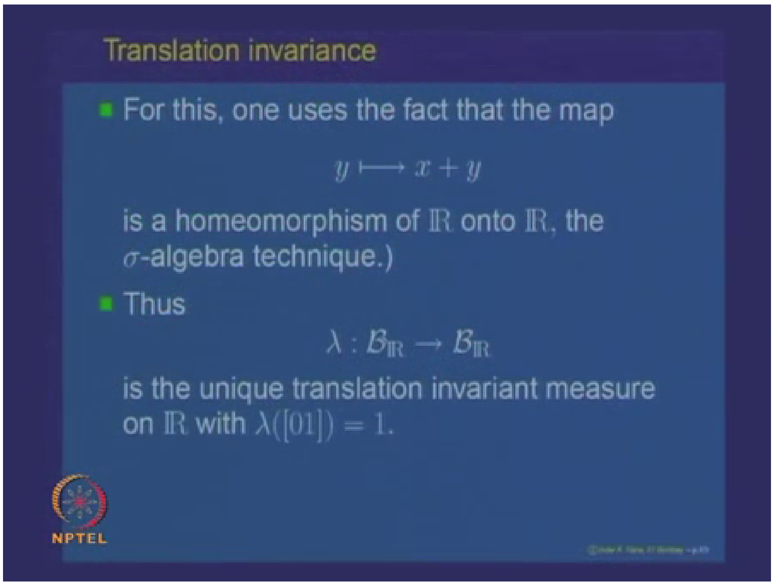

So, this gives us the fact that lambda the length function on we are the Borel sigma algebra of subsets of real line is the unique translation in variant measure such that the length of the interval 0 comma 1 here is comma missing. So, 0 comma 1 is equal to one and this is an very important property of the length function.

So, if you observe on the real line there is a notion of addition and we just now pointed out that the group operation x comma Y goes to x plus Y is a continuous map . And one can also easily check that x going to minus x that is the universe and the group operation is also a continuous map that is summarize by saying that the real line is a topological group. So, on the real line there is a topological structure, there is a metric ,there is a topology on the real line there is a group structure on the real line and that two behave very well with respect to each other saying that there the group operations x comma Y goes to x plus Y and x goes to x inverse both are continuous maps.

So, one says such a thing is called it topological group. So, the real line with addition and the usual metric the normal metric forms what is called it topological group and we are shown on this topological group there exist a translation in variant measure on the sigma algebra of subsets of it there exists a translation in variant measure. This is a very important fact and that can be analyze to what are called a locally compact topological groups that on every locally compact topological groups there exist a invariant measure because the group may not be Abelian. So, one has to make a specific thing there against a left invariant or a right invariant measure on every locally compact Abelian group, and that plays important role in doing analysis on such groups. So, we just a pointer that you may come across in your other courses in higher studies of higher mathematics that on every locally compact topological group there exist a invariant measure and that actually is called hour measure on the topological group.

So, Lebesgue measure on the real line is an example of harm measure on the locally compact topological group, real line under addition and the usual multiplication usual metric space topology. So, these were the properties of Lebesgue measurable sets these have being groups structure.

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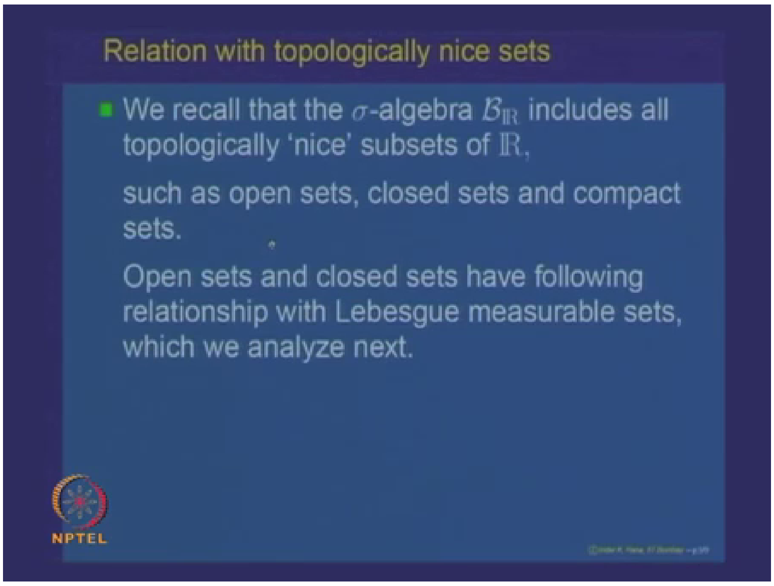

Now, let us look at properties of topology of the Lebesgue measurable sets with is set to topologically nice subsets of the real line. Namely topologically nice sets are open sets and closed sets. So, will prove will actually analyze and characterize measurability of sets in terms of open sets and closed sets.

