Measure & Integration
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Lecture - 12 B
Lebesgue Measure and It's Properties

So, what is axiom of choice is basically saying very (Refer Time: 00:20) saying given a non-empty collection of non-empty sets, you can pick up one element from each set and form a new set. So, it is how sets can be constructed when the sets are not indexed by a family which is finite in members essentially. So, it says given any indexed family of non-empty sets and that indexing set also is non-empty, from each one of this sets you can pick up one element and form a new set. So, using this one can show there exist sets in the real line which are not Lebesgue measureable.

So, we will prove this result. So, assuming axiom of choice there exists non-Lebesgue measurable sets in the real line, so let us prove existence of non-measurable sets by assuming axiom of choice.

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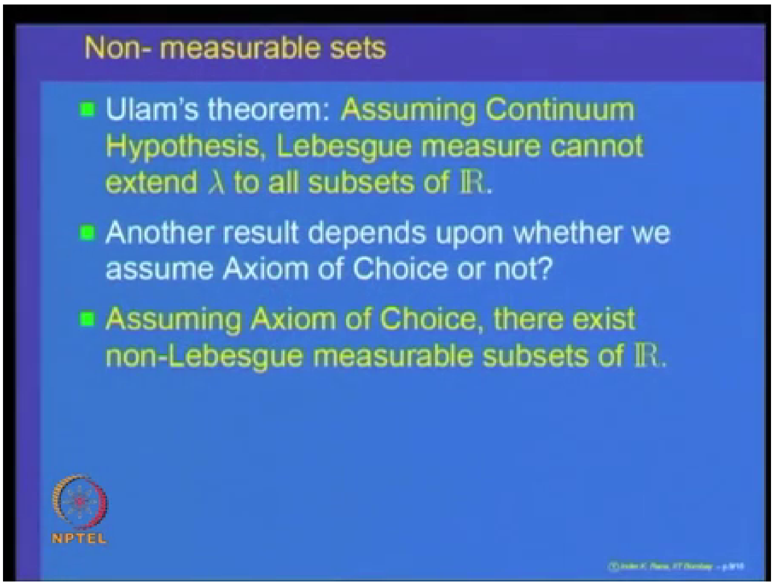

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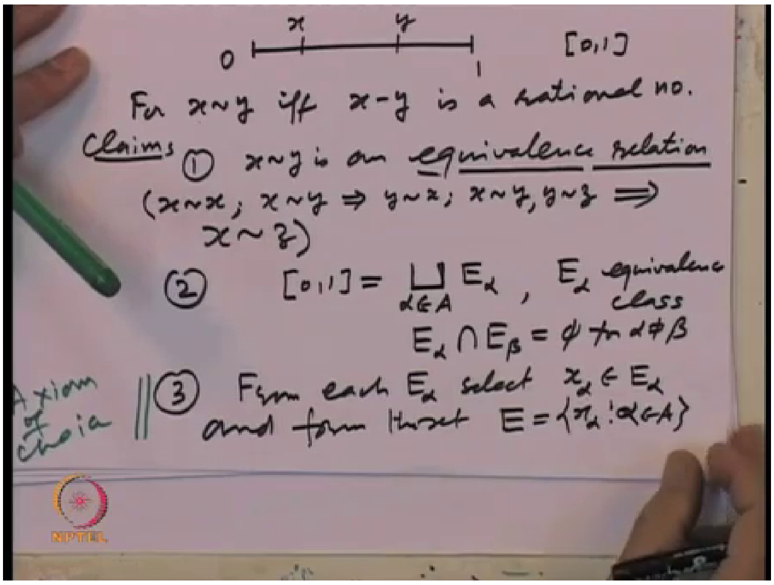

So, let us start. So, what we are doing is existence of non-measurable sets. So, that is what we are discussing. So, we want to construct a subset of the real line which is not Lebesgue measurable. So, to start with consider once again the interval 0 to 1. So, consider the interval 0 to 1. So, this is the interval 0 to 1 on this I am going to define a relation so far x related to y if x minus y is a rational number.

