Measure & Integration
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Lecture – 10 B
Outer Measure and It's Properties

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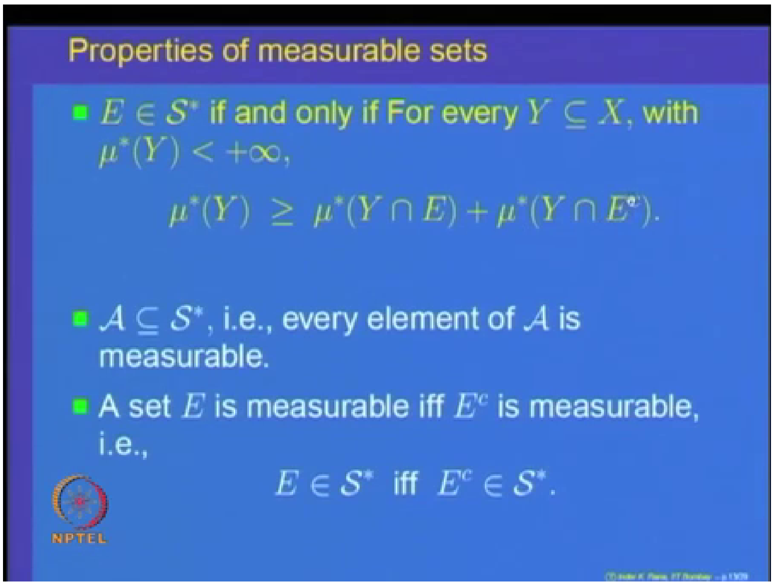

So, we are going to now understand the properties of this class S star; what are the properties of this collection of measurable sets? So, the first observation is that every set in the given algebra is measurable; that means, if A belongs to the algebra A; then this condition is always going to be true; so let us verify that.

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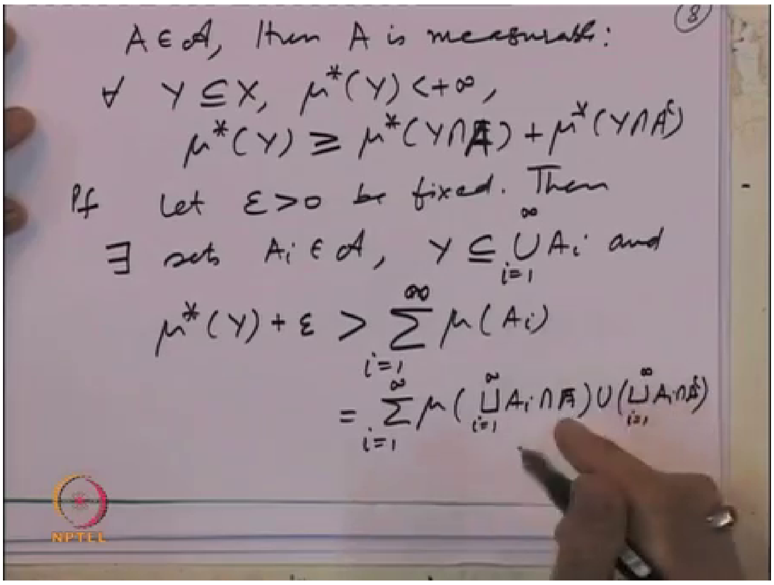

So, let us show that if a belongs to the algebra then A is measurable and that is for every Y is sub set of X with mu star of Y finite; we should have mu star of Y is bigger than or equal to mu star of Y intersection A; the set is A. So, Y intersection A; plus mu star of Y intersection A compliment.

So, this is what we have to show, so let us look at the proof of this. Now, we are going to use the fact that mu star of Y is finite and mu star of Y finite means that it is a infimum of some quantities. So, that crucial definition what is definition of infimum; we are going to use. So, let epsilon greater than 0 be fixed; then by definition of infimum; there exists a covering. So, there exist sets A i belonging to A with the fact that the set Y is covered by union of A i's; i equal to 1 to infinity.

