Measure & Integration
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Lecture - 12 A
Lebesgue Measure and It's Properties

Welcome to lecture 12 on Measure and Integration. If you recall the last time we looked at the extension of a measure from an algebra to the sigma algebra generated by it and slightly beyond the class of all outer measurable subsets. Today we are going to look at some special applications of this a particular case of that extension theory for the real line and that is the topic for today's discussion namely Lebesgue measure and its properties.

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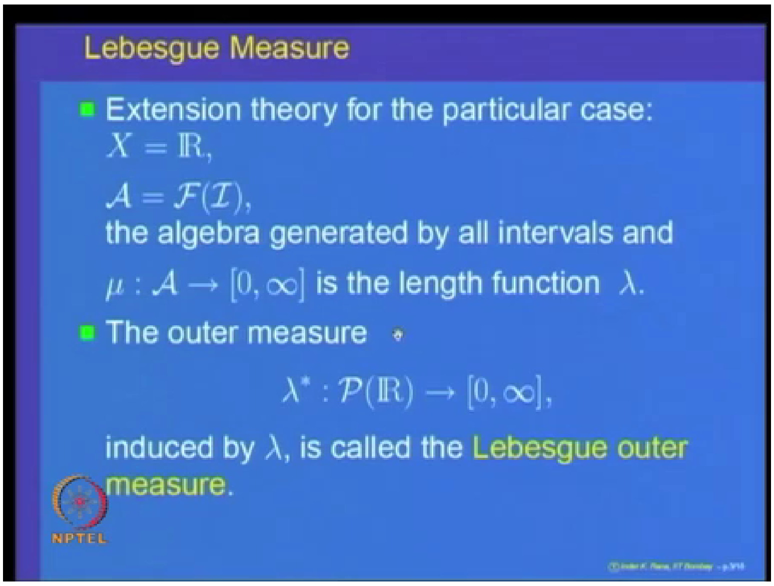

So, for the extension theory we are going to applied for the case x is equal to real line, the set is the real line the algebra a means algebra generated by all intervals in the real line and mu on this algebra is the length function that we are defined and we had seen that the length function on the algebra generated by all intervals is a accountably additive set function. The outer measure induced by this length function which is denoted by lambda star is on all subsets of the real line and that is called the Lebesgue outer measure. So, the outer measure induced by the length function is called the Lebesgue outer measure.

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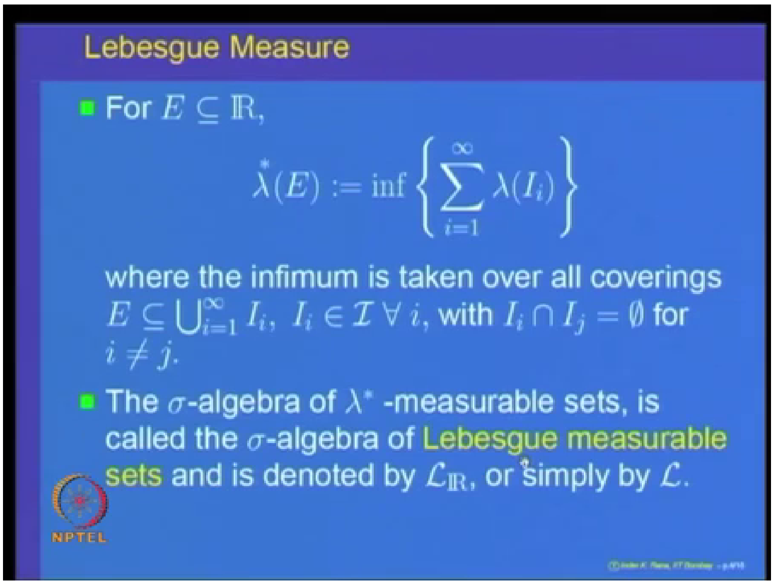

Let us just look at what is the Lebesgue outer measure for is subset of the real line. So, if you recall we defined it as outer measure of a set E is look at all possible coverings on the set E by elements in the algebra, but here the algebra being algebra generated by intervals it is finite disjoint union of interval. So, we can write this lay outer Lebesgue measure as the infimum over summation lambda of the intervals I where the intervals I i is form a covering of the set E and these intervals are pair wise disjoint.

So, lambda star of E is the infimum of the sums of the lengths of the intervals which form a covering of E, and we can take this intervals to be disjoint because if not then you can make them disjoint. So, that is the Lebesgue outer measure for a set E. The class of all Lebesgue outer measurable sets, lambda star measurable sets is called the sigma algebra of Lebesgue measurable sets. So, the sets which are outer measurable with respect to lambda star is called with respect to lambda star is called the sigma algebra of outer measurable or Lebesgue measurable sets and is denoted by L suffix R just to indicate L for the labesgue and R for the real line, in case there is no confusion we will just denote LR by simply L, so is the class of all Lebesgue measurable sets.