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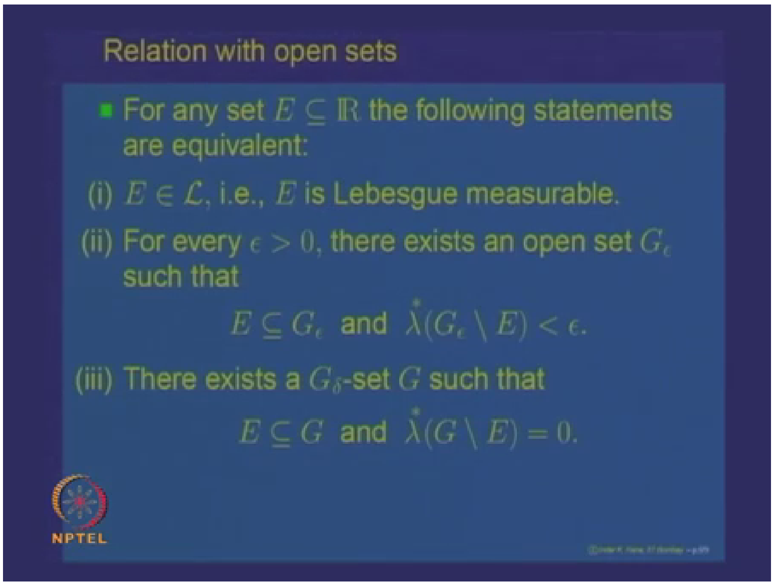

So, this is the precise theorem that we are going to prove namely if E is any subset of the real line then the following statements are equivalent saying that i a set E is Lebesgue measurable is equivalent to saying that for every epsilon bigger than zero, there exists and open set g lower epsilon such that E is a subset of G epsilon; that means, that open set includes E and the difference between the two sets has got outer measure small so; that means, there is a little difference between a Lebesgue measurable set an open set which covers it.

So, we will show that for every a set E is Lebesgue measurable if and only if for every epsilon bigger than 0, there exist an open cover of it such that the difference between the cover and the set. So, lambda star of g epsilon minus E is small and will show that this is equivalent to saying that there exists a g delta set g such that the set includes E and the difference such got measure 0 a set G delta set what is the Gdelta set a G delta set is nothing, but intersection of open countable intersection of open sets. So, subset of the real line or any metric space which are countable intersections of open sets are called G delta sets. So, let us prove this theorem.

So, saying that these three statements are equivalent is saying that if one of them holds than the other one also holds. So, what we are going to do is we will assume one and show that one implies two and then will show that statement two if you assume statement two that implies statement three, and if you assume statement three then that implies one and that will implied that all these statements are equivalence. So, if one of them is true then the other two statements are also true. So, that gives you a characterization of Lebesgue measurable sets in terms of open sets or g delta sets.

So, let us prove this theorem. So, will start with looking at. So, let us assume.

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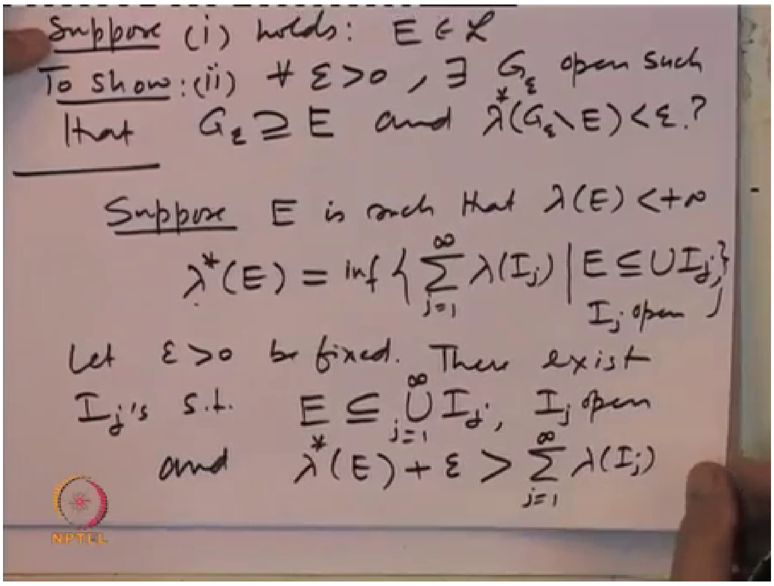

So, let suppose the statement one holds. So, that is E is Lebesgue measurable to show to and that is for every epsilon bigger than zero there exist a set g epsilon open such that g epsilon includes the set E and the Lebesgue outer measure of g epsilon minus the set E is less than epsilon. So, this is what we have to show.

So, let us start with something this is regarding outer measure. So, let us start looking at the set E. So, let us first suppose E is such that E is Lebesgue measurable. So, let us suppose that Lebesgue measure of E is finite. So, what is Lebesgue measure of E recall Lebesgue measure of E is same as its outer Lebesgue measure with is same as infimum over sigma lambda of I j one to infinity where the set E is covered by union of i js intervals and each i j open. So, recall we had made a observation that in the definition of Lebesgue outer measure you can assume that all the intervals involved are open.

So, now. So, let us fix be fix and says this number is finite and it is the infimum. So, when the definition of infimum there exists recovering. So, the resists intervals j's such that E is contained in union of Ij's, each i j open and we have got the property namely lambda star of E which is same as. So, lambda star of E which is same as lambda of E because it is measurable plus the small number epsilon is bigger than sigma lambda of I j j equal to one to infinity right.