And mu star of Y which is infimum plus a small number does not remain the infimum. So, it is bigger than or equal to mu of A i; i equal to 1 to infinity. So, here we are using the fact that mu star of Y is infimum and that is a finite quantity. Now, A i's are in the algebra; A is in the algebra. So, I can write this as this is equal to sigma i equal to 1 to infinity; mu of; union of A i’s intersection A; the set is A; 1 to infinity union of the sets; union i equal to 1 to infinity; A i intersection a compliment.

So, what I am saying is; not the union this is wrong. So, let me just simply write it as; so, let us observe what we are saying; we are saying because of this each set A i.

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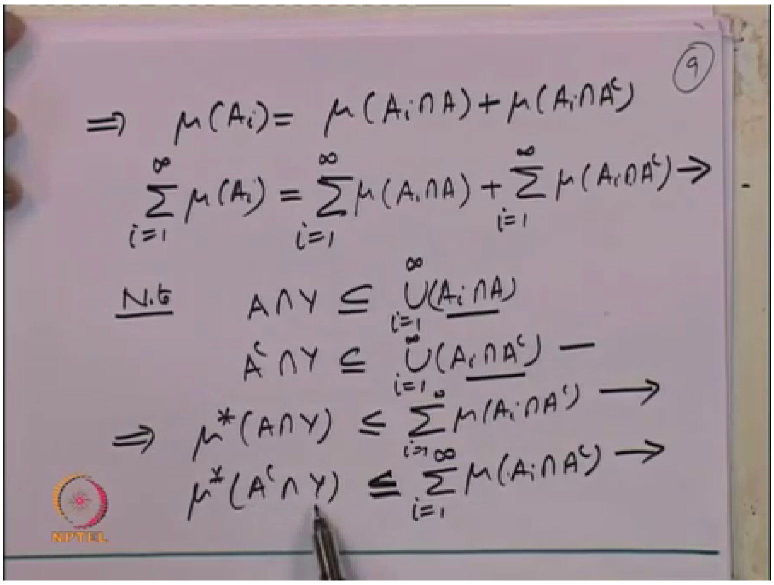

So, let us note A i; I can write as A i intersection A; union A i intersection A compliment. And these are two disjoint sets and mu is a measure; all the A i’s, A everything is in the algebra. So, using the fact that mu is a measure; I can write it as. So, implies that mu of A i is equal to mu of A i intersection A; plus mu of A i intersection A compliment.

So, now mu of A i intersection A compliment so; that means, summation mu of A i; i equal to 1 to infinity is equal to summation i equal to 1 to infinity, mu of A i intersection A plus summation i equal to 1 to infinity; mu of A i intersection A compliment. And now let us note that the set A intersection; Y is covered by union of A i intersection A; i equal to 1 to infinity.

Because Y is covered by union of A i's; so, A intersection Y is covered by this and A compliment intersection Y is covered by union of i equal to 1 to infinity; A i intersection A compliment. And these are sets in the algebra because A belongs to the algebra that is a crucial thing. So, this will imply that mu star of A intersection Y is less than or equal to summation of mu A i intersection; A compliment i equal to 1 to infinity and mu star of the second one gives me, A compliment intersection Y is also a sub is less than or equal to using this summation i equal to 1 to infinity; mu of A i intersection A compliment.

So, look at this equation, look at this equation, look at this equation, so mu star or summation mu star of A i's is bigger than this sum and that sum is bigger than or equal to mu star of A intersection Y; and this sum is bigger than or equal to mu star of A intersection Y; Y A compliment intersection Y.

And we had mu star of Y plus epsilon was bigger than this summation. So, that summation; so, putting this three equations together; so, if we call that earlier equation as 1, call this equation as 2, call this equation as 3 and call this equation as 4.