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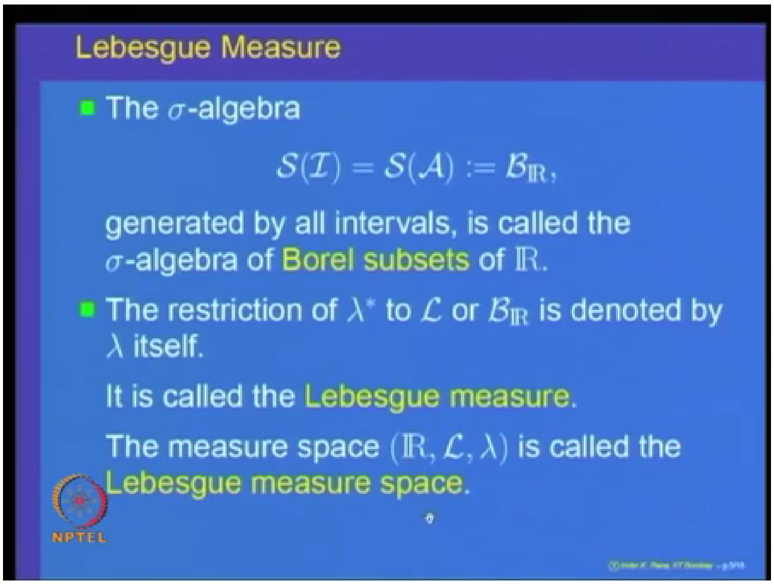

If we recall we had also defined the sigma algebra or Borel subset of real line and that was the sigma algebra generated by all intervals and A being the algebra generated by intervals. So, the sigma algebra generated by in finite disjoint union of intervals is same as the sigma algebra generated by all intervals and that is same as the definition of the Borel sigma algebra of the real line. So, this is the properties we have already seen. So, the length function in particular is also a define for all Borel subsets, because the sigma algebra generated by A is inside the class of all outer measurable sets that is L.

So, we have got that s of a there is a Borel sigma algebra is inside the class of all Lebesgue measurable sets. So, for all Borel subsets the notion of length is defined. So, this is called the Lebesgue measure. So, let us just summarize what we are we are saying we are saying that the extension theory when applied to the particular case of the real line gives us the notion of length for a class of subsets of the real line which are nothing, but the class of outer Lebesgue measurable sets and that includes the class of all Borel subsets. So, that is also the gives us the notion of length for all Borel subsets of the real line.

So, the triple are Lebesgue measurable sets the length function I as extended by the extension theory this triple is called the Lebesgue measurable space. So, the extension theory applied to the real line gives us the notion of the Lebesgue measurable space and sh it extends the notion of length from intervals to the class L of all outer Lebesgue measurable sets.

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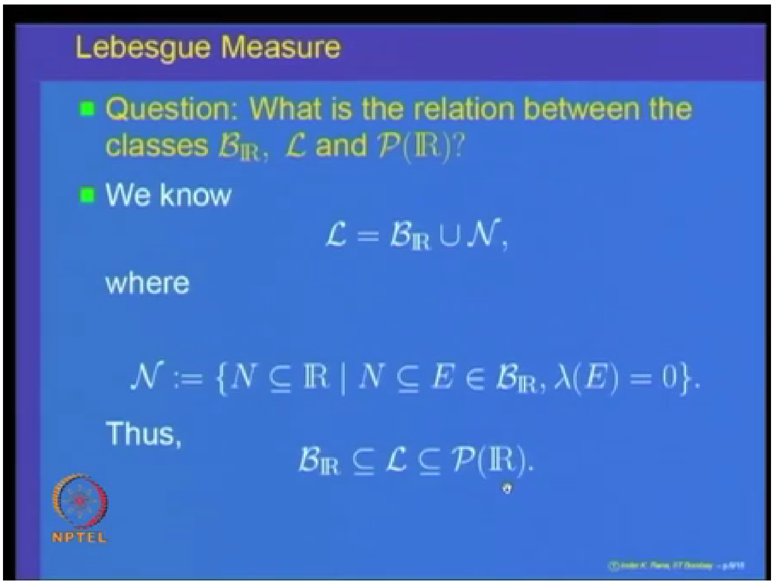

Let us is recall that the sets Borel subsets form a subset of class of all Borel sets is a sub class of the class of all Lebesgue measurable sets and of course, Lebesgue measurable sets is a sub class of all subsets of real line. So, the question is can be say something more regarding this 3 class is namely Borel subset, Lebesgue measurable sets and P R.

