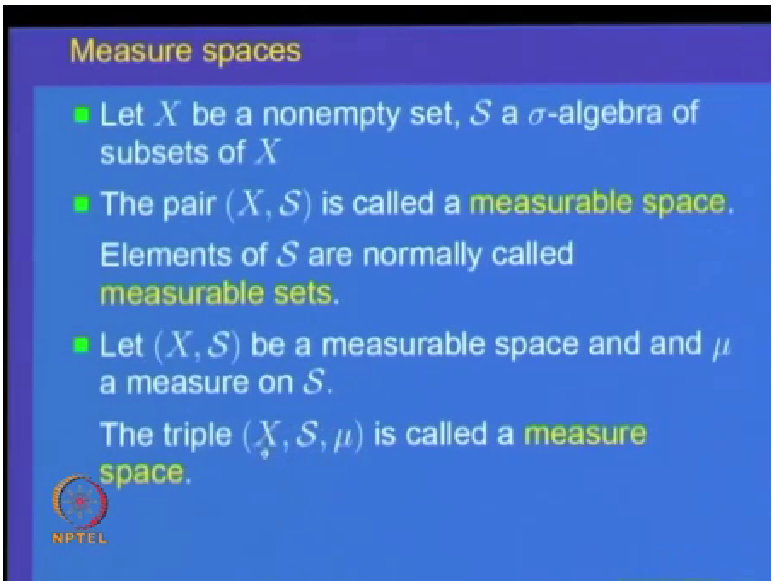
Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 11 B
Measurable Sets

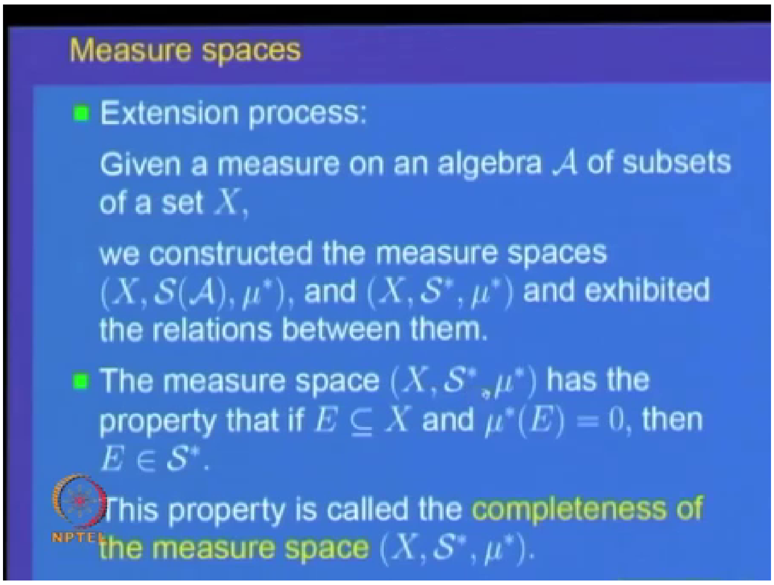
(Refer Slide Time: 00:17)



So, but we will this gives us a new notion. So, let us define that. So, let X be a non empty set, S a sigma algebra of sub sets of the set X, the pair X, S. Now onwards, we will be called a measurable space. So, a measurable space is a pair where X is a set and s is a sigma algebra of sub sets of it and suppose we are given. So, this elements of S normal, we are called measurable sets and. So, let us next suppose that, we are given a measurable space X S and we are given a measure on the sigma algebra S then we get a triple X S and mu that is called a measure space.

So, a measure space signifies a triple, a ordered triple where the first element X is a set X, a second one is a sigma algebra of sub sets of the set X and mu is a function defined on the sigma algebra taking non negative values and it is countably additive, it is a measure.