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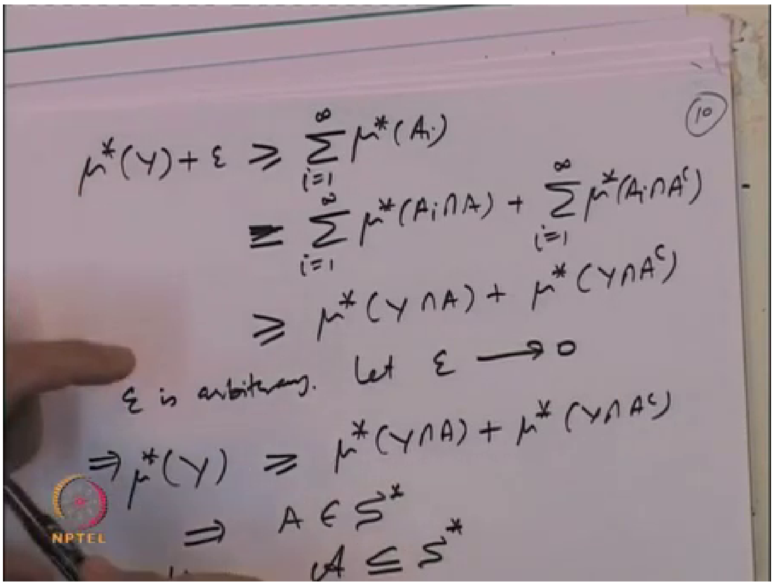

Then putting all this four equations together; what we have is the following that mu star of Y plus epsilon; which was bigger than or equal to summation mu star of A i; i equal to 1 to infinity, that is equal to actually summation of mu star of A i intersection; A i equal to 1 to infinity, plus 1 to infinity mu star of A i intersection; A compliment and that is bigger than or equal to mu star of Y intersection A; plus mu star of Y intersection A compliment.

And now epsilon is arbitrary; so let epsilon go to 0. So, this inequality will be still maintained will imply that; mu star of Y is bigger than or equal to mu star of Y intersection A; plus mu star of Y intersection A compliment and that will imply that A belongs to S star that is A; is a measurable set. So, hence we have proved that the algebra A is included in the collection S star; that is what we wanted to prove. So, this is the proof of the fact that the algebra A is contained in S star; every element of A is measurable.

The next property that the class of measurable sets is closed under complementation namely; if E is measurable then E compliment is may also measurable, that is obvious because in this criteria if you want to check if E is measurable; then this is what we required. And to check E compliment is measurable the same thing is required because this will become E compliment and E compliment of compliment is E. So, it is the same criteria; same equation to be verified.

So; obviously, because the definition has inbuilt E and E compliment symmetric with respect to E and E compliment; that says the set E is a set is measurable if and only if its compliment is measurable. Or the collection S star of measurable sets is closed under compliments.

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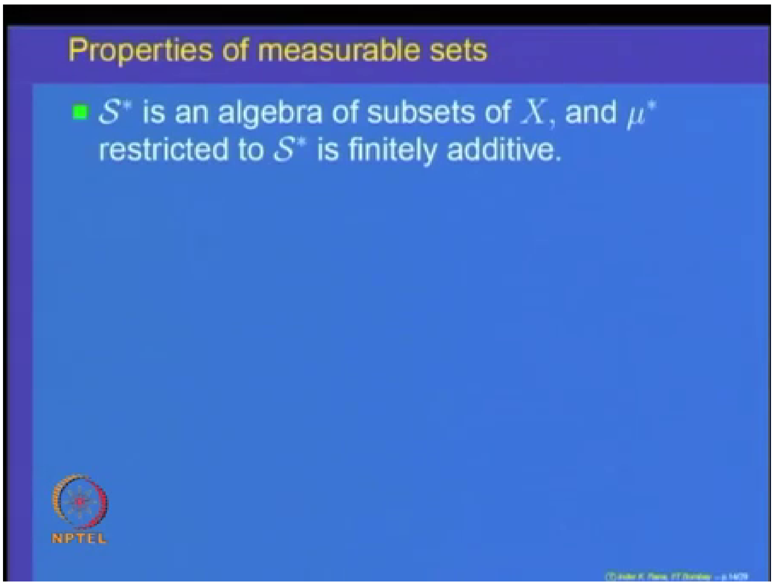

