Measure & Integration
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Lecture – 01 B
Introduction, Extended Real Numbers

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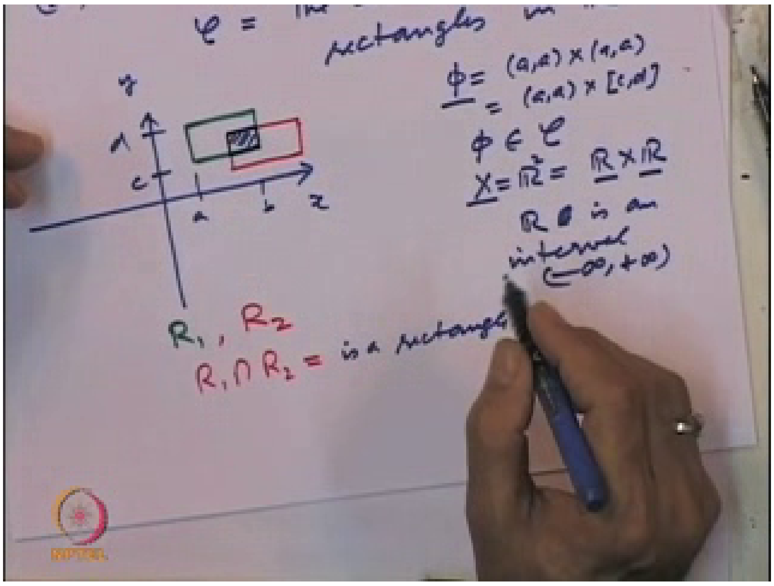

And let us look at some more examples. So, let us look at example number 4. Let us take the set X to be equal to R 2; and c is the collection of all rectangles in R 2. So, let us just look at the picture and try to understand. So, here is R 2. So, let us take a can I say empty set is a rectangle. Of course, empty set can be written as a comma a cross a comma a if you like, it does not matter or you can also write as a comma a cross c comma d in both cases it is the empty set or any other such representation. So, empty set is an element of the collection of all rectangles in the plane.

What about the whole space X that is R 2 of course that is R cross R, and R is a interval. So, R is an interval which is which we write normally as minus infinity to plus infinity. So, empty set is an element of it the whole space is an element of it. Let us take two rectangles and see whether the intersection of these two rectangles is also. So, let us take one rectangle here, and another rectangle the possibilities are they do not intersect in case they do not intersect then there is nothing to prove because then intersection is an empty set which is already a rectangle. So, let us take a rectangle, which intersects with earlier rectangle. So, we are taking two rectangles say R 1 and R 2 and R 1 intersection R 2 we want to check whether that is a rectangle or not a rectangle. But let us from the picture it is quite clear that R 1 intersection R 2 is this rectangle. So, is a rectangle one can write down a formal proof by writing this to be equal to a, b, c and d and so on, but that is not necessary once we understand from the picture that intersection of two rectangles is again a rectangle.

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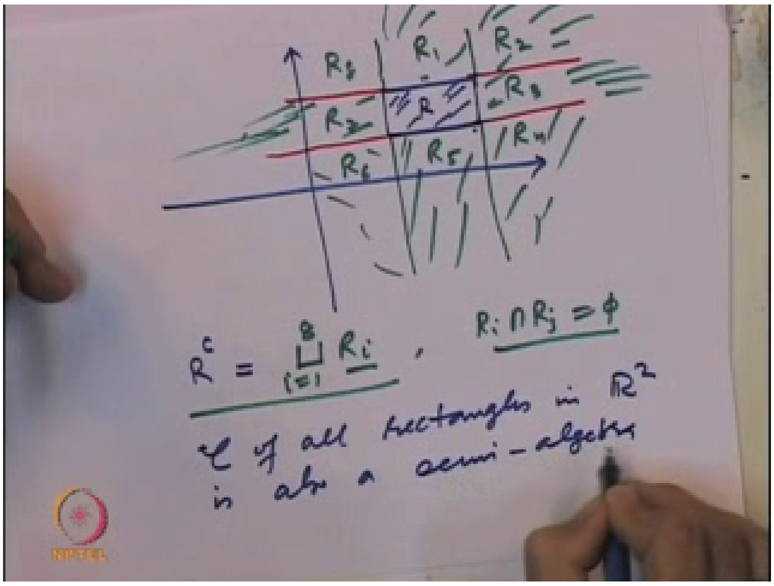

