Measure and Integration
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Lecture - 10 A
Outer Measure and It is Properties

Welcome to lecture number 10 on measure and integration, and let us just recall that in the previous lecture we are started looking at the notion of outer measure. So, let us recall how the outer measure was defined, and what are the properties the outer measure has a given a measure mu on a algebra A of sub set sub set X. the outer measure induced by this measure is a set function defined on the class of all sub sets of the set X.

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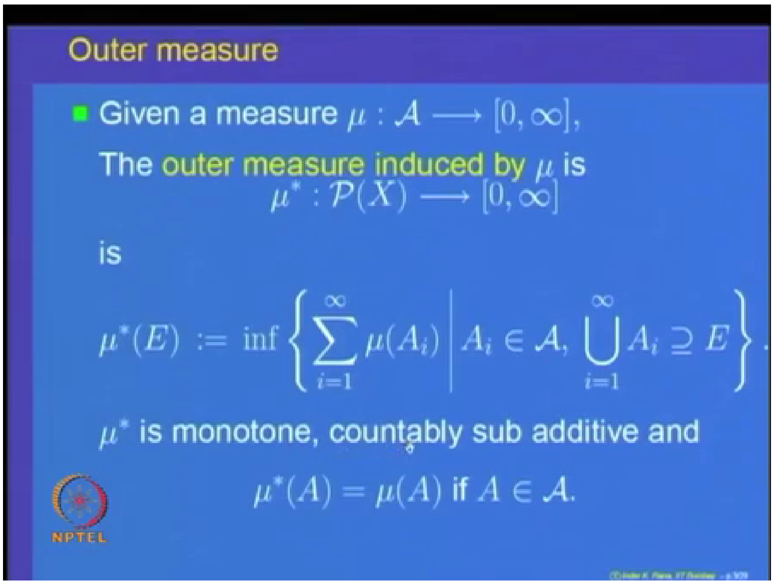

So, it is a function defined to the power set of X. Of course, taking non negative values, and it is defined as for a set E in sub set of X, look at a countable disjoint countable covering of the set E by elements of the algebra A, now, and look at the size the measure of the set A i. So, that is mu of A i add up the measures of all the sets A i; say that this union covers. So, that gives you a number which we can think of as an approximate measure of the set E. So, look at the infimum or all such possible coverings of E. So, mu star of E is the infimum of all summation mu of A i; such that union of A i is cover E, and we proved properties of this set function, the outer measure namely mu star is monotone.

It is countably sub additive, and on the sets in the algebra mu star is same as A. So, mu star extends the measure mu, but it is only monotone and countably sub additive.

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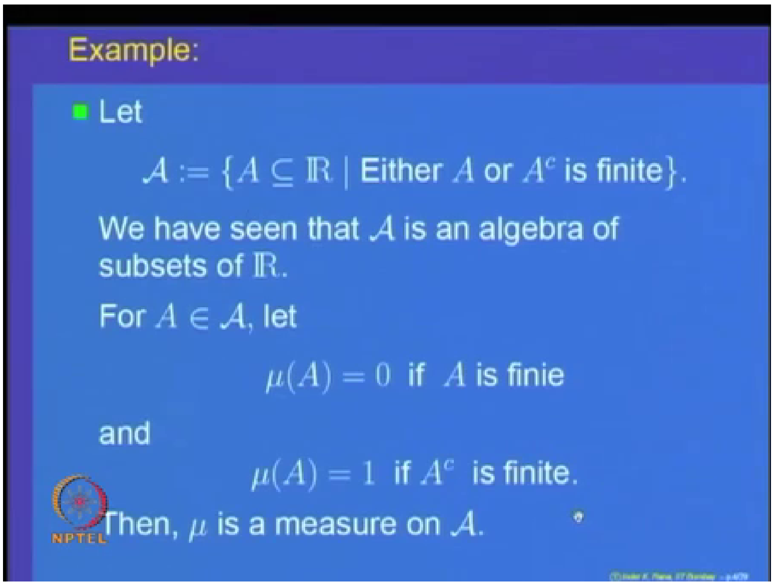

So, let us look at an example of this outer measure. So, in this example we will start with the collection A of all sub sets of the real line, which are either finite or their complements are finite. So, recall in the beginning of the lectures we have shown, that is collection A is an algebra of sub sets of A. So, this collection A forms an algebra of sub sets of the set R. Let us define a set function mu on this mu of the set A is equal to 0. If the set is finite, and if the set in the algebra is not finite, then we know it is A compliment is A finite. So, in that case we define mu of A to B equal to 1. If A compliment is finite, and we also checked this for this example, that mu is a measure on this algebra A. So, i would strongly says that you try to prove it yourself. Once again that this mu is a measure; that is mu on the algebra A is countably additive.