Next, we want to check the property. So, the collection of all measurable sets 1; it includes the class of all sub sets in the original algebra A and we want to check now that it is an algebra of sub sets of X; that means, and mu star restricted to S star is finitely additive. So, two things we want to check; one S star is an algebra and mu star restricted to S star is finitely additive. So, let us see what we have to check for this.

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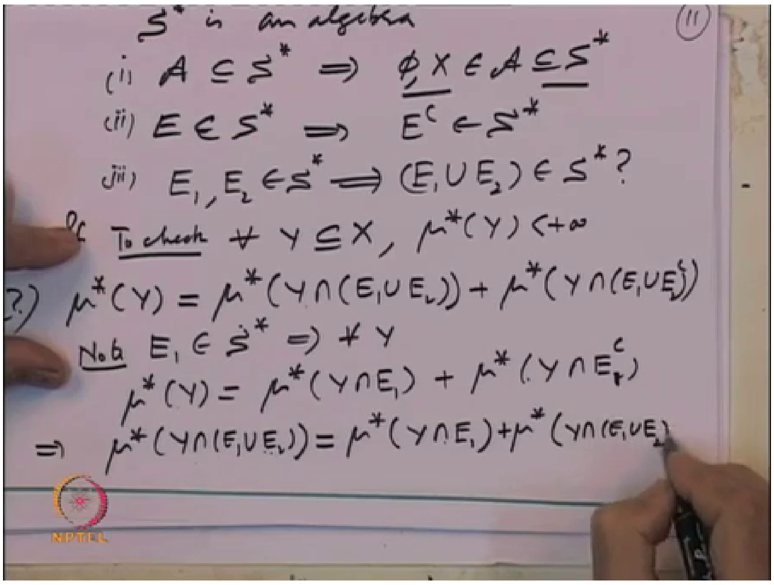

So, first of all we want to check that S star is an algebra. We have already shown A is inside S star, so that implies implying the empty set and the whole space that belong to A and hence A is in S star. So, empty set and the whole space belong to it; we just now observed that E belonging to S star; implies E compliment belongs to S star. So, if E is measurable; E compliment is measurable that also we have checked.

So, let us check the third property namely; if E 1 and E 2 belong to S star, we want to check this implies E 1 union E 2; also belongs to S star. That means union of measurable sets is again measurable, so this is what we want to check. So, let us look at a proof of this to; so, to check that E 1, E 2 is measurable; we have to check for every Y contained in X mu star of Y finite.

We have to check that mu star of Y can be written as mu star of Y intersection; the set that is E 1 union E 2; plus mu star of Y intersection E 1 union E 2 compliment. So, this is the property that we have to check. So, what we will do is we will compute each one of the term and show it is equal to mu star of Y. So, for that we start; so, note E 1 is measurable so, that implies that mu star of Y; we can write as mu star implies, for every Y; mu star of Y is mu star of Y intersection E 1 plus; mu star of Y intersection E 2.

And now this is important that this happens for every Y. So, I can change Y according to my requirements. So, what I want to do is; I will change this Y to Y intersection E 1; see I want to compute Y intersection E 1 union E 2. So, let us change it this Y to that. So, that implies that mu star of Y intersection E 1; union E 2 is equal to here I should replace Y by Y intersection.