Of course, let us verify the third property namely can I represent the complement of a rectangle as a finite disjoint union of rectangles. So, let us take a rectangle. So, let us take a rectangle in the plane this one. So, this is a rectangle R, and I want to write R compliment I want to see what does it look like, can I represent this as a finite disjoint union of rectangles again. Well, let us obviously, in the picture I can try to do as the following I can draw lines passing through the sides, and I can draw another line passing through this, then it is quite clear that is complement of R. So, this was the set rectangle R and its complement is nothing but a rectangle R 1, a rectangle R 2, a rectangle R 3, rectangle R 4, rectangle R 5, R 6, R 7 and R 8. And of course, these rectangles R 1 R 2 R 3, R 4, R 5, R 6, R 7 and R 8. So, this is R 5, this is and this part is there are many ways of. So, I am looking at this whole infinite I can look at this whole infinite, this side and this corner as a rectangle, and this part as a rectangle.

So, I can write it as union of R i, i equal to 1 to 8 where R i intersection R j is empty. So, it is matter of writing down the details that depending on in R whether which part of the boundaries included or excluded accordingly I can make this rectangles R i's to be disjoint. So, this is true that the complement of a rectangle in the plane is also a rectangle, so that means, what that says that the collection or c of all rectangles in R 2 is also a semi-algebra of subsets of X. So, we have given lot of examples of a objects which are semi-algebras of a subsets of X.

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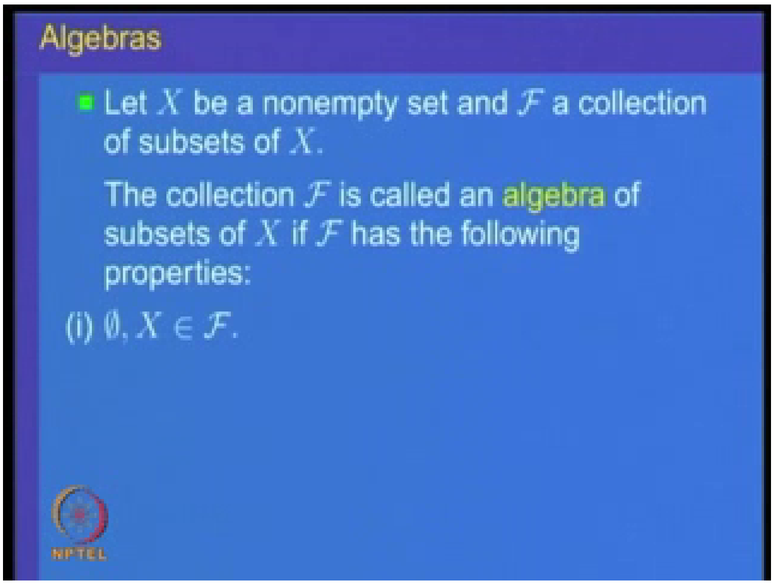

Now let us go to a next stage of understanding extending this concept of semi-algebra. So, what is called an algebra of subsets of a set X. So, let X be a nonempty, set and a collection f of subsets of X with the following properties. One like semi-algebra the empty set belongs to it, the whole space belongs to it. And of course, there is another property, so better let us write this.

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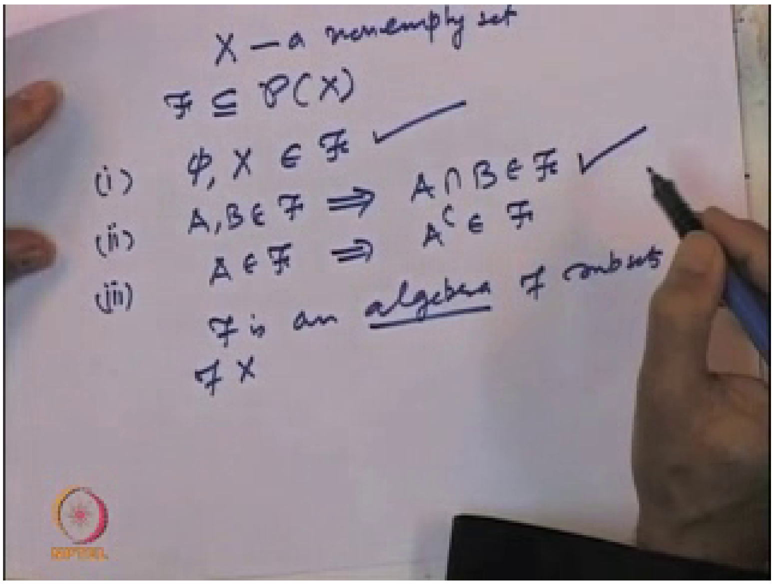