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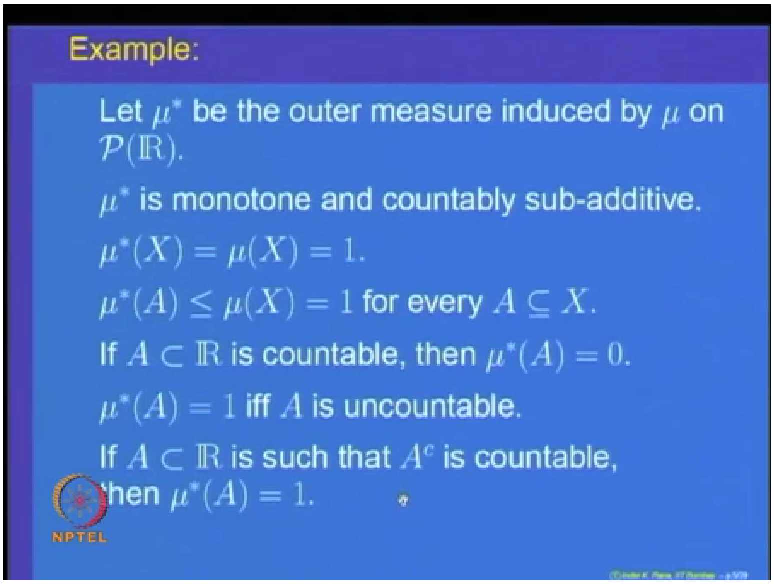

So, let us denote by mu star, the outer measure induced by mu on all sub sets of the real line. So, mu star is the outer measure given by this particular measure, and we want to correct. find some properties of this mu R this outer measure mu, and if you recall just now we said outer measure always is monotone, and it is countably sub additive. So, this properties are true for any outer measure. So, in particular this outer measure also, we want to do something more and obvious. Let us note that mu star of X, the whole space is same as mu of X, and that is equal to 1, because mu star extends. So, and X belongs to the algebra actually, X compliment is empty set, which is finite. So, by definition mu star of X should be equal to mu of X, which is equal to 1.

And if A is any sub set of X, and mu star being monotone. So, we know that mu star of X is less than or equal to mu star of X, and that is less than or, that is equal to one. So, mu star of every set is going to be between 0 and 1. So, this is a property we have deduced from the general facts, that mu is monotone, and mu star of the whole space is equal to 1. We want to show that if A is a countable set in real line, then mu star of A is equal to 0. So, let us see how we show that.

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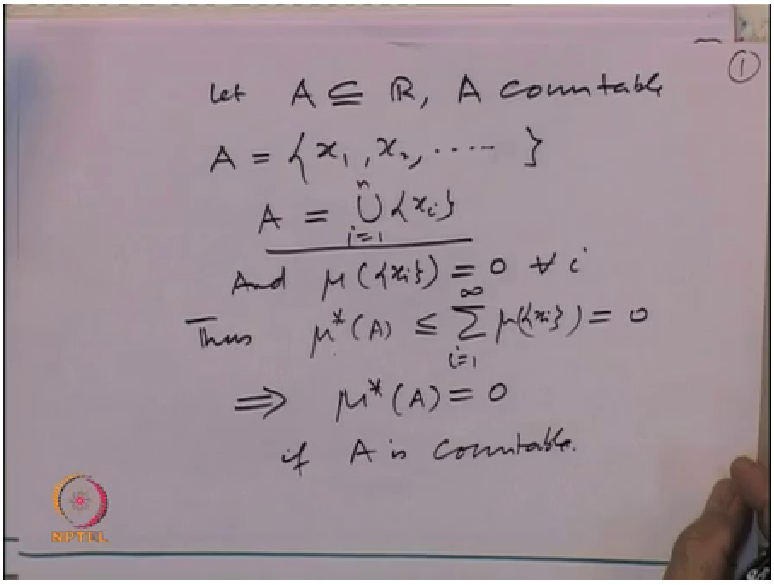