So, mu star of Y intersection E 1 intersection E 1 union E 2, but E is a sub set of E 1 union E 2. So, that is just Y intersection E 1 is that clear? Because if I replace Y by Y intersection E 1 union E 2, then this intersection with Y 1 now with E 1 is just Y intersection E 1 plus what is the second thing let us write. So, mu star of sorry this one is E 1 compliment, I am sorry we made a mistake saying it is measurable mu star of Y is mu star of Y intersection E 1 plus mu star of Y intersection E 1 compliment.

And now when we replace Y by Y intersection E 1 union E 2; so, this is same as this plus the second term is Y intersection E 1 union E 2 intersection E 1 compliment. So, let us simplify that, so what we have gotten is the following.

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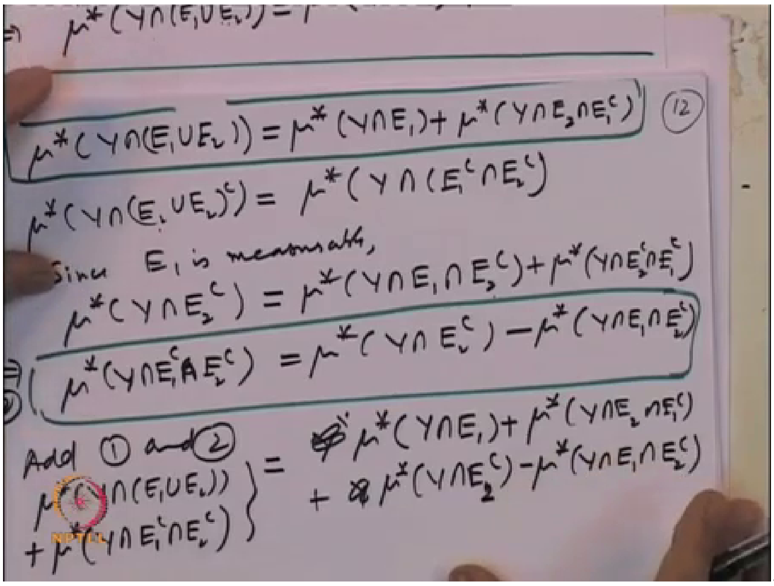

That mu star of Y intersection E 1; union E 2 that was the left hand side is equal to mu star of Y intersection E 1 plus; what is this? Now E 1 union E 2; intersection E 1 compliment. So, when I take E 1; E 1 compliment that is going to be empty set; so, this set is nothing, but mu star of Y intersection, E 2 intersection E 1 compliment. So, we have computed mu of Y intersection E 1 union E 2 to be equal to this.

And now I also want to compute; what is mu star of Y intersection compliment of this. So, what is the compliment of this? E 1 union E 2 compliment. So, what is that going to be? That is going to be mu star of Y intersection by using our popular Demorgan's laws for set theory; this is E 1 compliment intersection E 2 compliment. So, I want to compute mu star of E 1 compliment intersection E 2 compliment. How can we compute that?

Recall saying that E 1 was measurable we had that. So, if I replace Y by; Y intersection E 2 compliment, then I will get the required set here. So, use this equation; so, since E 1 is measurable we have mu star of Y. So, we will just keep it here to follow. So, mu star of Y intersection I want; instead of this we want Y intersection E 2 compliment. So, let us look at Y intersection E 2 compliment is equal to mu star of Y intersection; E 1 intersection E 2 compliment plus mu star of what will be this set Y intersection E 2 compliment intersection E 1 compliment; so, that is what we will have.

So, this is what I wanted; now let us observe in this equation; all the numbers are real numbers. Because of the assumption that mu star of Y is finite; so this is a sub set. So, this is finite, this is finite, this is finite all are finite numbers. So, I can interchange them; I can take one term on the other side if required, so let us do that.

So, from here we compute; so, implies mu star of Y intersection E 1 compliment; E 1 compliment intersection E 2 compliment. This set is equal to mu star of Y intersection E 2 compliment minus; take it on the other side it is mu star of Y intersection E 1 intersection E 2 compliment. So, we have gotten the required quantities; so, we wanted mu star of; we wanted what is mu star of Y intersection E 1 union E 2. So, that is lying here and we wanted that is lying here the second term.