So, let us observe which we have done during outer measure that Lebesgue measurable sets are characterized by the Borel subsets of the real line union the null sets. So, what are the null sets:? Sets in the R subsets of R sets that an is contained in a Borel set of measure 0 or equality one can also definite as sets of outer Lebesgue measure 0. So, B R is a subset of L we know that outer measure 0 sets also measurable. So, this and we set this class is nothing, but this form the sigma algebra and that is equal to the Lebesgue measurable sets.

So; that means, the B R union N is equal to L. So, all null sets are p[art of L, but we want to characterize what is the relation between B R and L and what is the relation between L and P R. So, at present we only know that the Borel sets all subsets of all Lebesgue measurable sets which is a subset of P R. To say something more we need to look at what is called a special subset of the real line called Cantor's ternary set.

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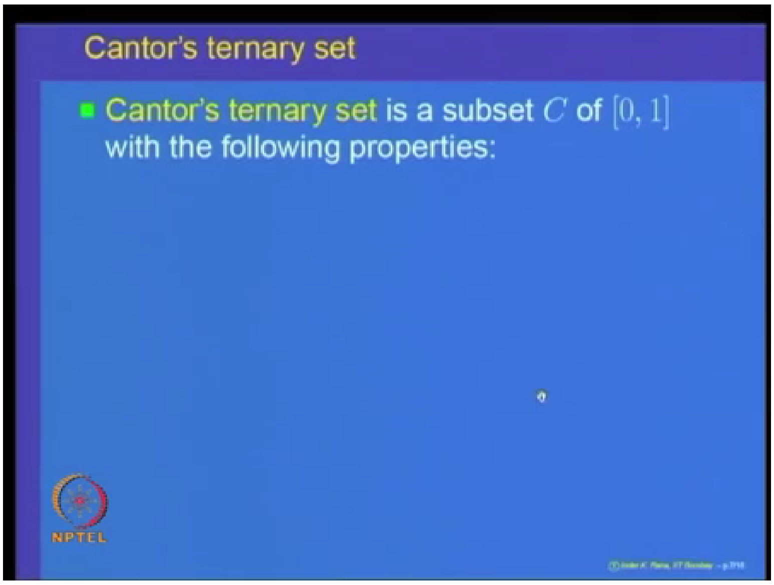

So, we are going to discuss spend some time on a special subset of real line which is called Cantor's ternary set. And Cantor's ternary set is an example of a set which is very nice properties and it is useful both from the technological point of you as well as measure theoretic point of you.

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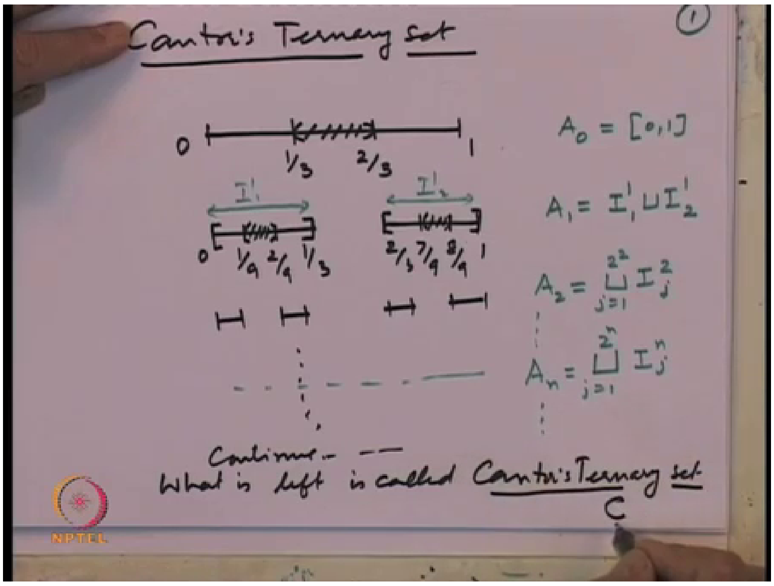

So, let us look at what is called Cantor's ternary set. Ternary it is called can Cantor's ternary set because it was given by the mathematician George Cantor, first defined by George Cantor and ternary set because it involves ternary expansions of real numbers.

So, what we are going to do it is a construction we are going to construct Cantor's ternary set. So, as a step 1 latest look at the interval 0 to 1. So, well first describe this process of Cantor's ternary set construction and then will analyze its properties. So, what is the first step? The step is divides into 3 equal parts. So, that is 1by 3 and 2 by 3 and remove the middle open position. So, this operation is removed from the interval 01. So, what it gives it gives as 2 peas is 0 to 1by 3 and from 2 by 3to 1. So, it gives us 2 close interval.