So, let us take the set A. So, let A contained R A countable; that means, i can write A as a sequence. So, X 1 X 2 and so on. So, i can write A is actually equal to union of single tons X i i equal to 1 to n, and by definition mu of the single ton X i, it is a finite set. So, that is equal to 0 for every i. Thus mu star of A which is less than or equal to summation mu of X is this. This is one covering and measure of each one of them being equal to 0. So, this is equal to 0. So, mu star of A is less than or equal to 0. We know it is always bigger than or equal to 0. So, that implies that mu star of A equal to 0. If A is countable. So, we have shown that for a countable set sub set A in the real line, the outer measure which we define is going to be 0, whenever A is countable. Let us go further and look at some other properties of this outer measure, we want to show that mu star of A is equal to 1 if and only if A is uncountable. So, let us prove that.

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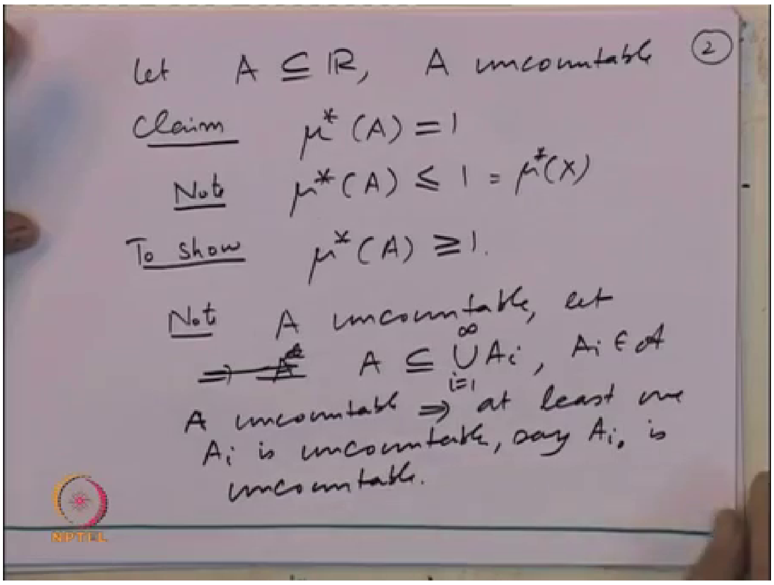

So, let A contained, then R A uncountable claim mu of mu star of A is equal to 1 right. So, note mu star of A is less than or equal to 1; that is obvious right; that is Y. The definition we said mu star is monotone which is equal to mu of X anyway mu star of X

So, mu star of A is always less than or equal to 1 to show that mu star of A is also bigger than or equal to 1. Now let us note a uncountable. So, look at A is uncountable. So, what can you say about A compliment of the set, it cannot, if A is uncountable, then we want to, if A is, sorry if A is uncountable, let us take a covering of A i i equal to 1 to infinity. Let us take a covering where A is belong to the algebra A right. Now note in this covering A uncountable implies at least one of this A is at least one A i is uncountable.

So, this is the observation, which is going to be crucial A is a sub set of unions A is, which are in the algebra, and each one of them, if A is uncountable, then each of them cannot be countable, because if each one of them is countable, then countable union of countable sets will be countable. So, A will be countable. So; that means, there is at least one A i which is uncountable; say A i naught is uncountable, but that will be in what A i naught is uncountable means. So, thus we have got that mu of mu star is A i naught is uncountable.

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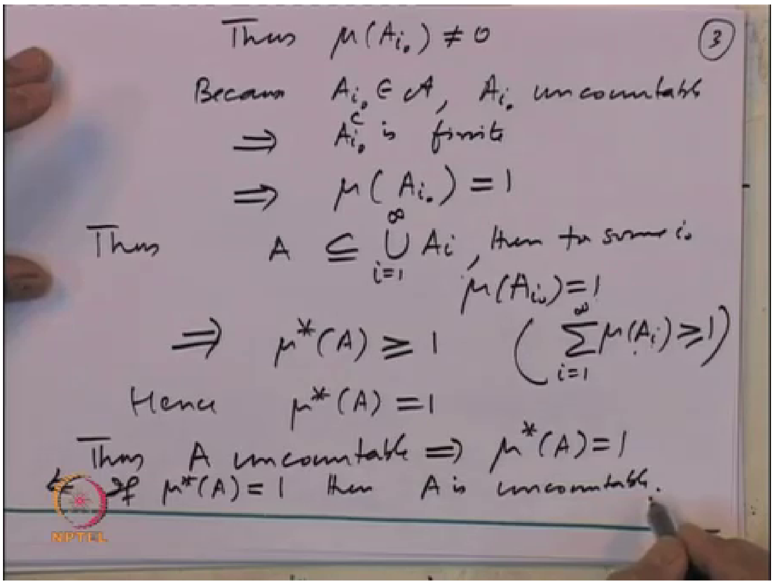