So, let us add these two terms. So, add call it as this equation as 1, call this equation as 2; add 1 and 2 and that will give you; that mu star of Y intersection; E 1 union E 2 plus mu star of Y intersection; E 1 compliment intersection E 2 compliment. So, this is equal to; there we have got mu star of Y intersection E 1 plus mu star of Y intersection E 2 intersection E 1 compliment plus mu star of Y intersection E 2 compliment; minus mu star of Y intersection E 1, intersection E 2 compliment.

So, this is what we have gotten and we want to check that this should come out to be equal to mu star of Y. Now, let us again try to use; so, this is mu of intersection E 2 compliment here and that is E 1 intersection E 2 compliment. Now, let us observe till now we have not use any where the fact that E 2 is measurable. So, let us try to use that fact that E 2 is also measurable and so, that we can simplify this quantity.

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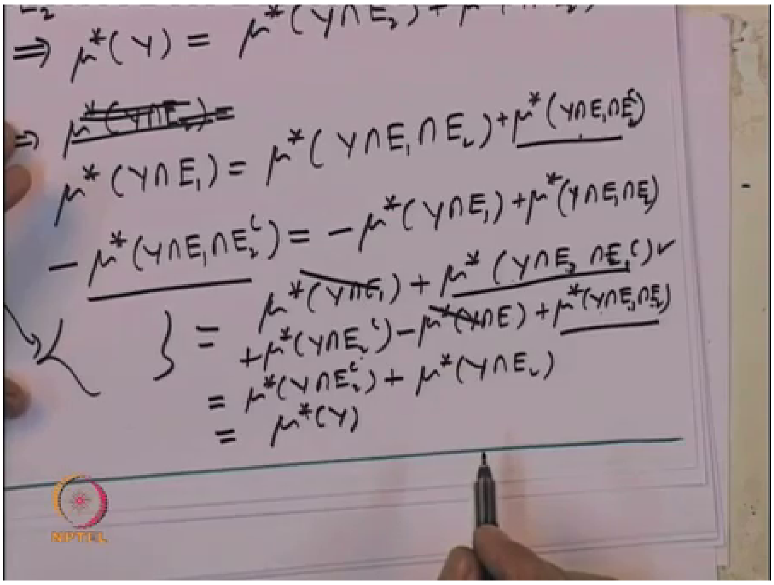

So, now observe E 2 measurable implies the following fact; we want to simplify this. So, let us look at what is going to be E 2; Y intersection E 2 and Y intersection E 2 compliment. So, let us try to E 2 measurable means so, for every Y; we have got mu star of Y is equal to mu star of Y intersection E 2 plus mu star of Y intersection E 2 compliment.

Because, as a measurability; now I want to use this to compute one of the terms here. So, let us replace Y by Y intersection E 2. So, that implies I can replace this by mu star of Y intersection E 2 will be equal to; that will not give us anything. Let us replace this by Y intersection E 1, so implies mu star of Y intersection E 1 is equal to mu star of Y intersection E 1, intersection E 2; plus mu star of Y intersection E 1, intersection E 2 compliment.

So, what is mu star of Y intersection; E 1 intersection, E 2 compliment that term is here. So, that we want with negative sign; so, if I take it on the other side so; that means, minus mu star of Y intersection E 1, intersection E 2 compliment is equal to; I bring it on the other side. So, that is minus mu star of Y intersection E 1 plus mu star of this term which is Y intersection E 1; intersection E 2. And now this; so, this is what we have reached here.

So, this is the value that I was looking for; so, let us put in this value. So, this required quantity; I will just take it here is equal to, so this required quantity is equal to mu star of

Y and here is minus mu star of Y; so, those two terms will cancel out. So, let me just write that is mu star of Y intersection E 1 plus mu star of Y intersection E 2; intersection E 1 compliment; that we already had.