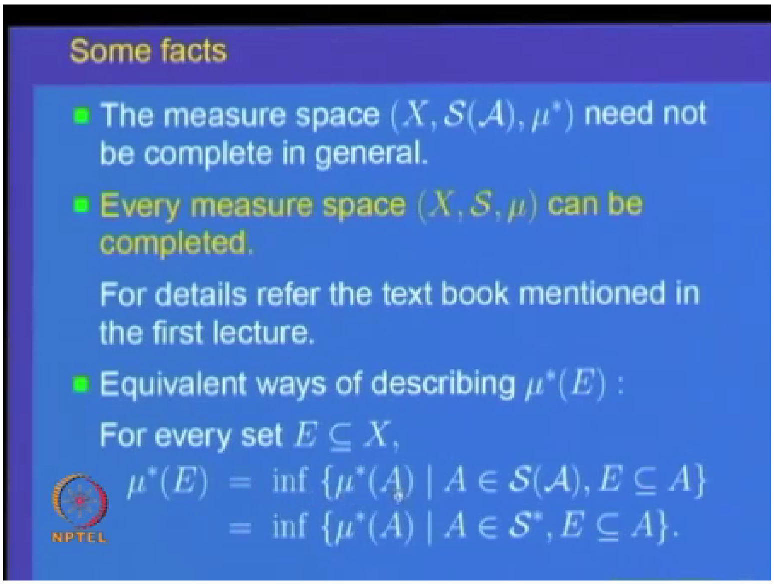
(Refer Slide Time: 01:47)



So, this triple is called a measure space. So, what we have done our extension process, we can now summarize it as follows given a measure on an algebra a of sub sets of a set X; what we did we constructed; two measure spaces one was X S of a the sigma algebra generated by it mu star, which is the outer measure induced by mu and we know mu star on S of A is a measure and we also have the measure space X S star and mu star mu star on S star, the class of all outer measurable sets. So, we get these two measures spaces keep in mind S of A is a sub set of S star and we gave the relation between these two namely the measures space. This is in some sense, we can say it is a bigger measure space because the algebra S star is bigger than S of A and this measure space as a special property namely that if you take any set E in X and mu star of E is 0 then e belongs to S star.

So, not only; so for example, this is a very special thing. Suppose, you take any sub set A of E then by monotone property mu star of A also will be 0. So, that also will be inside S star. So, S star includes all mu star, null sets, such a measure normally is called a complete measure space. So, our construction as given the measure space X S star mu star and it is a complete measure space namely all sets of outer measure 0 are elements of S star, that is a nice condition to have. We will see it a bit later on. So, this is called a complete measure space.

(Refer Slide Time: 03:53)



So, a complete measure space in this space. So, is that the sigma algebra A star. Sigma algebra includes all null sets. All sets whose measure is 0 in general a measure space need not be complete. So, for example, in particular for example, this measures space need not be complete in general. So, there is a theorem which says every measure space X S mu can be completed and this process of completion of a measure space is a slightly technical, one the basic idea is given a measure mu on a algebra S of sub sets of a set X collect together all sets, whose outer measure mu star is 0 and adjoin them, add them to the sigma algebra S; that means, generate a new sigma algebra by looking, taking together S and the sets, which are null sets.

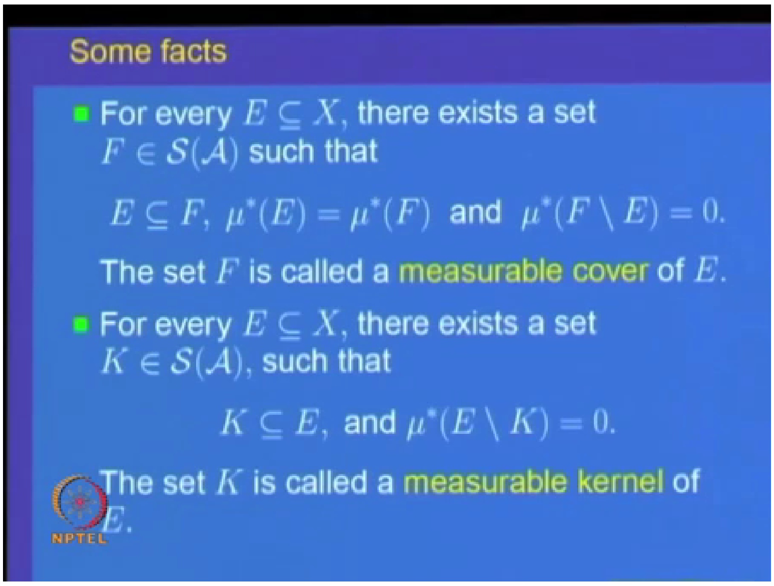
So, that gives a bigger sigma algebra and on that bigger sigma algebra 1 can show. We can extend that measure mu to the sigma algebra and the new measure space becomes complete. So, at the process is very much similar to looking at X S of A and mu star and viza V X S star and mu star. So, we will assume this theorem that every measure space X S mu can be completed. So, if you are interested in looking at the technical details for this look at the text book, which we mentioned in the first lecture, namely an introduction to measure an integration by me.

