Measure & Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 11 A
Measurable Sets

Welcome to lecture 11 on measure and integration, in the previous lecture we had defined what is called a outer measurable sub set and we had started looking at the properties of the outer measurable sets. We will continue that study of properties of the outer measurable sets today and if time permits in the end we will specialize the case when this space is the real line. So, let us recall what we have been doing. So, properties of measurable sets we were looking at.

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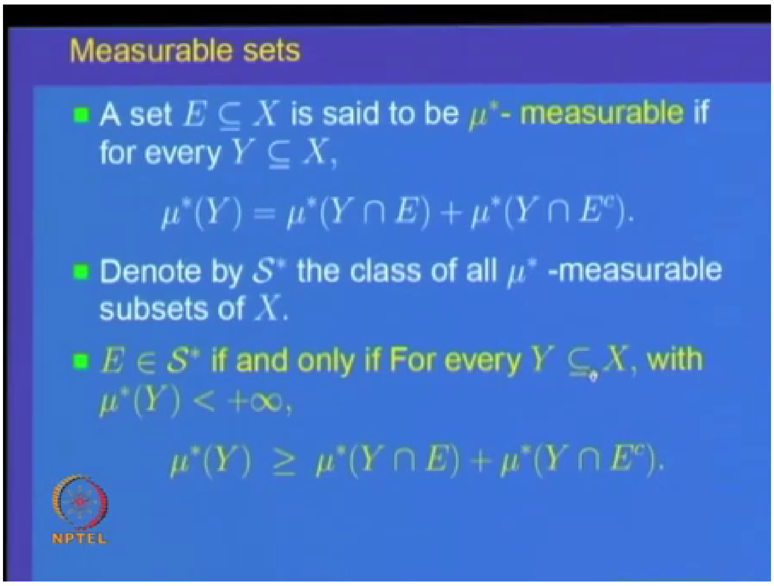

Let us just recall what is outer measurable set, A sub set E of X was set to be outer measurable or mu star measurable, if mu star of any set Y is written as can be written as mu star of Y intersection E plus mu star of Y intersection E compliment. So, this condition must be satisfied for every sub set Y of X and then we said let us denote by S star the class of all mu star measurable sets and we gave A equivalent way of verifying when A set is outer measurable. So, the condition is that A set E is measurable if and only if for every sub set Y in X with mu star of Y finite, we have the condition that mu star of Y is bigger than or equal to mu star of Y intersection E, plus mu star of Y intersection E compliment. So, instead of just saying that for every sub set Y this equality must be true we have to only verify for those sub sets Y of X for which mu star of Y is finite and instead of equality we have to verify only bigger than or equal to one way in equality because the other way around is always true for mu star being countabaly sub additive.

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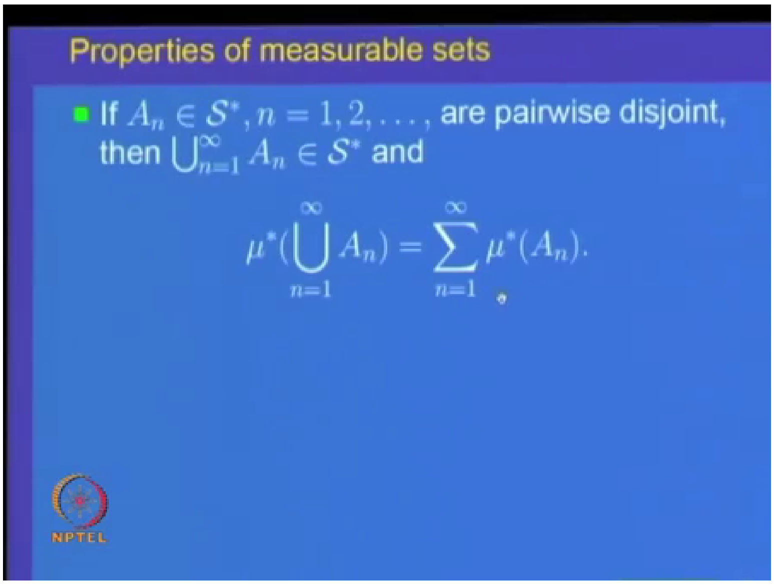