So, X is a nonempty set, f is a collection of subsets of X with the following properties. One, if empty set and the whole space belong to f like that in the semi-algebra. And secondly, we are going to look at the intersection property if A and B belong to f their elements of f then that implies their intersection also belongs to f. And of course, third property namely in the case of semi-algebra if I take an element f in f then its complement need not be in f, but we were able to represent it as finite disjoint union of elements of that class. But in an algebra, in the new concept, we are demanding this implies A compliment also belongs to f; in this case is we say f is an algebra of subsets of X. So, this is called an algebra of subsets of X.

So, how does the algebra differ from a semi-algebra. This property that is all true for algebra as well as semi-algebra, this property true for algebra as well as semi-algebra. This property may not be true for a semi-algebra namely in that case we will call we said A complement is a finite disjoint union of elements of f. And here we are saying A complement itself is an element of f. So, let us look at some examples of this again to understand.

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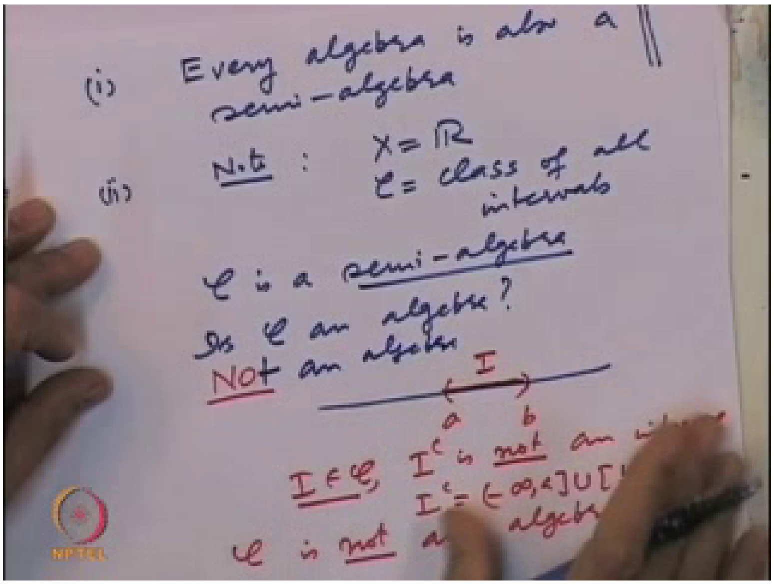

So, first observation of course, every algebra is also a semi-algebra, because the third property that we looked at namely in a semi-algebra, one would like to have a complement to be a disjoint union of elements of f in an algebra it is itself in f. So, this is much stronger. So, every l algebra is also a semi-algebra. Let us note when x was equal to R and the c is equal to class of all intervals, we showed that c is a semi-algebra that question is c an algebra? Obviously the answer is no. For example, I can take an interval any non-degenerate rate or any interval say a to b lets take this interval a to b that is my I. So, I belongs to c, but I complement is not an interval, because I complement is nothing, but minus infinity to a union b to plus infinity. So, when I belongs to c, I is an interval, its complement need not be an interval in general, so that implies that this collection c is not an algebra.

So, let me emphasize again property here one says every algebra is also a semi-algebra, and this says every semi-algebra need not be an algebra means there are examples of collection of subsets of X. For example, in the real line, the collection c of all intervals is a semi-algebra, but it is not an algebra. So, a collection of subsets is an algebra is a much stronger property easiest concept than that of a semi-algebra.

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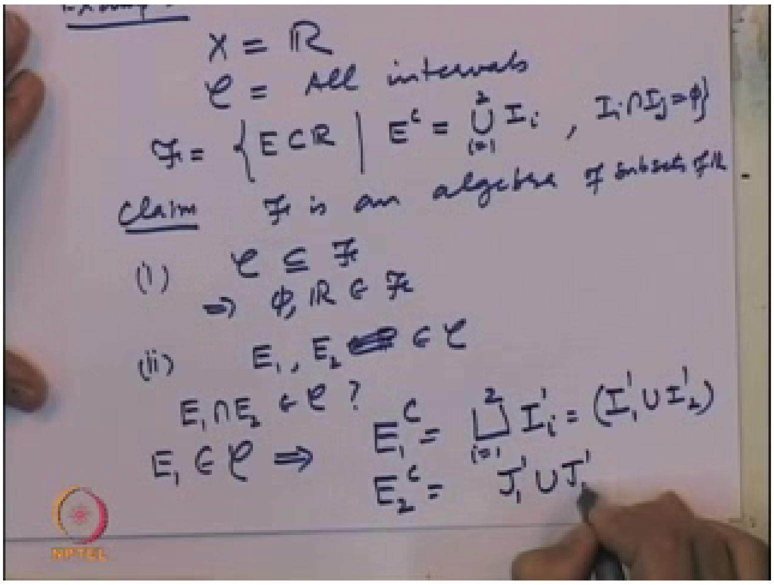