So, we will leave these details for those, who feel more interested in looking at the details. Next, we will give the, there are some equivalent ways of describing the set mu star of E and that is that mu star of E can be also written as infimum of mu star of A.

Where A belongs to S of A and all E are inside a right. So, look at all elements from the sigma algebra generated by A, which include the set E and look at the mu star of A and take the infimum of them. So, in some sense mu star of a set can be approximated by sets from mu star from elements of S of A and a similar result is true for belonging to elements which are measurable sets.

So, these are technical things. So, these are some facts, which will not prove and most probably will not be using them in our course, but they are nice to know that relation between mu star of E and mu star of sets in the sigma algebra S of A and S star of A.

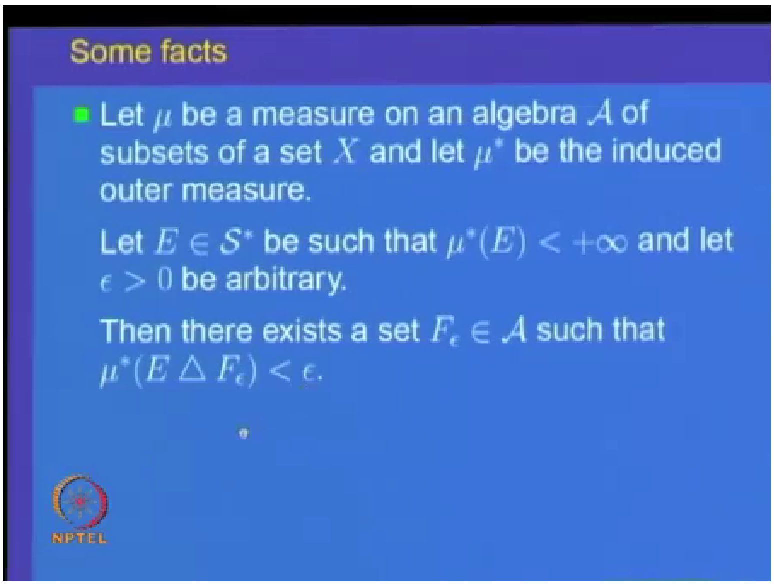
(Refer Slide Time: 06:39)



Here is another fact, which again we will not be proving and most probably we will not be using namely that for every sub set E in X, you can find the set in the sigma algebra S of A, the sigma algebra generated by a such that the set E is a sub set of F. So, F is which includes E and the outer measure of the 2 are same and that also in turn implies that outer measure of F minus E is 0. So, essentially, it says for every set E contained in X, there is a set in the sigma algebra S of a such that the difference has got outer measure 0 such a set is called a measurable cover of E; so such a set, because F covers E and a similar results for a set inside.

So, if E is in X, then you can find a set K inside E, such that the difference mu star of E minus K is 0 and such a set is called a measurable kernel of E. So, given any set E, there is a cover by a measurable set and there is a smaller set inside which is a kernel. So, and difference is of sets of measure 0.

(Refer Slide Time: 08:08)