So, this condition we will use whenever we required and so means the first observation that we proved last time was that A is the given algebra on which the measure is defined. So, the first claim we proved that every element in the algebra is also a measurable set. So, A is sub set of S star, the second property that we were looking at was that if S star is, that S star is an algebra of sub sets of X and mu star restricted to S star is finitely additive, we had already observed that a set E is measurable if and only if its compliment is measurable. So, S star A is closed under compliments only we have to verify that it is closed under unions and that proof we are working out in the last time and we are d1 it, let us just revise it again because we are going to nu need those inequalities.

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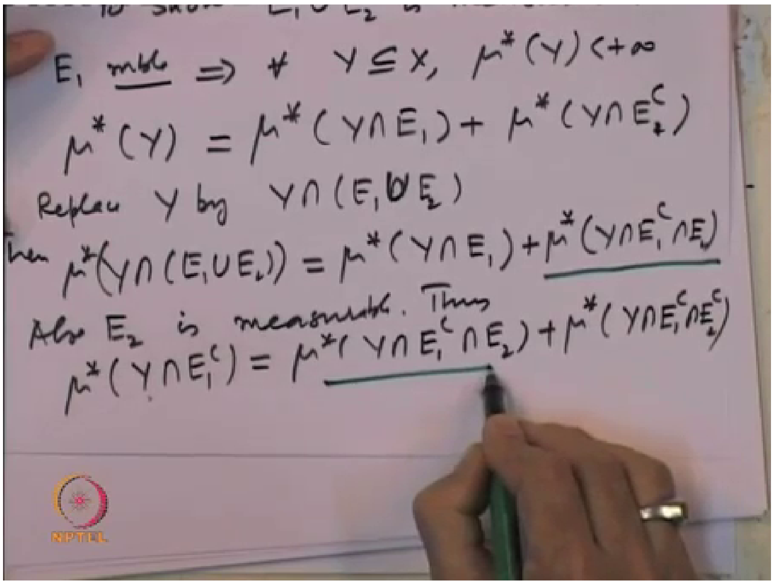

So, let us take let E1 and E2 be measurable sets to show that E1 union E2 is measurable. So, E1 measurable implies that for every sub set Y contained in X and let us also have the expressional condition less than finite, we know that mu star of Y is equal to mu star of Y intersection E1 plus mu star of Y intersection E2 sorry E1 compliment, this is true for every sub set Y with that property let us replace Y by Y intersection E1 intersection E1 union E2.

So, replace this y so then we get so then what do we have? We have mu star of Y intersection E1 union E2 is equal to. So, Y is replaced by Y intersection E1 union E2, but E1 is a sub set of it. So, the first term is just mu star of Y intersection E1 plus the second term becomes mu star of E1 union E2 intersection E1 compliment, the first term will give me only empty set, union Y intersection E2 intersection E1 compliment. So, E1 compliment intersection E2 so that is what we get by using the fact that E1 is measurable also E2 is measurable.

So, thus for every set Y a corresponding equation holds for E2 compliment, but will replace Y by Y intersection E1. So, so mu star of Y intersection E1 compliment is equal to mu star of Y intersection E1 compliment intersection E2 plus mu star of Y intersection E1 compliment intersection E2 compliment. So, using the fact that E2 is measurable we have written mu star of Y intersection E1 as the set intersection E2 the set intersection E2 compliment now. So, in this 2 equations look at this set this term Y intersection E1 compliment intersection E2 that is also sitting here. So, will compute the value of this and put it that equation. So, let us do that so from this second equation we put the value there so we have got so from this 2 equations.