So, for x and y take two points x and y in 0 1, and you say that they are related with each other if and only if x their difference is a rational number. So, the first observation, claims let I will just write claims one that this x related to y is an equivalence relation. So, what does equivalence relation mean? It means it is reflexive symmetric and transitive. So, what is reflexive? X related to x that is obvious because x minus x is 0 and that is a rational number and secondly, if x is related to y; that means, x minus y is a rational number and so, the difference.

So, the negative of that that is y minus x are also is a rational number. So, that implies that y related to x. So, if x is related to y then y is related to x that is called symmetry that the relation is symmetric and the third one is let us suppose x is related to y and y is related to z. So, x related to y means x minus y is rational and y related to z means y minus z is rational. So, if you take the difference that implies that x minus z is a rational. So, that implies that x is related to z also. So, it is an equivalence relation it is a reflexive symmetric and transitive and every equivalence relation given on a set partitions the set into equivalence classes. So, that is the basic idea that 0 1 can be partitioned into equivalence classes by this relation.

So, that implies. So, second that implies. So, let us write that 0 1 can be written as a disjoint union of equivalence classes. So, let us write it as E alpha, alpha belonging to some indexing set let us call it as A. E alpha equivalence class and recall equivalence class means E alpha intersection E beta is empty for alpha not equal to beta that is why I have written as a union with a this sign; that means, equivalence classes they cover 0 1 and they are disjoint. So, there is there is a partition of the set on which equivalence classes are defined. So, that is and the third step is from each E alpha select some element x alpha and form the set call let us call it as E which is x alpha x alpha belonging to the indexing set A.

So, what we are saying is using this equivalence relation partition does interval 0 1 into equivalence classes, and from each equivalence class pick up one element exactly one element x alpha select one element x alpha, choose one element x alpha from each equivalence class and put them together in a box call that E and claim is that E is a set and this is a place this is the place we are using axiom of choice.

So; that means, E alpha is a collection of non-empty collection of non-empty sets from each we can pick up one element and form this set this is possible only if we assume axiom of choice. So, here is a place where we are using axiom of choice. So, from each equivalence class we are picked up one element and constructed a set E.

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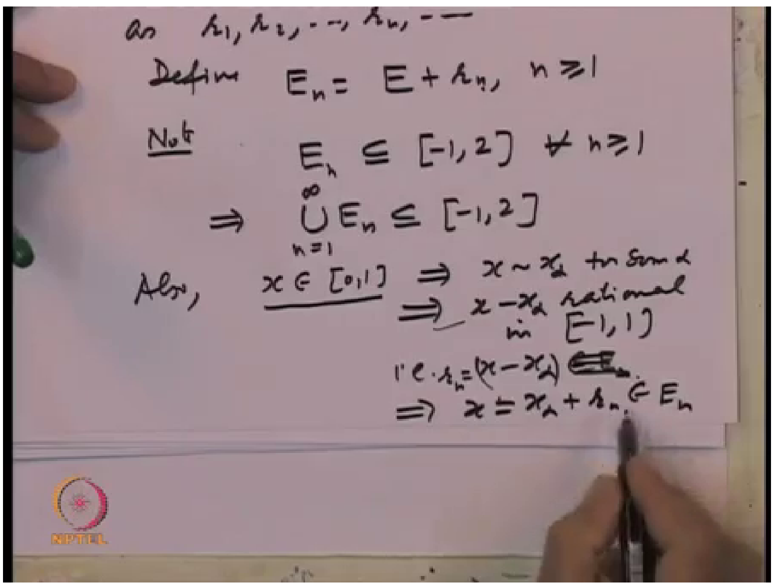

So; obviously, this set E e is a subset of 0 1 right because each equivalence classes subset of 0 1 and from each we have picked up one element. So, this is a subset of 0 1 ok.

Let us write. So, let rationals in minus 1 2 1 be written as r 1 r 2 r n and so on rationals in the interval minus 121 is a countable set. So, they can be enumerated they can be written in the form of a sequence we are not saying r 1 is smaller than r 2 or anything we are just giving a numeration of the rationals they are countably many. So, we can write them as a sequence and construct define a set E n which is E plus r n, n bigger than or equal to one construct a set E n this. So let us observe where is the set E n E is in 0 and each r n is between minus 1 to 1. So, what can you say about the set E plus r n. So, E can be 0 to 1 r n could be minus 1 to 1.