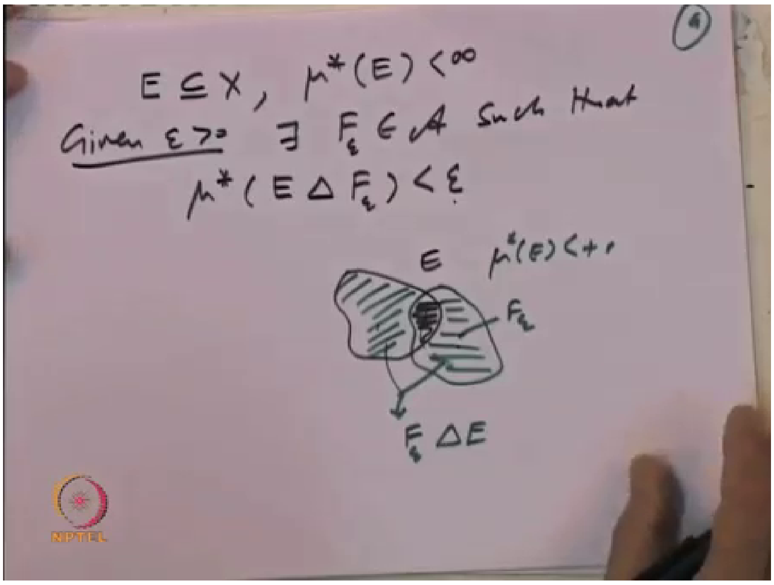


So, these things we will not prove. We will prove result which will be needing electron and that relates the outer measure with measure of the set inside the algebra that we have started with. So, mu start with a measure mu on an algebra A of sub sets of A set X, let mu star be the induced outer measure. So, suppose we have got a set E such that mu star of E is finite. This set need not be in the sigma algebra. So, take any set E, such that mu star of E is. So, we do not need this condition that E should be measurable set. So, take any set whose outer measure is finite then given any epsilon you can find a set in the algebra A. Such that mu star of E symmetric difference that set F epsilon is less than epsilon.

So, this is a very nice result which says any set of finite outer measure.

As I said this condition is not there, it is not needed for any. It is a type for any set of finite outer measure you can find a set in the algebra, such that mu star of E symmetric difference. So, the measure of the symmetric difference is small.

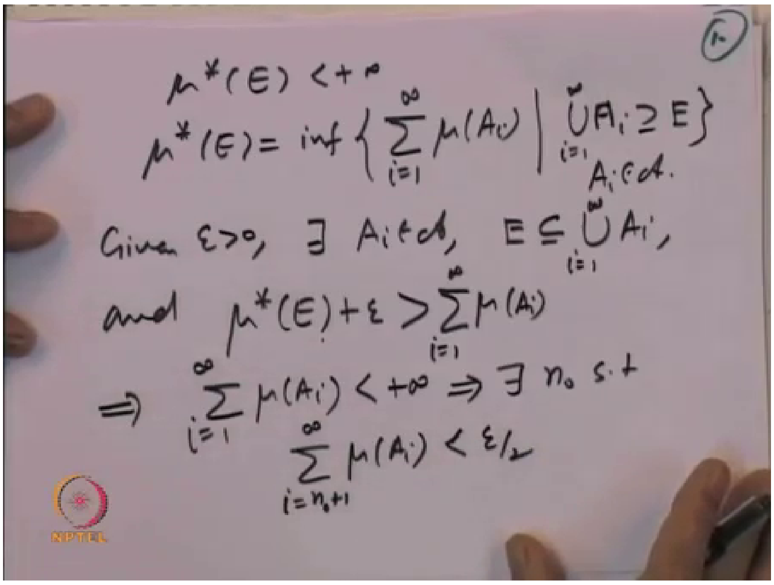
(Refer Slide Time: 09:36)



So, let us look at a proof of this result. So, what we are saying is. So, let us take a set E contained in X with the condition that mu star of E is finite. So, it says given epsilon bigger than 0, there exists a set F epsilon belonging to algebra, such that mu star of E symmetric difference with F epsilon is less than epsilon.

Let us see what we are saying. So, we are saying that, this is a set E, it says given a set E with the condition that mu star of E is finite. I can find a set call this as F epsilon, such that what is the symmetric difference; symmetric difference is E minus and F minus. So, that is the portion. So, this portion is F epsilon, symmetric difference E. So, it says that the sets. So, this is the common portion. So, it says that the measure outer measure of the sets were which are outside the common portion is small. So, essentially almost we can say that E and F epsilon are same.

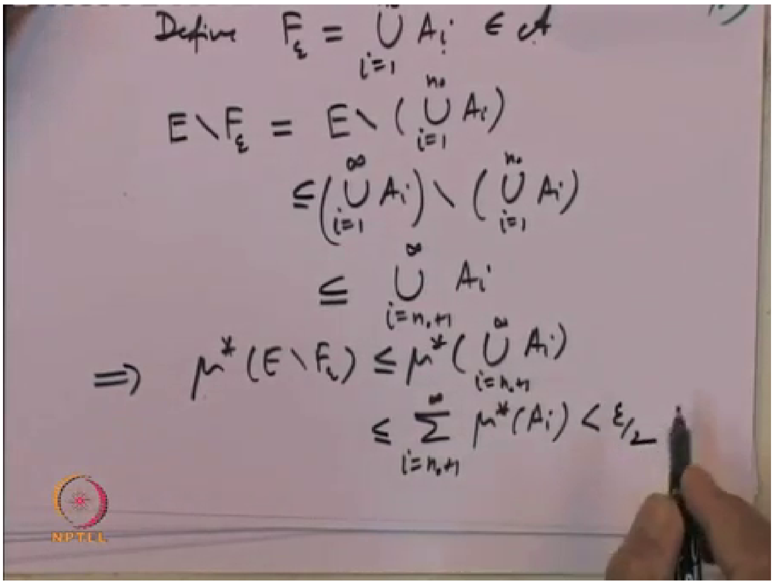
(Refer Slide Time: 11:18)