Let us look at some more examples. Let us look at the example same example of a axis real line, c is all intervals. We know that this collection is not an algebra, it is a semi-algebra. Let us write f to be the collection of all subsets of the real line such that look at this collection of all intervals was not an algebra because the complement of an interval need not be a interval, but it look like is union of two disjoint intervals. So, let us write where E such that E compliment is equal to a union of intervals I i i, i equal to 1 to 2, where I i intersection I j is equal to empty.

So, what we are saying look at all those subsets of real line which can be represented as fine as disjoint union of two intervals. So, claim f is an algebra, we claim that this is an algebra of subsets of R. So, let us first observe make some observations namely c is a subset of f, because if I take an interval then it has this property namely it is a disjoint its complement is a disjoint union of two intervals. So, c is part of subsets of X. So, this as a consequence implies empty set and the whole space belong to f.

Let us look at the second property. Let us look at two elements. So, let us call E 1 and E 2 belongs to R sorry E 1 and E 2 are elements of c. We want to check whether E 1 intersection E 2 belong to c or not. So, what is E 1, because E 1 belongs to c implies E 1 compliment can be written as a disjoint union. So, this square bracket normally indicates that I am writing something as a disjoint union. So, I 1 1 I i equal to 1 to 2 or let us just simply write it as this is union of 2 interval, so I 1 1 union I 1 2, where both are these are disjoint. Similarly, E 2 compliment can be written as disjoint union let us call a J 1,1 union J 1,2, where J 1 and J 2 are disjoint.

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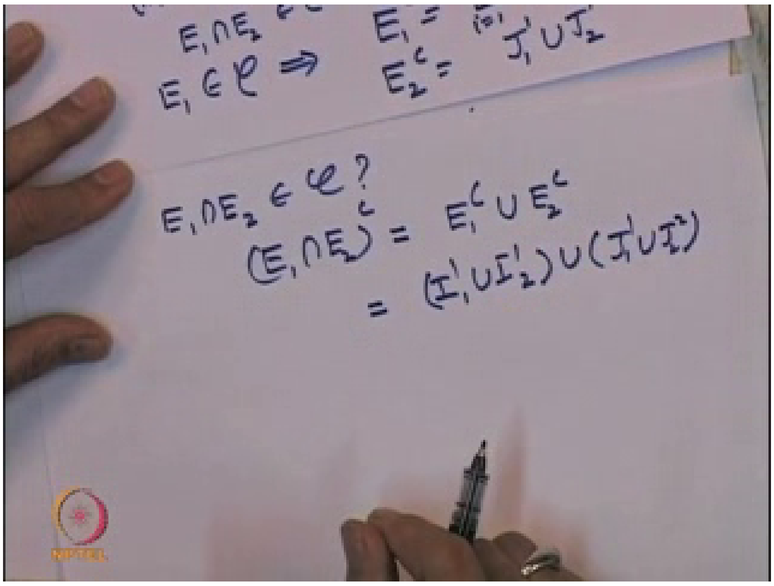

But now we want to look at E 1 intersection E 2, we want to look at E 1 intersection E 2. And we want to check whether this belongs to c or not, that means, what I should look at E 1 intersection E 2 compliment and try to represent that as a union of two intervals. So, let us look at this is equal to E 1 complement here is intersection. So, by De Morgan laws that becomes E 2 complement. Now, this is same as E 1 compliment is nothing but I 1 1 union I 1 2 union E 2 compliment that is J 1 1 union J 2 2. Now, from here I 1 and these two are disjoint, these two are disjoint, but all of these four may not be disjoint. So, this set of ideas seems not leasing us do a claim that f is an algebra of subsets of X. So, let us modify our arguments. So, instead of checking this second one, whether the intersection belongs to it right.