So; that means, each one of them is a subset of minus 1 to t2 at the most this sum can become minus 1 when elements of E are smaller smallest one is a 0 and the possibility here is a minus o1 and the largest possible is r n is equal to 1 and E also element is one, so 1 plus 1 2. So, for every n E n is a subset of 0 1, of minus 1 to 2. So, this implies that the union of E ns is also contained in minus 1 to 2. So, that is one observation also if I take x belonging to 0 1, if I take an element x in 0 1 that implies x is related to x alpha for some alpha right because the equivalence classes cover 0 1. So, every element x in 0 1 has to belong to one of the equivalence class. So, say it belongs to E alpha so; that means, it is related to x alpha the element that we have picked.

So, that implies that x minus x alpha is a rational, x minus x alpha related means the difference is a rational and. So, where will that rational be x is in 0 1 x alpha is in 0 1 this is a rational is a rational in minus 1 to 1 right because both could be 1 and that means, that is x minus x alpha belongs to E n because if it is a rational line minus 1 to 1 that must be equal two sum r n right and that means, x is equal to x alpha plus r n and that means, it is in E n.

So, what we are saying is for every x in 0 1 x minus x alpha belongs to is in r n. So, sorry is not that that implies that x is equal to x alpha plus r n and that belongs to E n. So, x belongs to E n. So, the second observation is that 0 1 is inside the union of E ns. So, that is what we have gotten.

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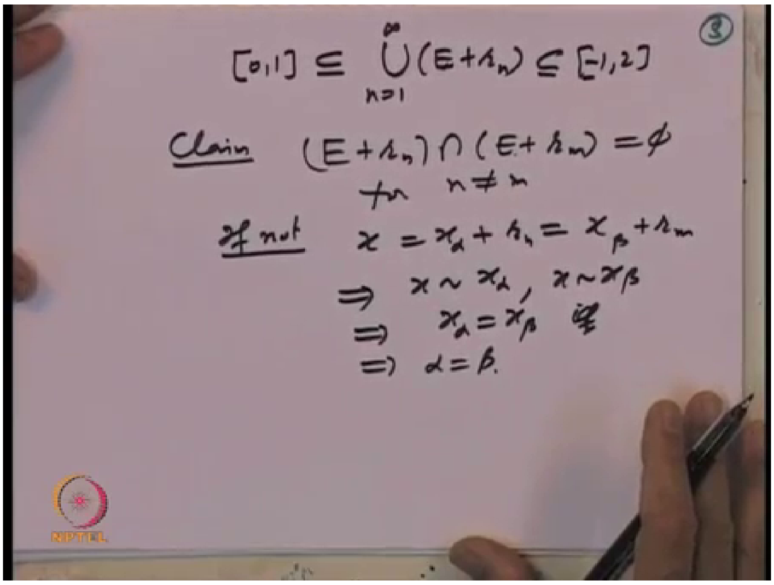

So, this construction we have got is the following that 0 1 is contained in union of E plus r n that is E n, n equal to 1 to infinity and that is contained in minus 1 to 2 and in this construction of the set E we have use axiom of choice. Now here is one observation that let us we want to observe claim that this sets E plus r n intersection E plus r n are disjoint sets for n not equal to m to prove this. So, let us take an element x which is common. So, if not x belongs to E plus r n; that means, x is equal to x alpha plus r n is also equal to it is also n E plus r m. So, it is also equal to some E beta plus r m and that implies that. So, that implies x is related to x alpha and x is related to x beta that is x related x l (Refer Time: 12:04) that implies either x alpha is equal to x beta right a if right that is that should be same and that is possible implies that alpha is equal to beta.

So, if alpha is not equal to beta right then this is not possible. So, that says that means, that these two sets are disjoint. So, this is what we have got.