So, let us prove this property. So, to prove this, let us observe that mu star of, so, mu star of E is finite and what is mu star; mu star of E, if you recall mu star of E is equal to infimum of sigma mu of A I. I equal to 1 to infinity where this A i S union of A i s cover. So, union of A i S cover the set E and A i S in the algebra.

So, this being finite; so given epsilon a small quantity bigger than zero there exists a covering, so that this sets A i belonging to the algebra, such that E is contained in union of A i's and mu star of E, which is infimum plus the small number is bigger than sigma mu of A i's right, that is by the definition of the infimum is finite. So, note this implies, because this is finite. So, this implies that the series I equal to 1 to infinity mu of A i is finite right. So, there exist as a consequence of this. There exist some n naught such that the tail of the series. So, n naught plus 1 to infinity mu of A i is less than say epsilon by 2 that is because of the series this is convergent. So, once that is done we define.

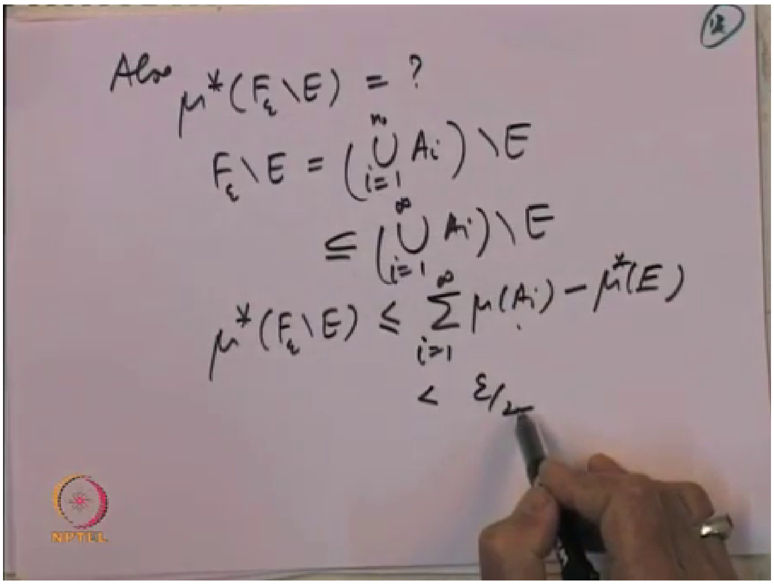
(Refer Slide Time: 13:13)



So, let us define the set F epsilon to be equal to union of A i I equal to 1, to the stage N naught. So, note this set belongs to the algebra, because it is a finite union of elements in the algebra. So, it belongs to the algebra. So, let us calculate, look at the set E minus F epsilon. So, what is that? So, that is E minus union I equal to 1 to n naught A i right and now the set E is contained union I equal to 1 to infinity A I. So, this is contained in this minus union I equal to 1 to n naught A I. So, I can say this is contained in union I equal to n naught plus 1 to infinity of A I. So, that implies that mu star of E minus F epsilon is less than or equal to mu star of this set union I, n naught plus 1 to infinity A i and which by sub additive.

So, this was sub set of this. So, mu star of this is sub less than or equal to by monotone property and by sub additive property. This is less than or equal to sigma I equal to n naught plus 1 to infinity mu star of A i and that if you recal,l we have less than epsilon by 2. So, this is less than epsilon by 2. So, we get that mu star of E minus F epsilon is less than epsilon by 2.