So, and that implies. So, this implies.

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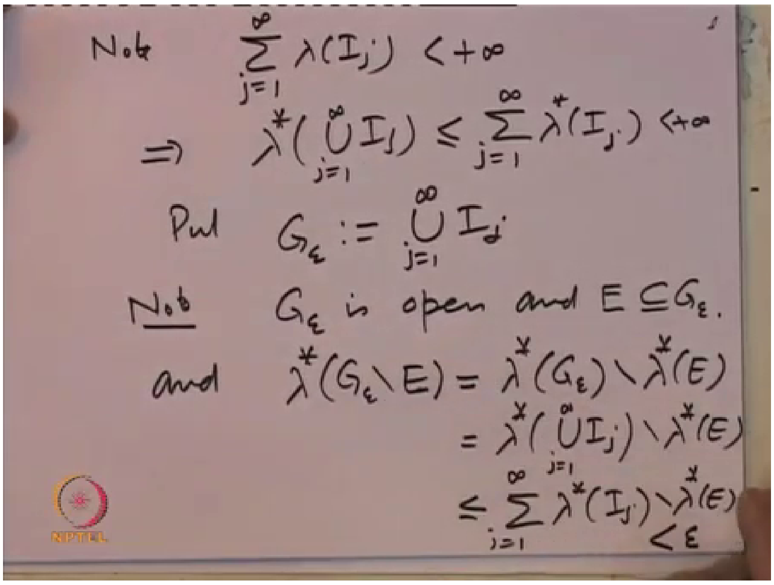

So, note lambda star of E is finite. So, this is finite quantity. So, that implies sigma lambda of i j j equal to one to infinity is finite. So, all the sets iI have got. So, this implies that if i take quite look at lambda star of union I j ,j equal to 1 to infinity that will be less than or equal to summation j equal to 1 to infinity lambda of i j by the sub additive property of the length function and that is finite. So, this set union of i j's is a set of finite outer measure.

So, let us define. So, put g epsilon to be the set which is union of I j's and now let us note that first of all G epsilon is open why it is an open set? bBcause each i j is an open interval. So, a countable union of open intervals is an open set is open and E is inside union of i j's. So, E is contained in g epsilon right.

So, we have got the required property that we have got a cover of E by an open set and let us note and what is the difference. So, lambda star of G epsilon minus E, now note g epsilon is an open set. So, it is a Borel set. So, it is a Lebesgue measurable set and E is given to be Lebesgue measurable and it is subset of if it and everything is finite that we are just now observed. So, we can say this is equal to lambda star of g epsilon minus lambda star of E. So, here we are using the finite additive property of the length function and lambda star of G epsilon that is here is that is same as lambda star of union I j because j epsilon is union i j minus lambda star of E and that is less than or equal to sigma by sub additive proper countable sub additive property one to infinity lambda star of i j minus lambda star of E and that by our choice of our intervals i j's if you recall this was the choice so; that means, lambda star of the summation minus this is less than epsilon

So, which is less than epsilon. So, lambda star of g minus g epsilon minus E is less than epsilon so; that means, what; that means, we have proved the required property when lambda star of E is a finite set.

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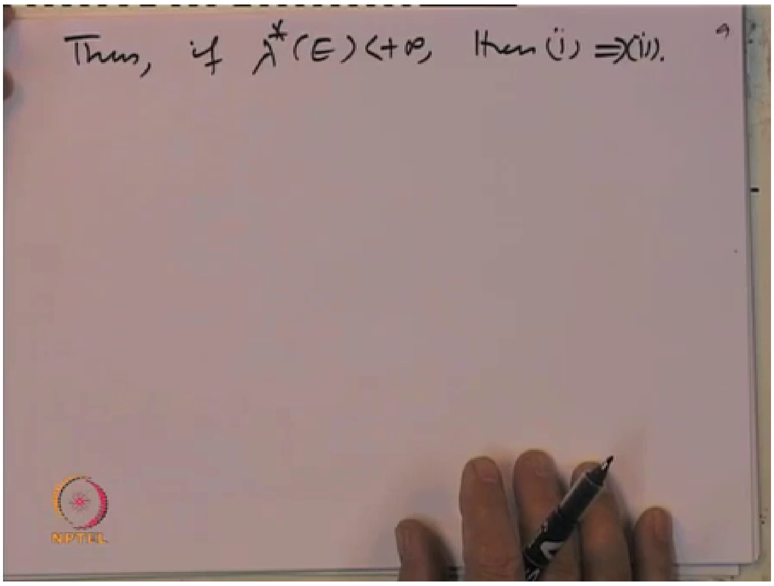

So, thus if lambda star of E is finite than 1 implies 2. Let us remove this condition and that condition here is a step where is a important step we should keep in observation that we first proved a property about the length function for finite sets of finite measure and now we are going to extend this using the fact that lambda is sigma finite

So, whenever you want to one wants to prove a property about the length function or about sigma finite measures, many a times it is easier to prove it when the underline set is of finite measure and then extended to sets general sets of sigma finite measure.