So, at the first step at the first stage having remove the middle one-third of the closed interval 01 middle one-third open interval we get the store. And now we repeat that process again with these to sub intervals. So, from each of this subintervals remove the middle one-third portion. So, that is middle one-third is 1 by 9, 2 by 9 and here the middle one-third will be equal to 7 by 9 and 8 by 9. So, this is a middle one-third portion which we are going to removed the second stage. So, that will give us four subintervals and we continue this process.

So, eventually something will be left. So, continue. So, question is what is left; what is left is called Cantor's Ternary set. So, let us analyze and let us note this set Cantor's ternary set by the let us see. So, how do we mathematically set this? So, that is a question. So, for that start with the first stage that is A 0 that is the closed interval 0 1. After having perform the first stage what is left i write it as A1. So, that consists of 2 disjoint intervals 0 to 1 by 3 and 2 by 3 to 1. So, it consists of 2 disjoint intervals thirds write them as at the first stage one union the second one first stage the second one ok.

So, this portion is the first interval and this portion is the second interval there is I11 and this one is I12. So, at the second stage will be left with four disjoint closed intervals. So, let us write them as union I second stage j, j equal to 1 to 4. So, that is going to be 2 raise to power 2; and let us see what will be at the end stage if you continues prove of this process at the end stage how many intervals will be here. So, there will be intervals how many of them we start with one at the next stage 2, at the next stage 4 and so on.

So, they will be disjoint intervals j equal to 1 to 2 to the power and the repeat 2 to the power and closed subintervals of 01, let us write them as I n j. So, these are the intervals. So, what is An? A n is the union of those intervals which are left at the stage at the end stage and what you want and we continue this process we want what is c. So, how do we write mathematically c?

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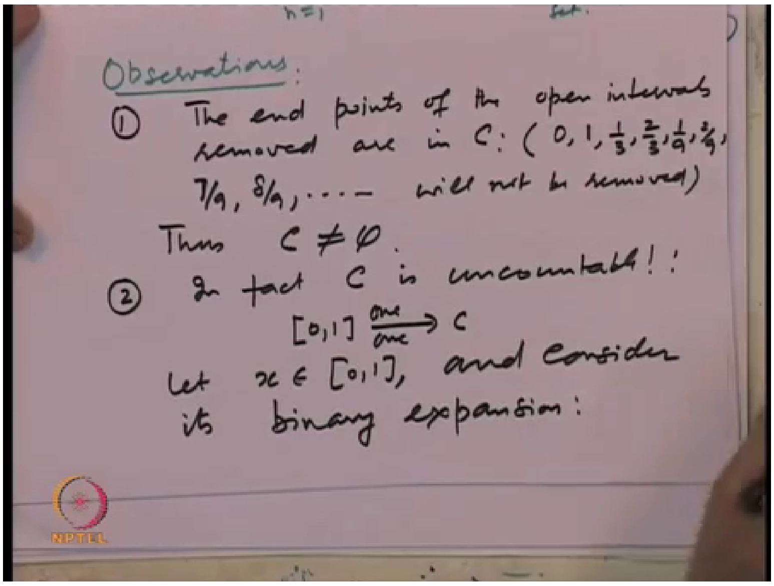

So, the Cantor set we can write it has. So, the Cantor set C we can define it has intersection of A ns n equal to 1 to infinity right.

So, each A n is a subset of the previous one right. So, let us write what is left eventually as intersection of all this A ns. So, this is what is called Cantor's Ternary set. So, let us make some observations about this observations about this Cantor's ternary set, the first observation is that the end points of the open intervals removed are in C say for example, 0 is. So, 0 is not removed is not going to be removed one is not going to be removed at the first stage we removed the open middle one-third.

So, 1 by 3 is not going to be removed 2 by 3 is not going to be removed and it the next stage 1 by 9 will not be removed 2 by 9 will not be removed and similarly 1 by 3 we already listed then 7 by 9 will not be removed 8 by 9 will not be removed and so on will not be. So, for example, this points will not be removed they will stay in this process of removing middle one-third open interval from each some interval at every stage.

So; that means,. So, thus the class C the sets C is a non empty set it is non empty is non empty. So, that is first observation there is something left behind and the second observation we want to show that in fact, C is uncountable that it is an uncountable set.

So, how do you prove c is uncountable, what we are going to do is we are going to define a map from the closed interval 01 to C. So, to prove this will define a map with is 11; will define a 11 map from 01 to the Cantor's ternary set and that will prove that the cardinality of the set C is at least as much as 01. And c being a subset of 01 it cannot be more than that of 01. So, cardinality will of c will be same as cardinality of 01 that may c memories strange a observation to you that from c we have removed from the interval 01 we have removed so, many pieces and still what is left is as much as the points in 01. So, this these are properties of infinite sets it will they are the characterizing properties of infinite set the interval 01 is an uncountable set, and from that we are removing subintervals and still what is left behind is as much as 01.