So, may mu of A i naught, what can we say about that. So, that cannot be equal to 0. Can we say that cannot be equal to 0. Let us see one of the A i is uncountable right. So; that means, it is not finite right. So, this is because A i naught belongs to the algebra A that is the crucial thing. So; that means, either A i naught is finite or its compliment is finite, but we know that A i naught is uncountable. So, that means

A i naught uncountable. So, that implies that A i naught compliment is finite, because it belongs to the algebra. So, either A i naught has to be finite or its compliment has to be finite. So, this is finite. So, that implies that mu of A i naught by definition is equal to 1. So, thus what we have shown. So, thus A is contained in, if A is contained in union A i 1 to infinity, then for some i naught mu of A i naught is equal to 1. So, that implies that automatically implies the fact that mu star of A, the mu star of. So, any covering will have at least one of the elements.

So, implies, this is bigger than or equal to 1 right, because mu star of A is the infimum of, because mu star of A the summation i equal to 1 to infinity mu of A i is going to be bigger than or equal to 1 right, because A i naught is 1, at least one of the term is 1. So, for every covering A i of a sigma mu of A i is bigger than or equal to 1. So, the infimum has to be bigger than or equal to 1. So, hence mu star of A is equal to 1. So, we have proved. So, therefore, thus a uncountable implies mu star of A is equal to 1, and the converse is obvious, because conversely right, conversely if. So, conversely if mu star of

A is equal to 1, then A is uncountable, because for countable set we have already shown mu star of A is equal to 0. So, A is uncountable if and only if. So, we have shown, we have characterized all sets for which outer measure is going to be equal to 1. So, we have shown the fact that mu star, the outer measure induced by the measure that we are looking at; namely mu. If mu of A is equal to 0, whenever A is finite and mu of A is equal to 1, when A complement is finite, and if you look at the outer measure induced by this measure, then that has the property that mu star of every countable set is equal to 0, and mu star of A set is equal to 1. If and only if the set is uncountable

So, that is the property that we have. So, mu star of A is equal to 1, if and only if A is uncountable. Now let us, we already know that mu star is going to be, is countable, is sub additive, we want to know, is mu star additive right.

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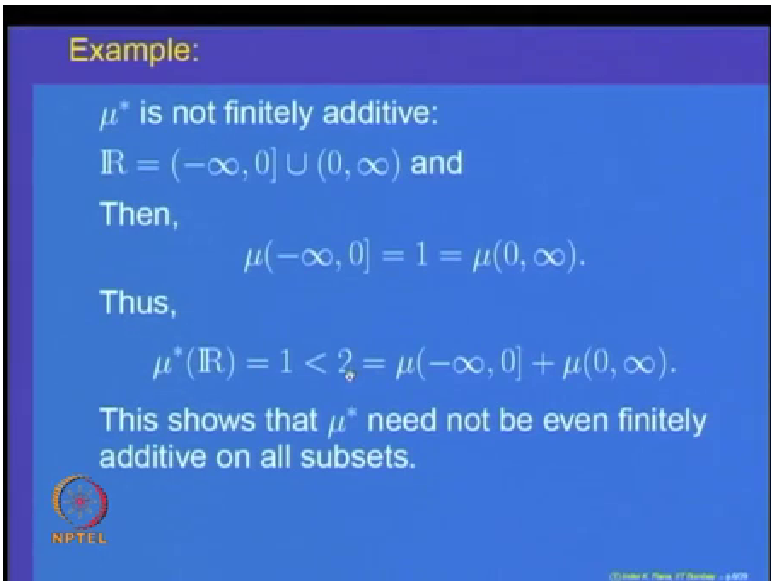