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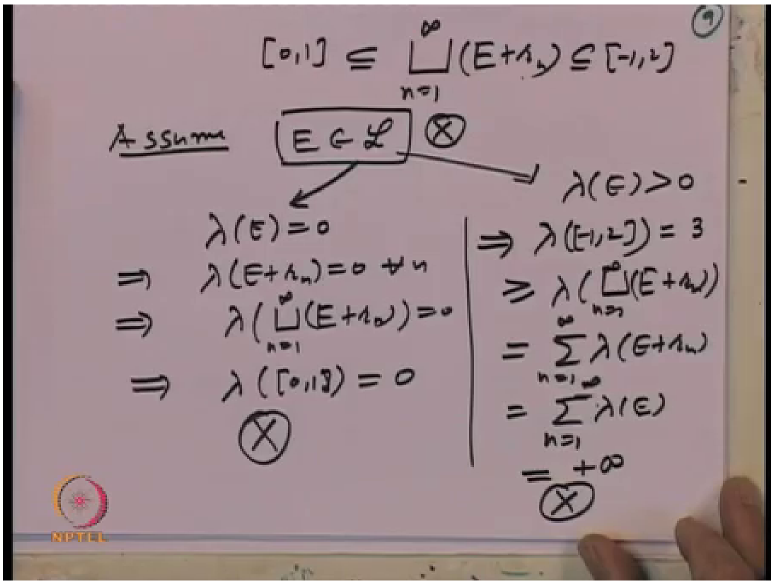

So, as a consequence let us write this as that 0 1 is contained in a disjoint union of E plus r n, n equal to 1 to infinity and that is contained in minus 1 to 2. So, till now we have not done anything except we defined a equivalence relation and using axiom of choice we constructed a set E and this has this property.

Now, suppose assume that E is Lebesgue measureable then there are two possibilities one Lebesgue measure of E is equal to 0, but that that implies Lebesgue measure of E plus r n is equal to 0 for every n because Lebesgue measure is translation in variant and that implies that the Lebesgue measure of the union E plus r n is equal to 0 and that implies because 0 1 is inside this. That means, Lebesgue of 0 1 equal to 0, which is a contradiction because Lebesgue measure of 0 to 1 is equal to 1. The second possibility is that the Lebesgue measure of E is strictly bigger than 0 then that implies Lebesgue measure of minus 1 to 2 this close interval is bigger than or equal to Lebesgue measure of this union because that is a subset of it and that is equal to sigma lambda of E plus r n and that is equal to sigma lambda of E because for every n it is same and this being a positive quantity added infinite number of times that is equal to plus infinity, which is again a contradiction because lambda of minus 1 to 2 actually is equal to 3 and 3 equal to infinity is a contradiction. So, either case this assumption cannot be true right.

So, this is a set which is in 0 1 and which is not measurable. So, what we have shown is the following that if we assume axiom of choice.

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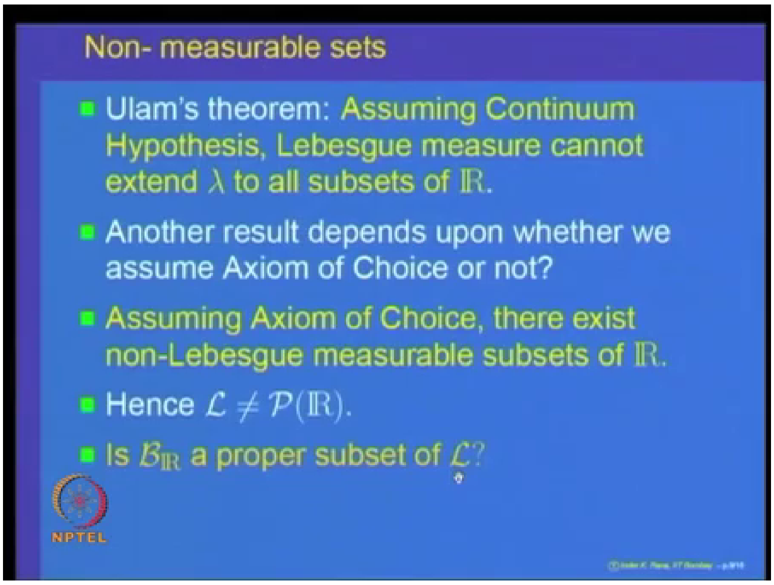

Then there exist non Lebesgue measurable sets in the real line; without axiom of choice or without continuum hypothesis it is not known that you can construct subsets of the real line which are not measurable non Lebesgue measurable. In fact, there is a theorem which says that the condition that assume axiom of choice right actually if you put this as an axiom in set theory, that every subset of the real line is Lebesgue measurable if you take that as an axiom and if your set theory axioms are already consistent then adding this new axiom to your set theory will not make any difference it will still leave it consistence.