So, let us prove this fact namely that there is a one to one map for this. So, it for this let us start let us take a point x belong into 01 and consider its binary expansion. So, what is the binary extension? The binary expansion of a point n 01 is written as. So, x can be written as point a 1 a 2 a 3 a n and so on.

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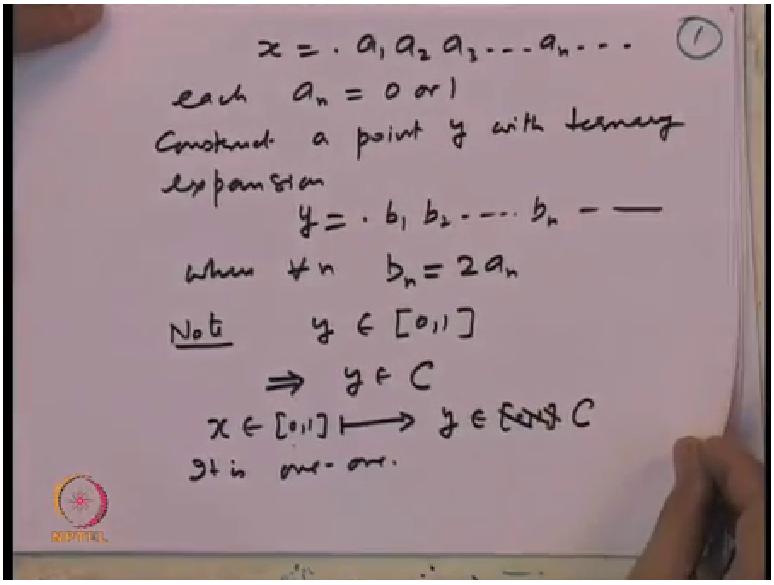

We are each a n is equal to 0 or 1.

So, that is a binary expansion of every pint essentially the idea is that the intervals 01 can be divided into 2 parts name first part as 0 second part as 1 and c at each stage what where it lines. So, that is 011 and let us assume see there are 2 different ways of writing for some points there are 2 different ways of writing binary expansions. So, will fix one of the ways and say there is a unique binary expansion for every point in 01.

So, will fix that binary expansion process and now what we do is the following construct a point y with ternary expansion be with generally expansion. So, y is equal to point b 1 b 2 and b n where for every n b n is nothing, but 2 times a n. So, a in the binary expansion look at the a nth place either it will be 0 or 1 dabble it and call that as b n. So, b n is twice as much as a n. So, each b i is either going to be 0 or it is going to be 2.

So, this is the ternary expansion. So, note y belongs to 0 1 because it is dash is dot b 1 b 2 b 3 so on, so no integral part. So, it is going to be part of its a point in 0 1 and it has in the ternary expansion the only numbers that come r 0 2 times a n a n is 0 or 1 it is 0 or 2. So, in the ternary in expansion of y which is in 0 1 only 0 or 2 appear that implies that y belongs to C; because in the construction of the Cantor ternary sets we have remove the middle one-third. So, in the ternary expansion the number one is not going to appear. So, each one is. So, this is a part of. So, this is observation we make that starting with a point x belong in to 0 1 with binary expansion a 1 a 2 a n construct a point y. So, send it to the point y. So, this x is sent to the point y which is again in 0 1. In fact, it belongs to. So, let us i more specifically it belongs to C.

So, we are got a map from 0 1 to C and the claim is that this map this is it is 11 and that is obvious because for every point x we got this binary expansion a 1 a 2 a 3 a n the unique binary expansion. So, if you take to different points x 1 and x 2. So, let us write try to write this mathematically that this is.

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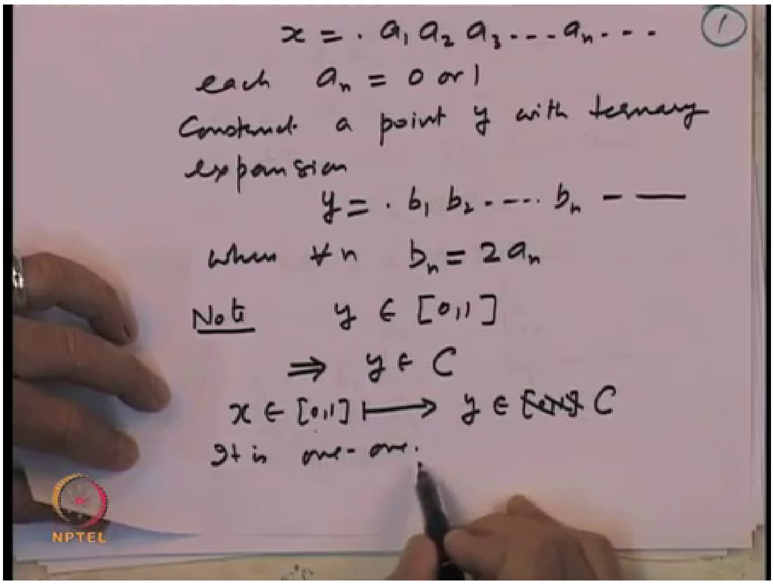

So, let us take a point x 1 with binary expansion a 11, a 1 2 a 1 n and so on.