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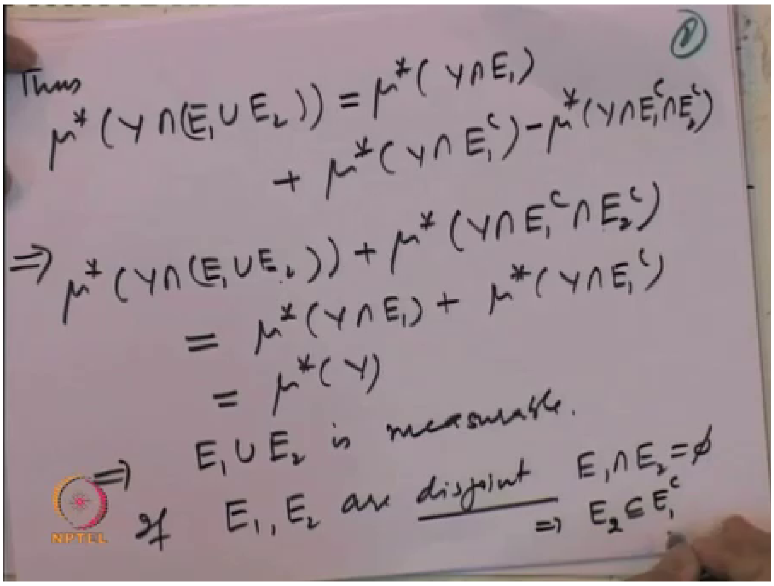

So, thus mu star of Y intersection E1 union E2 that is the left hand side of this equation, that is left hand side of this equation. So, that is equal to the first term. So, that is mu star of Y intersection E1 Plus Y intersection E1 compliment intersection E2 is equal to mu star of Y intersection E1 minus that thing.

So, mu star of Y intersection E1 compliment minus mu star of Y intersection E1 compliment intersection E2 compliment and now 1 should note down note here that we have taken 1 term on the on the other side. So, this is possible because all the sets involved have finite outer measure so this is the equation of real numbers. So, we can take one term on the other side and so on in general that will not be possible if Y, one of the terms is equal to plus infinity. So, the condition that mu star of Y is finite is being used here.

So, we get using the fact that E1 and E2 are measurable we get this equation. So, from here let us take this negative term on the other side. So, implies mu star of Y intersection E1 union E2 plus mu star of Y intersection E1 compliment intersection E2 compliment, is equal to mu star of this term Y intersection E1 and the second term plus mu star of Y intersection E1 compliment and now using the fact that E1 is measurable this is same as mu star of Y. So, we have shown that for every sub set Y with mu star of Y finite its measure mu star of Y can be written as mu star of E1, Y intersection E1 union E2 plus mu star of Y intersection E1 compliment, intersection E2 compliment, but note this set is nothing, but E1 union E2 compliment. So, this implies that E1 union E2 is measurable.

So that says E1 union E2 is measurable and now for the special case so if E1, E2 are disjoint; that means, E1 intersection E2 is empty set that is same also implies that E1 is contained in E2 compliment or E2 is contained in E1 compliment, either 1 is true so note this is true. So, in that case let us go back and look at the first equation that we had, we had because E1 and E2 are measurable. So, we had this condition right. So, in this equation this is true for every Y.

So, let us replace this Y by E1 union E2 they will give us the measure mu star of E1 union E2. So, in this we are going to replace Y by E1 union E2. So, let us just put that equation here and look at what we are doing. So, in this equation we are putting Y equal to so in this star.

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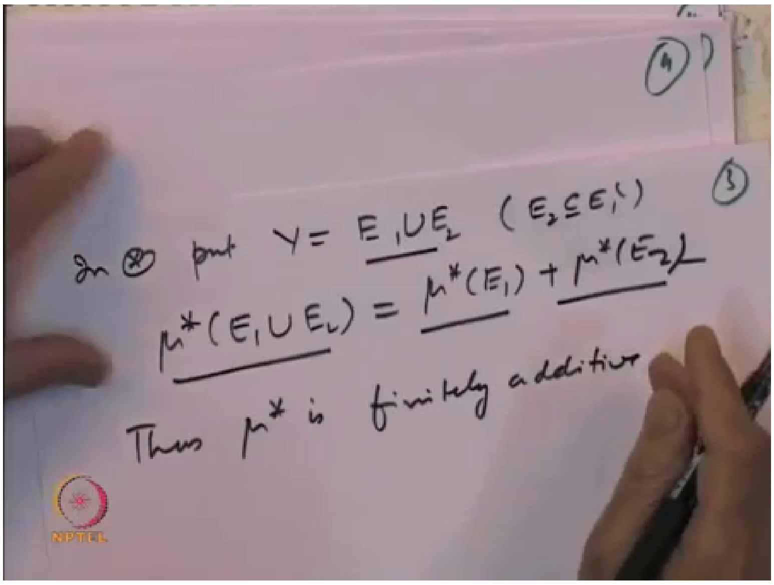