So, plus mu star of Y intersection E 2 compliment and minus; so, that is equal to minus mu star of; from here Y intersection E plus mu star of Y intersection E 1; intersection E 2 and now this two terms cancel out. So, what we are left with is; so, this is equal to mu star of Y intersection E 2; intersection E 1 compliment. And Y intersection E 2 intersection E 1, so look at these two terms. So, these two terms this Y intersection E 2; intersection E 1 compliment plus Y mu star of Y intersection E 1 and E 2; so; that means so these two terms are nothing, but mu star of Y intersection E 2.

So, and one term is here; so, this is mu star of Y intersection E 2 plus what I am saying is this plus this term is nothing, but mu star of Y. So, this is compliment mu star of Y intersection E 2; is that clear? This term as it is; now look at that fact that E 1 is measurable. So, mu star of Y intersection E 2 is mu star of Y intersection E 2; intersection E 1 compliment, plus mu star of Y intersection E 1 intersection E 2.

And now once again using the fact that E 2 is measurable that is equal to mu star of Y. So, we have proved the required condition that mu star of Y is equal to mu star of Y. So, we have proved that this is mu star of Y is equal to mu star of Y intersection E 1 union E 2 plus mu star of Y intersection E 1 compliment; intersection E 2 compliment. So; that means, we have proved the fact that S is an algebra of sub sets of the set X.

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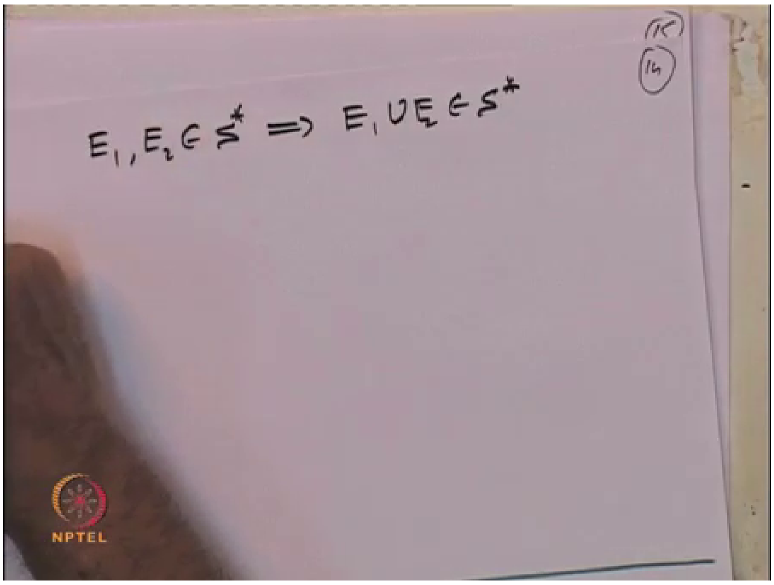

So, what we have shown is E 1 E 2 belonging to S star; imply E 1 union E 2 also belong to S star. Here, let me just comment that this proof looks a bit technical, but it is not so difficult. Even measurable gives you one condition; that mu star of Y is equal to something, E 2 measurable gives you mu star of Y is equal to something. Now, this sets Y are arbitrary and given E 1 and E 2 are measurable means mu star of Y is equal to mu star of Y intersection E 1 plus mu star of Y intersection E 2; E 1 compliment.

So, you can change this Y to Y intersection; E 1 Y intersection E 2 and so on. So, write down the equations which are given; write down the equation, the equality that we proved and just manipulate this is only a simple algebra which is required. So, today what we have done is; we have looked at; we have defined the concept of what is called a measurable set for a outer measure mu. And we have shown that the original elements of the algebra are already measurable sets and the class of all measurable sets form an algebra. So, we will continue the analysis of this class S star in our next lecture.

Thank you.