So, let us take another point with binary unique binary expansion that we have fix the methodology. So, a 2 1 a 2 2 and a 2 n and so on and x 1 not equal to x 2. So, that implies if x 1 is not equal to x 2 that implies the release some stage and not such that a 1 n naught will not be equal to a 2 n naught and that implies that 2 times a 1 n naught will not be equal to 2 times a 2 n naught and that is this is b 1 n naught and this is called that be 2 n naught; that means, y 1 if we have y 1 that is point b 11 b 1 2 up to b 1 n and so on and y 2 is the other point a image of x 2. So, that is b 2 1 b 2 t2 b 2 n and so on then So, if this is so, then y 1 is not equal to y 2.

So; that means, this process of a sending x taking x with binary expansion as this and constructing y with ternary expansion is this. So, if send x to y this gives us a map from 0 1 to c which is 11 and hence. So, this implies as a consequence.

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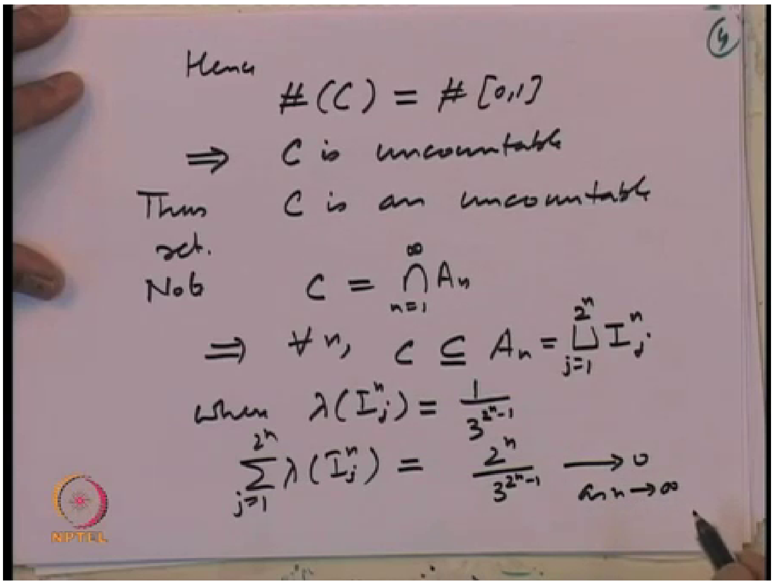

So, hence the cardinality of C is same as cardinality of 0 1, and if you recall cardinality of a 0 1 that is. So, let us write this implies that C is uncountable because 0 1 is uncountable. So, thus C is an uncountable set. So, this is followers from work instruction that she is uncountable set. So, this is follows are construction that C is an countable set. In fact, let us try to now calculate. So, note that C which is equal to intersection of a ns implies that for every n, C is a subset of A n and what was A n? That was a disjoint union of intervals I and j j equal to 1 to 2 to the power n right and at the end stage what will be the length of each where the length of each I n j.

So, what is the length of the intervals which are left at the end stage that is one over? So, let us just looked at the construction at the first stage when we removed 2 at a 1, 2 intervals were left each of length 1 by 3. So, this is 1 by 3 this is 1 by 3 this is 1 by 3 and this is 1 by three. So, 4 travels at the second stage of length 1 by 3. So, at the end stage how many 2 an intervals of each of length how many will be left that 2 to the power n intervals each of length how much, here the length of each I n j. So, will at the second stage it is 1 by 3 first second stage is 1 by 3 and. So, end stage will be one over. So, it will be 1 over 3 raise to power 2 two n minus 1 right.

So, that will be the length of each one of them and there are 2 to the power n of them. So, what is the total length? So, sigma lambda of I n j equal to j equal to o1 to 2 the power n. So, that is 2 to the power n intervals each as got the same length. So, divided by 3 raise power 2 raise power n minus 1 and observe that this number goes to 0 rights raise 2 to the power n by 1 or 3 raise power n that goes to 0 as n goes to infinity.

So; that means, what; that means, that C can be covered by for every n by 2 to the power n intervals whose length is this, and that can be made as small as a want so; that means, that the outer Lebesgue measurable.