So, in star put Y equal to E1 union E2 keep in mind they are disjoint. So, the left hand side will be mu star of E1 union E2 equal to right hand side the first term is mu star of E1 plus in the second term because E1, E2 are disjoint E2 is a sub set of E1 compliment. So, this implies E2 is a sub set of E1 compliment so; that means, this is nothing, but plus mu star of E2, A star of E2. So, when E1 and E2 are disjoint mu star of E1 union E2 is mu star of E1 plus mu star of E2. So, that is; that means, though thus mu star is finitely additive. So, we have proved that whenever our so we have proved this property namely S star is an algebra of sub sets of X and mu star restricted to S star is finitely additive.

Next step is to go a bit further and we want to prove that whenever we got a sequence of sets in S star which are pair wise disjoint then their union is also in S star and mu star of the union is equal to summation of mu stars of An; that means, we are going to show that S star is close under pair wise disjoint union of sets, even countably infinite and mu star is countably additive. So, let us prove this property so let us take.

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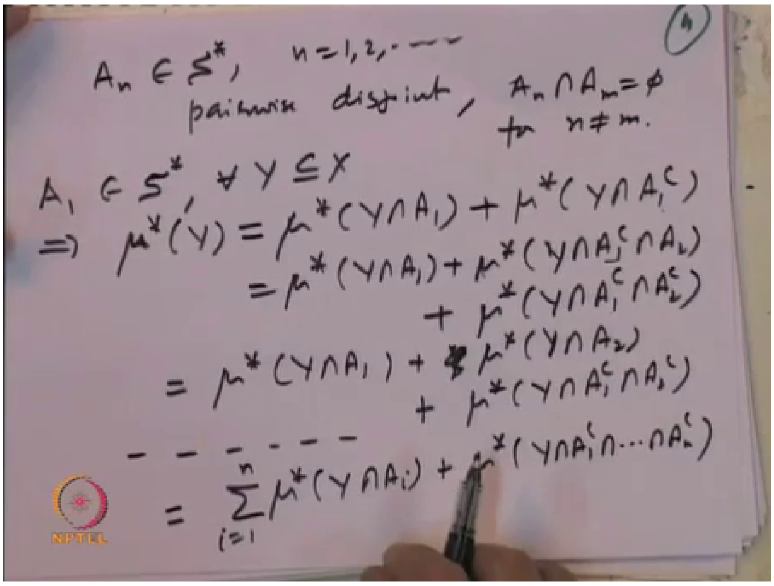

So, An belong to S star n equal to 12 and so on, pair wise disjoint that is An intersection Am is empty for n not equal to m right. So, we start A1 belonging to S star, A1 measurable implies that mu star of any set Y can be for every Y contained in X, I can write this to be equal to mu star of Y intersection A1 plus mu star of Y intersection A1 compliment and now use the fact that A2 is measurable. So, leave the first term as it is, Y intersection A1 plus A2 is measurable.

So, measure of mu star of this set can be written as mu star of A1 compliment, intersection A2 plus mu star of Y intersection A1 compliment, intersection A2 compliment. So, this term mu star of Y intersection A1 compliment is written as mu star of Y intersection A1 compliment plus intersection A2 plus mu star of Y intersection A1 compliment, intersection A2 compliment. So, here we have used the fact that A2 is measurable and now observe that A1 and A2 are disjoint. So, A2 will be a sub set of A1 compliment so this set is nothing, but Y intersection A2. So, I get this is same as mu star of first term A1 the second term is mu star of mu star of Y intersection A2 and the third term as it is mu star of Y intersection A1 compliment, intersection A2 compliment.