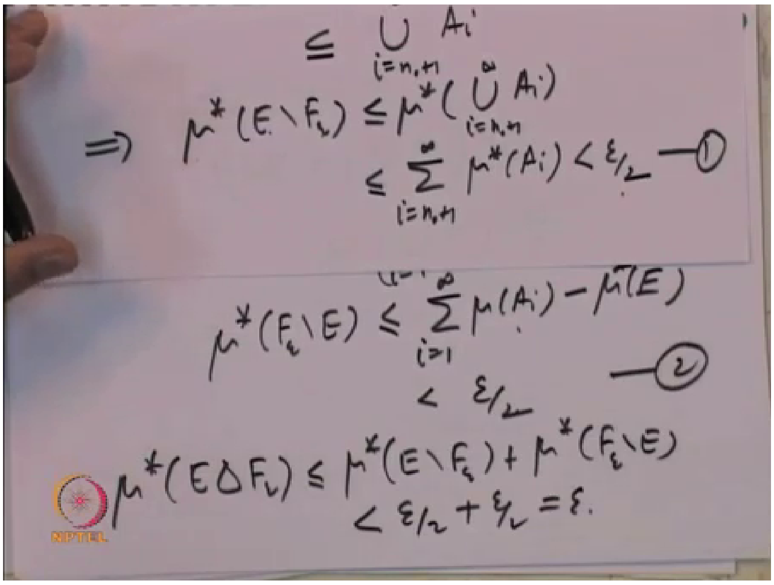
(Refer Slide Time: 15:15)



Let us also compute the measure of the other part namely, we want to compute also mu star of F epsilon minus E. Let us compute that. We want to compute, what is this equal to. So, F epsilon minus E is union I equal to 1 to n naught A i minus E and note. So, this is a sub set of union of I equal to 1 to infinity A i minus E and E is a sub set of this. So, that implies that mu star of F epsilon minus E is less than or equal to mu star of this. So, that is sigma I equal to 1 to infinity mu of A i minus mu of mu star of E and that, if you recall is by the well, the way we started, we had mu star of summation mu star of A i this relation.

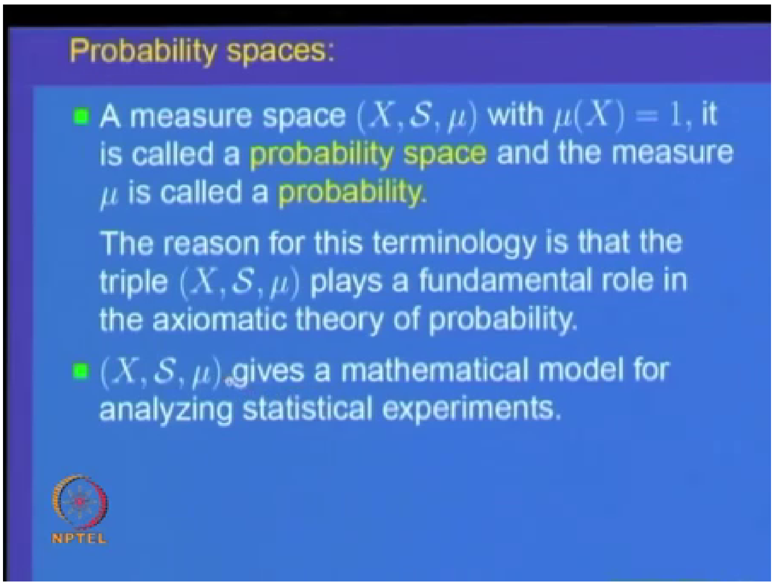
So, this says sigma mu of A i minus mu of E is less than epsilon. So, we could have started with if required by epsilon by 2. So, then we have gotten the, this is less than epsilon by 2. So, we are getting that mu star of F epsilon minus E is less than epsilon by 2 and we already shown that mu star of we already shown that mu star of F minus E epsilon is less than epsilon by 2; so putting this two together. So, call this as 1, call this as 2.

(Refer Slide Time: 17:03)



So, by putting 1 and 2 together mu star of E delta of epsilon is less than or equal to mu star of E minus F epsilon plus mu star of F epsilon minus E and both of them are less than epsilon by 2 plus epsilon by 2 which is equal to epsilon. So, that proves the required property which we wanted to prove namely that given epsilon bigger than 0, there is a set F epsilon, which is in the algebra A, such that mu star of E delta F epsilon is less than epsilon. So, this is a approximation property, which will be using electron to prove some facts.