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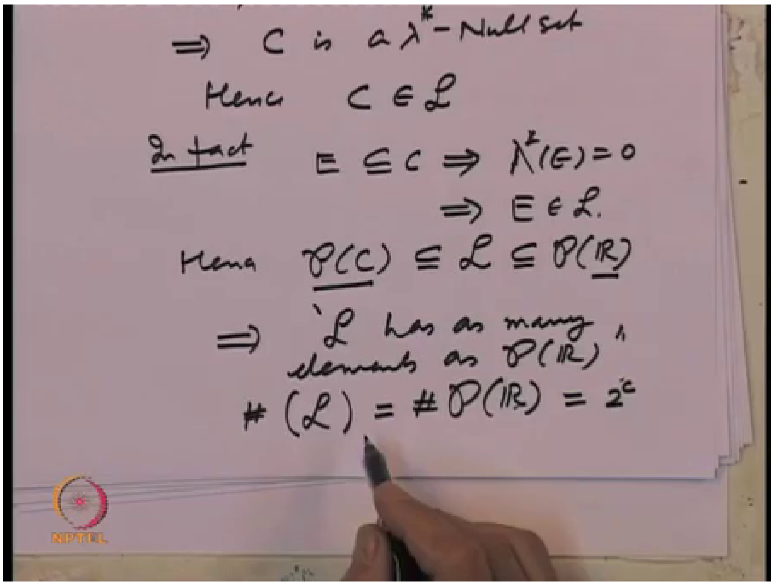

So, lambda star of c is equal to 0 right because what is lambda star of E? It is the infimum of the sums of the intervals which cover the set c and we are here we are just now show on that C is contained in A n which is a final decision of intervals and the total length of these interval is becoming smaller and smaller ok.

So, that is that length of C. So, that implies that C is a lambda star null set hence C belongs to is a Lebesgue measurable set. And not only that C is Lebesgue infect what we know is something more it if E is any subset of C then that implies that lambda star of E also equal to 0 because lambda star is more not on and that implies that E also belongs to L. So, that is hence all subsets of C power set of C is a sub subclass of Lebesgue measurable sets and of course, Lebesgue measurable sets are a subset of power set of real line now, but C is uncountable and R is uncountable. So, what does is implied; that means, this implies that L has as many elements as P of R.

So, what is the meaning of this has as many elements as P R that is same as saying the cardinality; if you know what is cardinality, cardinality of L is same as the cardinality of the power subset of real line and if you know that the cardinality of real line which is the will be called as cardinality of continuum is the noted by small letter c. So, and this is denoted by 2 to the power c. So, there if you look at so, that what does that prove. So, that proves that if you look at from the cardinality point of you, if you look at how many elements are there in the class of all Lebesgue measurable set then it says cardinality of Lebesgue measurable sets is as much as the cardinality of all subsets.

So, if you look from the cardinality point of you, you cannot say that the class of all Lebesgue measurable sets is a proper subset of the class of all subsets of the real line so, but that does not also imply that all subsets of real line are Lebesgue measurable. So, the question still remains and decided whether the class of all Lebesgue measurable sets is a proper sub class of all subsets of all real line. So, to decide this question is a, but difficult and leads to sum fundamental question in set theory.

So, let us look at. So, what where shown this now let us just recapitulate that lambda star of c is equal to 0 and that says that a power set of C is subset of L.

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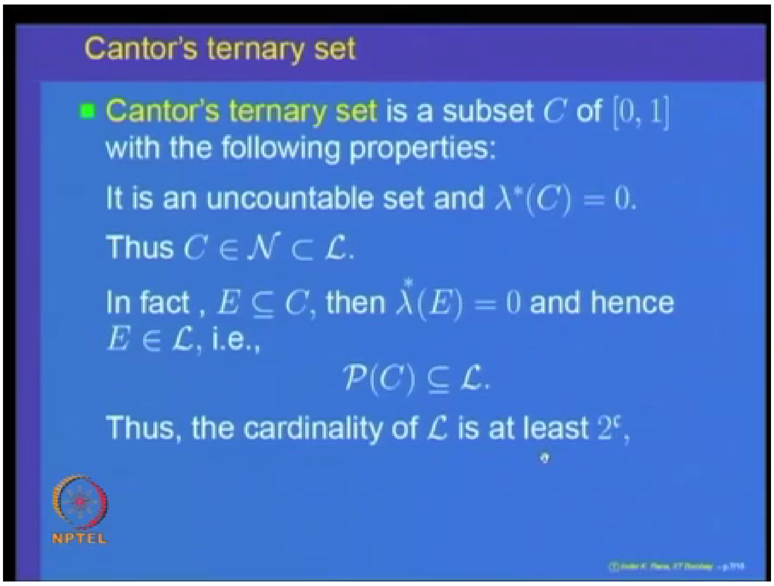

And hence there are at least as many elements in L as 2 to the power C.