So, in the first we use A1 is measurable in the second we use A2 is measurable and use A1 and A2 are disjoint, we continue this process. If we continue this process after n steps we will have this is equal to the second step gives you mu star of Y intersection A1 plus mu star of Y intersection A2. So, after n steps this will have mu star of Y intersection A i, i equal to 1 to n plus one term will be there which is mu star of Y intersection A1 compliment intersection up to An compliment right. So, let us write this last term in terms of union.

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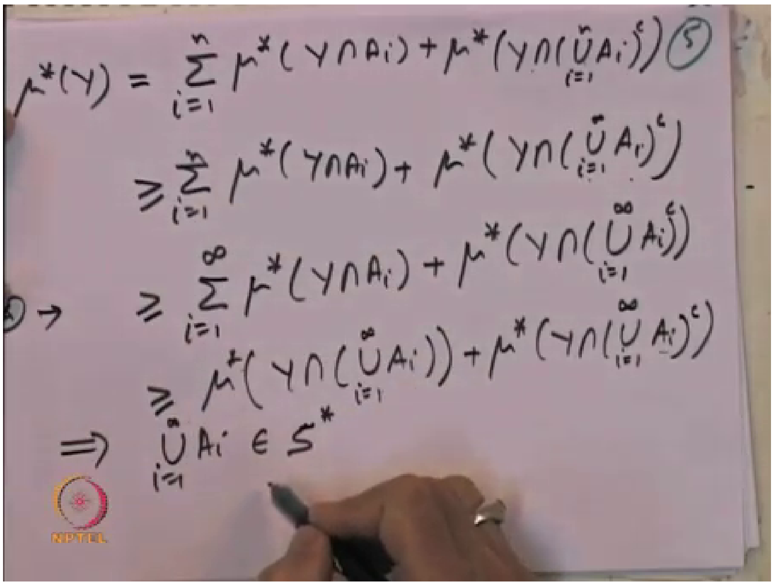

So, this is equal to summation of i equal to 1 to n mu star of Y intersection Ai plus mu star of Y intersection union Ai, i equal to 1 to n compliments right. So, this term is represented in terms of compliments of the union. So, this is after n steps so far every n we have got mu star of Y can be written as this and now this is true for every n. So, here I would like to write this union as 1 to infinity, if I do that I will be making this set bigger and hence the compliments will be a smaller set.

So, replacing this set so if I replace this by Y intersection union I equal to 1 to infinity of A i compliment this set is bigger than this, is smaller than this set. So, mu star of this will be bigger than mu star of this. So, if I write mu star of this. So, this term is bigger than this term. So, this will be bigger than or equal to summation i equal to 1 to n this term as it is Y intersection Ai plus this. So, what we had done in the second term where it has union 1 to n, I have taken union 1 to infinity and because of compliments this term will be smaller. So, instead of equality I have got the inequality and this happens for every n.

So, I can let n go to infinity so this will be bigger than or equal to summation I equal to 1 to infinity mu star of Y intersection Ai plus mu star of Y intersection union i equal to 1 to infinity Ai compliments and now mu star is countably sub additive. So, this term the first term is bigger than or equal to mu star of Y intersection union Ai, i equal to 1 to infinity second term as it is. So, mu star of Y intersection union 1 to infinity A is compliment.

So, using the fact that for every n, An is A measurable set we are able to say that mu star of Y is bigger than or equal to mu star of Y intersection the union Ai's plus mu star of Y intersection the compliment of the unions. So, that implies that union Ai 1 to infinity belongs to S star is a measurable set not only that we can say something more actually. So, let us in this equation so this equation star this one, let us put Y is equal to so in star let us put Y equal to union of Ai's. So, what will get. So, let us do that substitution and see what do we get.