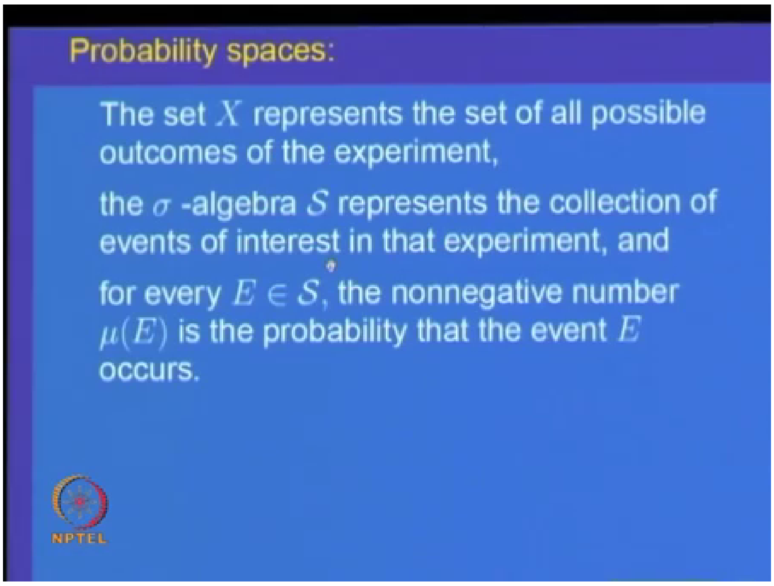
(Refer Slide Time: 17:54)



So, this is the process of extension theory. So, the process of extension theory gives us ways of constructing triples; which are measure spaces at this point. It is worth mentioning, there are measure spaces of importance in other subjects called probability theory; a measure space X S mu, where mu of X is 1. So, that is a totally finite measure and mu of the whole space is equal to 1 is called a probability space and the measure mu is called a probability. So, I measure space where mu of X is 1 is called a probability space and mu is called a probability.

The reason for this terminology is that, such triples play, a fundamental role in axiomatic theory of probability whenever you want to described a phenomena, a statistical phenomena which depends upon some randomness one has to construct a probability space to analyze it. So, this gives a mathematical model in the theory of probability to analyze statistical experiments.

(Refer Slide Time: 19:22)



So, let us let me just give you a few things more. The set X denotes in the triple X S mu X represents the set of all possible outcomes of the experiment. For example, you are tossing a coin. So, all possible outcomes are head or tail or you are throwing a die and there are 6 possible outcomes the number 1 2 3 4 5 and 6 or you are observing the temperature of a particular place; every day at say particular time. So, the observation will be a real number.

So, in any particular experiment, the all possible outcomes of that experiment are they constitute a set and that is the set x and all the sigma algebra S represents the collection of events of interest in that experiment. So, any sub set of the set of outcomes in the experiment is called an event. So, for example, when you are tossing a coin, there are 2 outcomes possible head and tail. So, if you look at the single ton H that is an event, when you toss head, can come or you toss a tail can come or if you throwing a dye then the outcomes possible are 1 2 3 4 5 and 6.

Look at the sub set 1 3 and 5 of X; the set of all odd outcomes. So, when you throw is possible to find out whether that event has occurred or not; that means, whether the outcome was a odd number or not. So, that is the sub set of the set of all possible outcomes. So, in general when you want to describe it is statistical experiment 2 has to construct a class of sub sets, of that set X of interest and 1 requires because of mathematical considerations, that class should be a sigma algebra.

So, the sigma algebra represents the collection of events of interest, in that particular experiment and finally, for every event E of interest, you want to assign some what is the possibility of that event happening a probability of that event taking place. So, a probability is a measure defined on the sigma algebra of all possible events of interest and taking non negative values and of course, probability of the whole space. The chance of the whole space happening is 1 and probability of the empty set is 0. So, the probability is a set function defined on the collection of all events of interest and we want that to be a measure. So, that is the reason that the triple x as mu is called a probability space and gives a mathematical model for analyzing statistical experiments, when mu of X is equal to 1.

So, in till today's lecture, we have constructed measure spaces and from the next lecture onwards. We will specialize. This measure space when X is real line and that gives the important. Example of a measure space and a measure called Lebesgue measure. So, we will do that in the next lecture.

Thank you.