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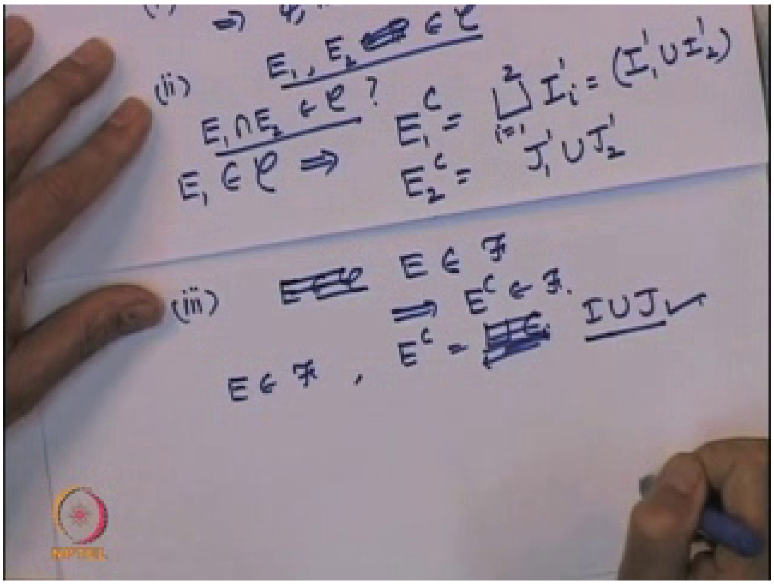

Let us look at the property, the third property that we want where that is true or not. So, what are the third property that property said if a set E belongs to c right or we want to check an algebra. So, let us look at E belongs to f does that imply E complement belongs to f right. So, let us take a set E belonging to f. Now, what is E complement, E complement looks like a finite disjoint union of elements, because E belongs to it. So, it is a disjoint union of two elements. So, E complement is equal to E complement is I union J. So, it seems to say as that if I can show that the collection of finite disjoint unions, this f is closed under union then I may be true. So, we modify all our arguments again and see how do we proceed this is how one does not get all the time a polished proof in mathematics one has to modify the arguments. So, I modify my arguments to prove that f is an algebra as follows.

So, the first step, so let us keep in mind what are we trying to do. So, here is a collection f of subsets of x now subsets of the real line which are union of two of them, but that seems to complicate the issue. So, instead of this let us modify this definition of f itself.

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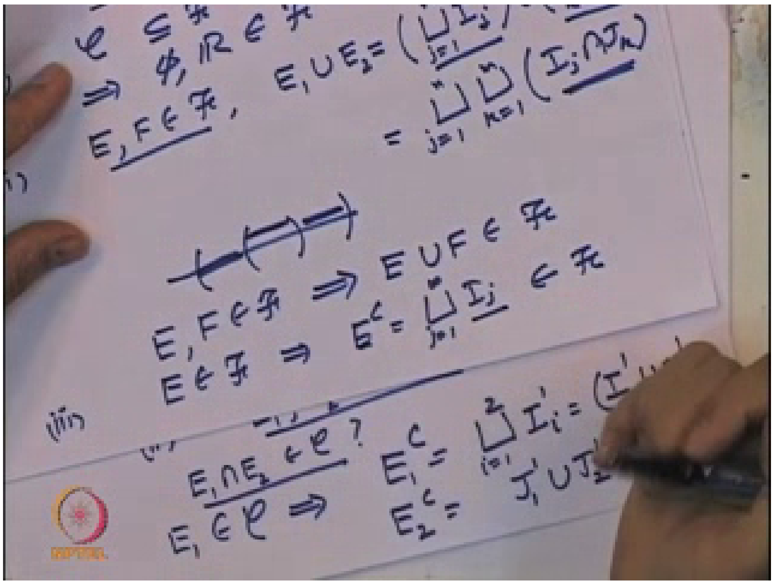

And let us look at the modified version of this example X is real line. Let us look at f the collection of all those subsets E of R such that instead of seeing E complement is a union of two disjoint intervals. Let me just write E compliment is equal to a finite disjoint union of intervals I j j equal to 1 to n where ijs are intervals and they are pair wise disjoint, so that is already indicated by writing this square bracket. So, now, let us observe keeping in mind our previous arguments that see the collection of all intervals is a part of f, so that implies that empty set and the whole space are members of f. Second now if E and F belong to this collection then what is E, E is a disjoint union of intervals. So, let us write E 1 union E 2 I can write this as E 1 is a disjoint union. So, let us write E 1 as union of ij j equal to 1 to n union of disjoint union of some I k, k equal to 1 to m.