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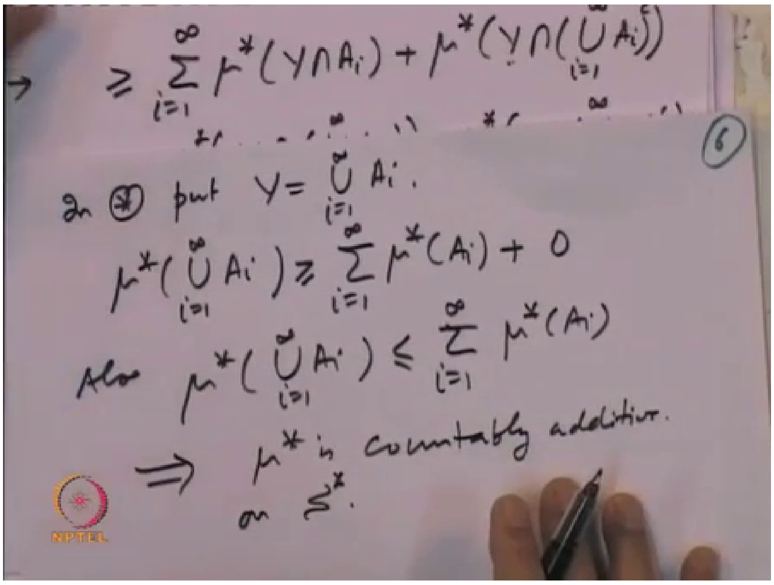

So, in this equation in star take Y equal to union of A is then left hand side is mu star of union Ai 1 to infinity is bigger than or equal to summation I equal to 1 to infinity mu star of Y is unions of A is. So, this is just Ai plus this is union and this is compliment that is empty set mu star of that is equal to 0 so that is equal to 0. So, what we get is mu star of the union of A is is bigger than or equal to this, also by sub additivity mu star of the union Ai's is less than or equal to summation 1 to infinity mu star of Ai's.

So, should, sorry imply that mu star is countably additive on S star, so this is very nice. So, we have got the following property that if. So, what we have done till now is we have shown that S star as a consequence of all this properties now we can say that S star the class of all measurable sets is a sigma algebra of sub sets of S, as of X and mu star on this is countably additive.

So, we started with a measure mu on a algebra A of sub sets of A set X, we defined an outer measure via this on all sub sets of sub set X then we picked up A sub class namely S star of sets which are mu star measurable and we have shown that mu star which in general is countably sub additive is actually countably additive, on S star the sigma algebra of measurable sets its. So, it is a sigma algebra, why it is a sigma algebra? Because we have already shown it is an algebra and it is closed under countable disjoint unions. So, any collection any algebra which is close under countable disjoint unions is automatically a sigma algebra that we have shown.

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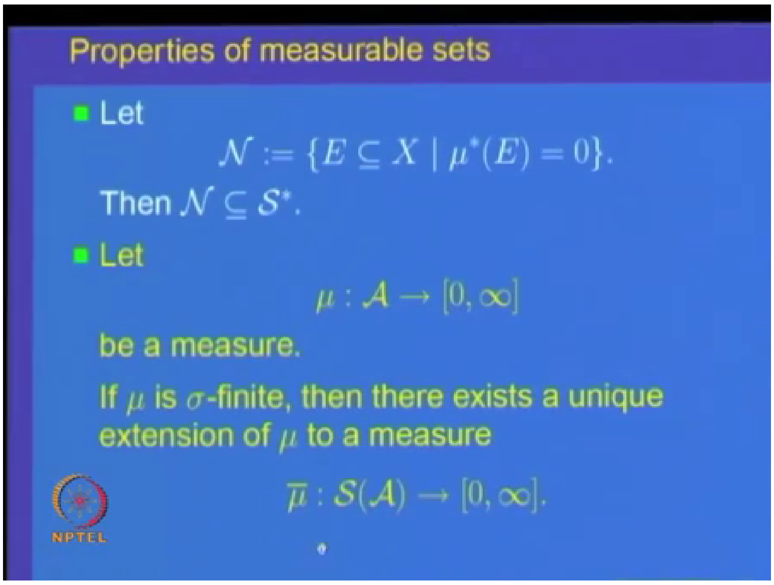

So, this gives us say way of defining measures on ascending measures, let before doing that let us observe one more thing let us look at sets E in X whose outer measure is 0 these are called sets of outer null outer measure measurable sets. So, the claim is every set whose outer measure is 0 is automatically measurable. So, let us check that.