So, existence of non-measurable sets get related to fundamental questions in set theory. So, on this side will leave it as it is saying that if you either assume continuum hypothesis or you assume axioms of choice then there exists sets which are not Lebsegue measureable. Let us tend to the other side can we say that the Borel sigma algebra the Borel subsets of real line they form a subset of this form is Laplace of Lebesgue measureable sets what is the relation between these two. Can we say that the Borel sets form a proper subset of the class of all Lebesgue measureable sets.

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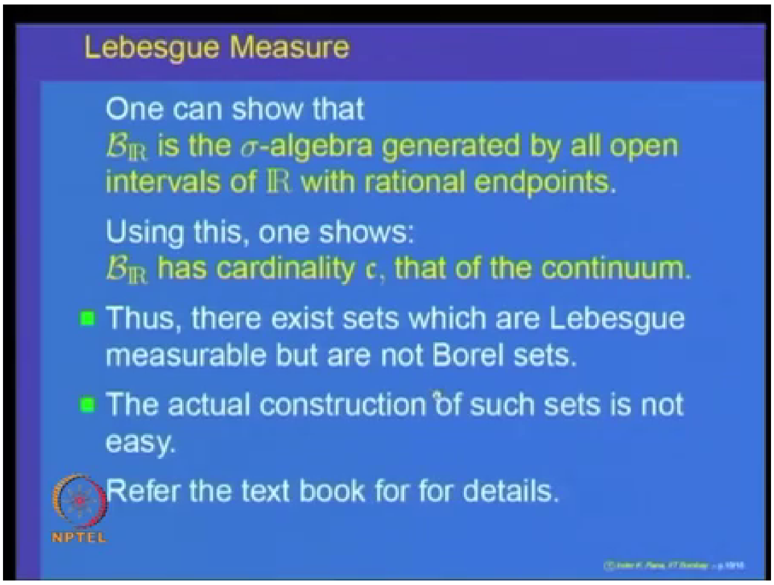

So, one can show we will not prove most of the things here because they are slightly technical. So, first observation is that the Borel sigma algebra of the real line which is the sigma algebra generated by all intervals is the same as the sigma algebra generated by all open intervals; because one can show that every open set in real line is a countable union of open intervals actually that is using the basic topology in the real line.

So, topological property of real line come into play. So, and not only that. In fact, you can take open intervals will only rational end points and if you generate the sigma algebra by them that is same as the Borel sigma algebra. So, that needs a prove as we will not prove it, but just indicate what is involved here. So, the Borel sigma algebra, this is a countable family open intervals with rational endpoints.

So, you take a countable family of intervals and generate the sigma algebra and that is B R, and one can show that the cardinality of this process of generating is exactly equal to C. So, one using this properties one shows using this construction one shows that the cardinality of the sigma algebra of a Borel sets is same as that of C that of the continuum and that is called the real line whereas, the cardinality of the Lebesgue measurable sets was t2 to the power C so; that means, there exists sets. So, cardinality looking at the cardinality says that there exists sets which are Lebesgue measurable where which are not Borel sets. But construct actual construction of this sets is not very easy it is possible to construct such sets which are Lebesgue measurable, but which is not Borel sets they are called analytical sets analytic sets and for that we refer the our text book, for more details those who are of you are interested they should refer the textbook for more details.

So, what we have shown today is that in the special case of extension theory we get the notion of length function on a class of sets which are called Lebesgue measurable sets which include the Borel sigma algebra of subsets of the real line and the cardinality of the lebesgue measurable sets is same as the cardinality of all subsets, and if you make some assumption like continuum hypothesis or axiom of choice you can show existence of sets which are not Lebesgue measurable otherwise you cannot show there is no such proof known, and on the other side the Borel sigma algebra has got cordiality C which is much stricter strictly less than the cardinality of Lebesgue measurable sets.

So, we will continue looking at the properties of Lebesgue measurable sets (Refer Time: 19:48) open sets close sets and the group sector on real line in the next lecture.

Thank you.