Now these collection of sets intervals are disjoint this collection of intervals are disjoint, but all of them may not be disjoint, but that does not matter much. I can write this as union over j equal to 1 to n union over k equal to 1 to m of I j intersection J k. So, what I am doing is I am intersecting, so the basic property is if two intervals are not disjoint then I can write them as a union of disjoint pieces. So, this collection of intervals the union of intervals which may be overlapping, but I can intersect one with another and write this as a disjoint union. Now, this pairs of intervals will be disjoint, so that implies this. So, this will imply that if E and F belong to f then that implies E union F also belongs to f. So, this collection of finite disjoint union of intervals is closed under unions.

And let us write finally that if E belongs to f then that implies by definition E complement is a disjoint union of intervals. And each I j is a interval, so it belongs to f and just now we prove it is closed under unions, so this implies this, this also belongs to f. So, the collection of sets of the real line which are finite disjoint union of intervals have the property, c is a subset of it if E and F belong to it then it is closed under unions and also closed under compliments.

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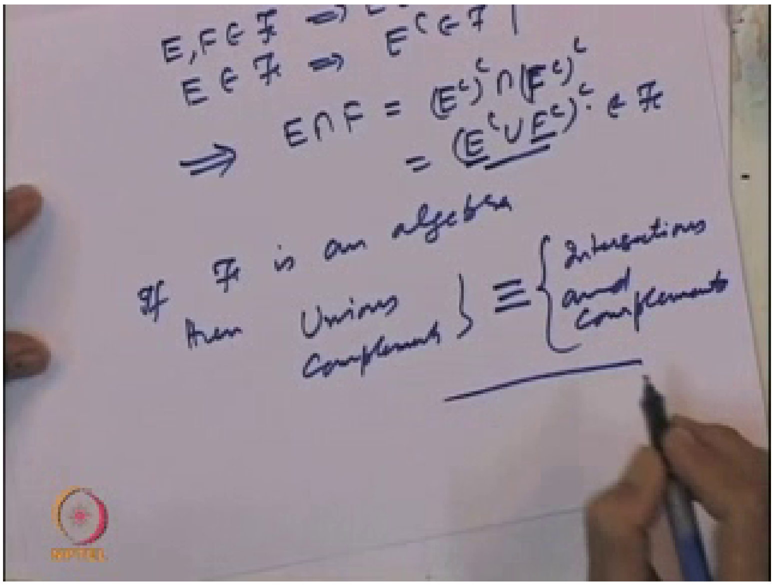

Now, it is a simple matter for us to check that these two properties whenever a collection of sets is closed under unions E, F belonging to f. We showed implies E union f belongs to it and also E belonging to f implies E compliment belongs to f. But these two properties imply that E intersection F also belongs, because I can write this as because I can write E as E compliment compliment union because if you like this intersection of E F compliment compliment and that I can write as E complement union F compliment compliment. Simply this is just saying that because this is a De Morgan's laws this will give you E complement complement intersection now E belongs to f, so this belongs to f this belongs to f this union belongs to f and complement belongs to f. So, this belongs to f. So, basically in an algebra if f is an algebra then saying that it is closed under unions and complements is equivalent to saying it is closed under intersections and complements.

So, let us just summarize what we have done today. We started by looking at our course measure and integration. And a set the underlying set of real numbers need to be extended to a larger class, namely set of extended real numbers. And there we defined the notion of order addition, and multiplication, and analyze how would a sequence is series supremum of sets behavior. Then we started looking at collection of subsets of a set X with some properties. We started at the first thing was we looked at what is called a semi-algebra of subsets of X, namely it is a collection of objects subsets of the set X with the property empty set and the whole space belong to it. It is closed under intersection and the complement of any set in this collection is representable as finite disjoint union of elements of that collection again. And a typical example was that of all intervals in the real line.

Then we looked at a slightly stronger concept namely the algebra an algebra of subsets of a set X, which we defined as the collection with the properties empty set and the whole space belong to it. It is closed under intersections and also closed under complements. And then we made a remark every algebra is a semi-algebra. The collection of intervals in the real line form as a semi-algebra, but they do not form an algebra. And we showed how to construct an algebra out of this intervals namely we looked at the collection of all finite disjoint unions of this intervals and that collection we proved it is an algebra of subsets of it. So, we will continue analyzing such collections of objects in our next lecture.

Thank you.