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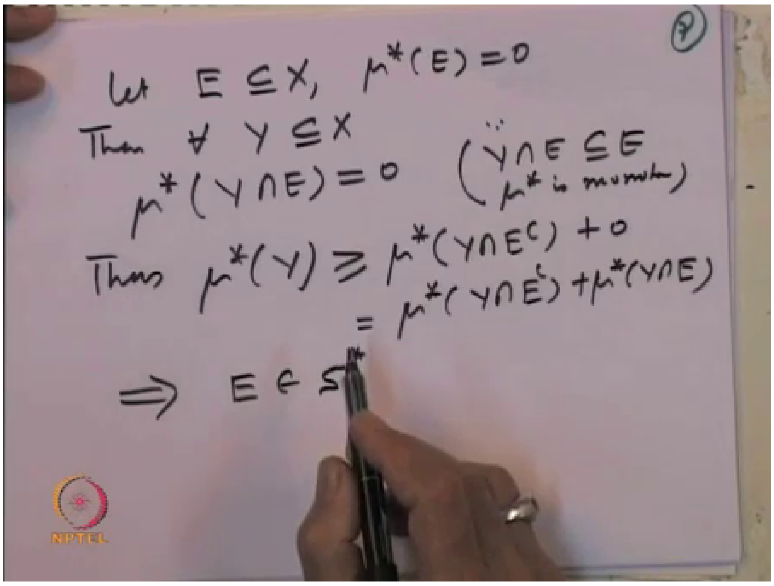

So, let E be a sub set of x and mu star of E equal to 0 then for every Y contained in X mu star of Y intersection E is equal to 0 because Y, because Y intersection E is contained in E and mu. So, because and mu is mu star is monotone. So, this is 0 so thus mu star of Y is given in bigger than or equal to mu star of Y intersection E compliment because again Y intersection E compliment is a sub set of A this and I can add 0 to 8. So, that is equal to mu star of Y intersection E compliment plus mu star of Y intersection E and that is precisely saying that the set E is measurable.

So, that is shows that the class of mu star null sets are also measurable. So, this class n is inside S star. So, let us summarize the process now what we have gotten. So, let us start we said that let us start with the measure mu on an algebra a mu is a measure, A is a algebra of sub sets of to the set X then if mu is sigma finite then there exists a unique extension of mu to the sigma algebra generated by it and how do we conclude that. So, that the conclusion for that is as follows.

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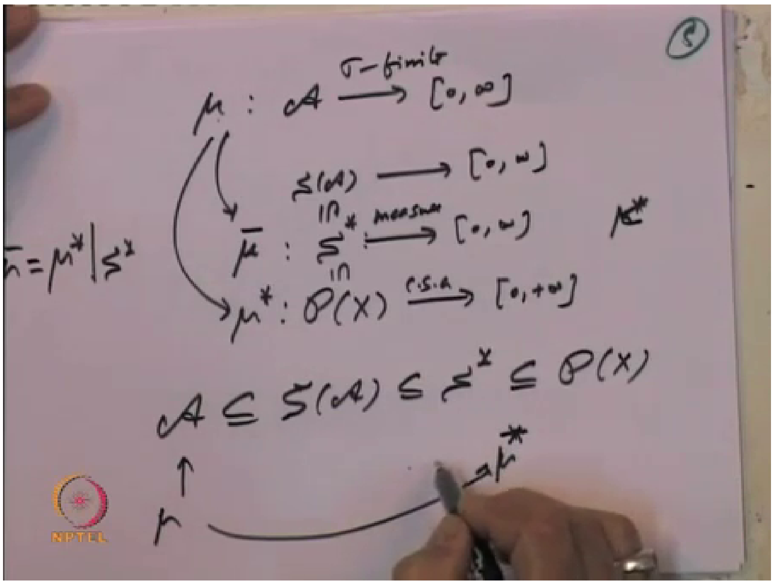

So, mu is on the algebra and it is sigma finite given. So, we define from it outer measure mu star which is defined on all sub sets of X it is countably sub additive and we picked up the class of measurable sets S star. So, if we restrict mu to this so let us call it as mu bar that is a restriction of mu to the smallest, smaller class S star keep in mind S star is right the class on measurable sets and so what is mu star?