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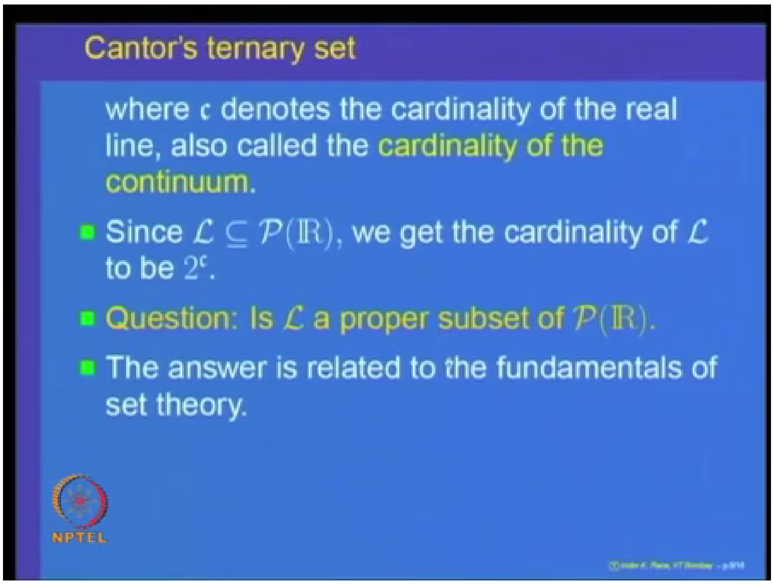

So, that is a cardinality of the continuum. So, so we get cardinality of L and power set is same both of got same cardinality. So, question still remains is L a proper subset of P R. So, if you recall the answer to this question is related to some of the fundamental questions in set theory.

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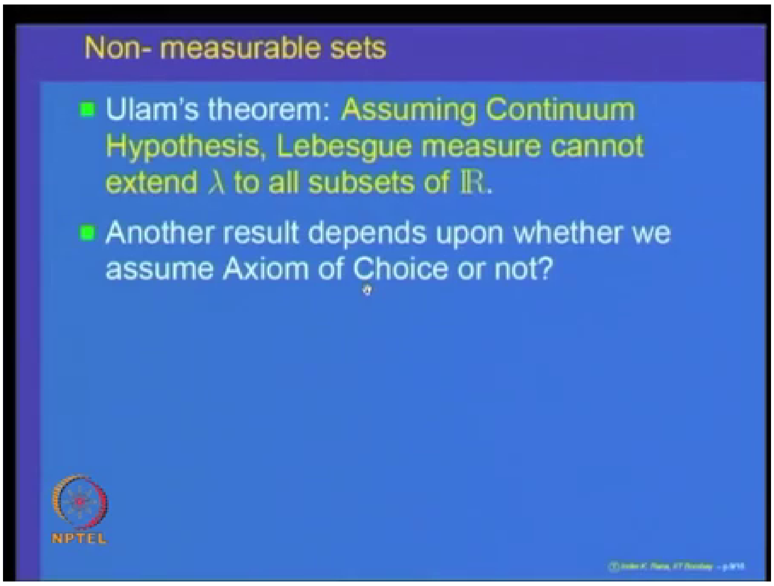

So, if you recall we proved what is called Ulams theorem we did not prove it really we mentioned what is called Ulams theorem and I said that one can read a proof of this in the textbook that we have mentioned and that is statement of the Ulams theorem says assuming continuum hypothesis. Lebesgue measurable cannot be extended to all subsets all of real line there is something called Continuum Hypothesis it set theory. I will not explain at this stage what is continuum hypothesis because will be going slightly of stream, but it is worth mentioning hear that the set theory is based on certain axioms.

So, whatever modern mathematics we are doing is based on axiomatic set theory and there is a which has some kind of some axioms on which we are we can deal with set theory, but there is something called continuum hypothesis which relates to the subsets of real line and so on and that is not part of the axioms of set theory that is why it is called continuum hypothesis. Some people believe in continuum hypothesis and do mathematics according to that and some people do not believe in it.

So, if you assuming continuum hypothesis and Ulams theorem says that you cannot extend the means not all subsets of real line or measurable. Another result which one can use which is again not part of the axiomatic set theory is the following which says that supposing you a assume what is called axiom of choice. Axiom of choice is another axiom which is not part of the axiomatic set theory and one can either accepted part of set theory and do mathematics or do not accept part of it and do mathematics the mathematician those who accept axiom of choice they are supposed to be doing.

What is called non constructive mathematics because there are some existence theorems which assume axiom of choice helps improving some theorems which are existentially in nature. For example, proving that every vector space has a base is a requires the need of using axiom of choice you cannot prove it if you do not assume axiom of choice there are many results in mathematics in which are which use axiom of choice and which are not true, if you do not assume maxima of choice.