Mu star, mu bar so mu bar is equal to mu star restricted to S star then this is A measure S star is A sigma algebra and this is A measure and we know that this is an extension. So, from mu we come to mu bar and extension of mu from the algebra to S star and note A is all sets in A are measurable. So, the sigma algebra is also inside here. So, A is inside S of A which is inside S star which is inside all sub sets of x.

So, mu is defined here we get mu star here and we restrict we get mu bar and that is same as mu bar on S of A. So, we get a measure mu bar on S of A. So, that is same as mu bar on S of A. So, what is mu bar? Mu bar is a restriction of the outer measure mu star to the sigma algebra is related by A and that is inside the class of measurable sets. So, it is A well defined measure and because mu is sigma finite supposing there were another extension by some other method to the sigma algebra then by the uniqueness some measures on the sigma algebras we know that there is only 1 possible extension that we have already proved that, in case an extension exist if 2 measures agree on the algebra they will also agree on the sigma algebra provided their sigma finite. So, uniqueness follows from that theorem. So, we have got that if mu is a sigma finite measure on an algebra then we can extended to the sigma algebra generated by it so this is the extension process. So, 1 has to start with a measure mu on an algebra and recall we already have extended from a semi algebra to the algebra generated right.

So, essentially it says that if we have A measure on the on A semi algebra of sub sets of set X and the measure mu is sigma finite then it can be uniquely extended to a sigma finite measure on the sigma algebra generated by that algebra. In fact, we have proved something more.

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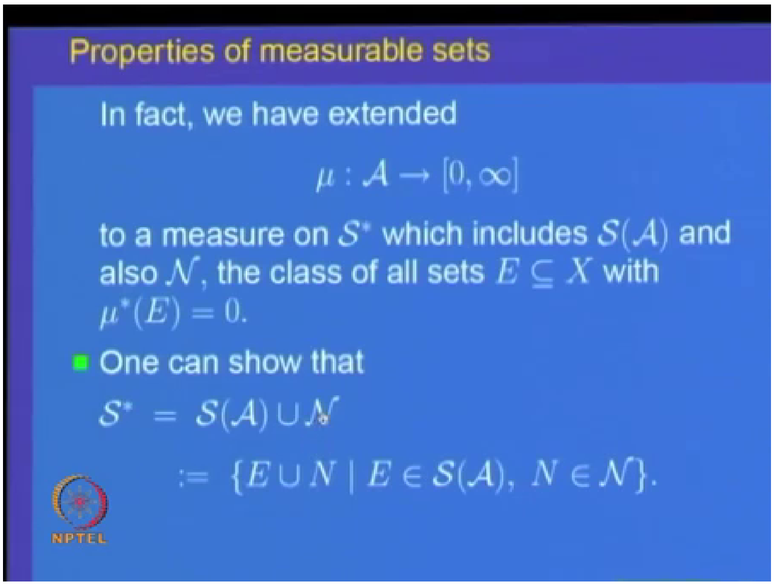

So, we have actually shown that not only mu which is defined on algebra extends to S of A, the sigma algebra generated by it actually it extends to a class S star which not only includes S of A it includes also the class of null sets mu star null sets, sets of outer measure mu 0. So, let us denote the class of the sigma algebra generated by S the sigma algebra generated by S of A and the null sets by a new name. So, what we are saying is 1 can show that this S star the class of all outer measurable sets which is a sigma algebra which includes S of A also if includes n. So, it includes this union. So, n so, this union we are writing it as, sets of the type. So, S A union n it is not the union of this 2 classes it denotes sets of the type E union n where E belongs to S of A and n is A null set. So, take sets which are in the sigma algebra generated by A adjoin to it any null set mu star null set. So, look at this new collection then one can show that S star is same as E union n.

So, it involves 2 things name one, namely this collection is a sigma algebra and this sigma algebra is same as S star, we will not go in to the details of this they are slightly technical we will assume this.