So, we will show, that is not the case that mu star is not even finitely additive. So, to show that, let us observe that the real line, I can decompose in to 2 disjoint sets minus infinity to 0 0 close union 0 to infinity. So, the real line is written as the union of 2 sub sets, and now the outer measure of the set minus infinity to 0; that is uncountable set. So, that is equal to 1, and the outer measure of 0 to infinity is also equal to 1. So, outer measure of both of this sets is equal to 1, their union is R and. So, we get mu star and mu star of R is equal to 1. So, which is strictly less than to equal to sum of the outer measures of each one of them so that says mu star of R is strictly less than mu star of minus infinity to 0 plus mu star of 0 to infinity. So, this should be mu star and this also should be mu star. So, that says that the sets that mu star, the set function mu star is not even finitely additive, but let us observes some more facts about this outer measure. This outer measure on all sub sets is not finitely additive, but it has some nice property, it is countably additive on a sub class.

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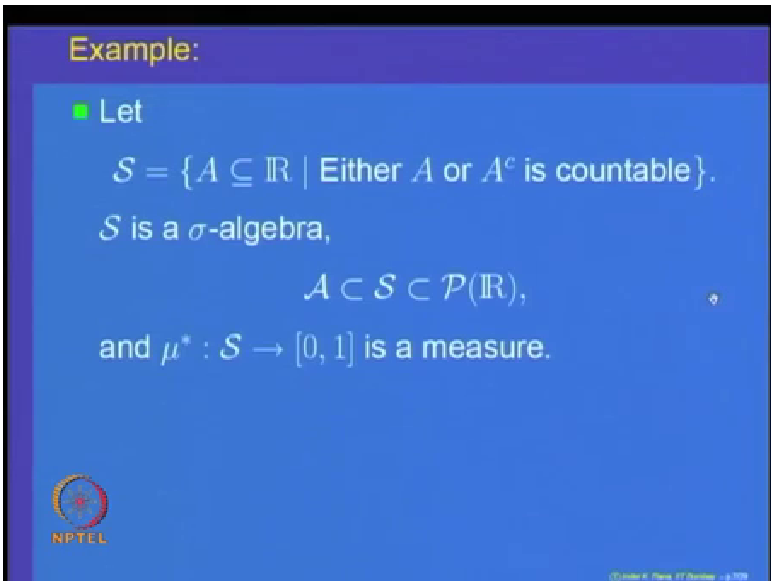

So, let us look at the sub class of the S, which is a sub sets of R, all those sub sets of R, where either A or A compliment is countable. So, keep in mind A or A compliment is countable, we had shown that this collection forms a sigma algebra, and our given algebra A which was the collection of all sets for which A or A compliment is finite; obviously, is a sub set of this class S, of sub sets which are A or A compliment countable

And that is of course, a sub set of all sub sets. So, algebra A is contained in S, this sigma algebra S and which is inside of P R, and its actually we have also shown that mu star. So, what is mu star on S, if I A, set A is countable then mu star of A is 0 right, if it is not countable, but then it is A compliment is countable right, then mu if A compliment is countable, then the set A cannot be countable, it has to be uncountable, and for uncountable sets mu star of A is equal to 1. So, mu star restricted to S is the set function, which is mu star of a set A is 0. If A is countable and mu star of a set A compliment A of A is equal to 1, if its compliment is countable. So, and that we have already shown is a measure on the class of an all sub set, all sets in the sigma algebra S.

So, given that measure in our example, given the measure on the algebra A, when we define the outer measure, which is defined on all sub sets of P R is not even finitely additive, but if we restrict it to the collection of sets S, which is A or A compliment countable, then on that class, it is a measure and it extends. So, the given measure on the algebra does not extend to all sub sets, but at least it extends to a collection A, sub collection of all sub sets and that includes the original algebra. This is a situation which we are going to see, is very common in our extension process. So, we are starting with a measure mu on an algebra, and just now we said in the process of extension, let us define outer measure on all sub sets, and here is an example which says to all on all sub sets, outer measure is not, it may not be even finitely additive

But at least this example says that we can probably find a sub class of all sub sets of that set X, which includes the given algebra, and on that probably it is countably additive. So, the problem is to look for a collection of sub sets on which it is going to be countably additive.

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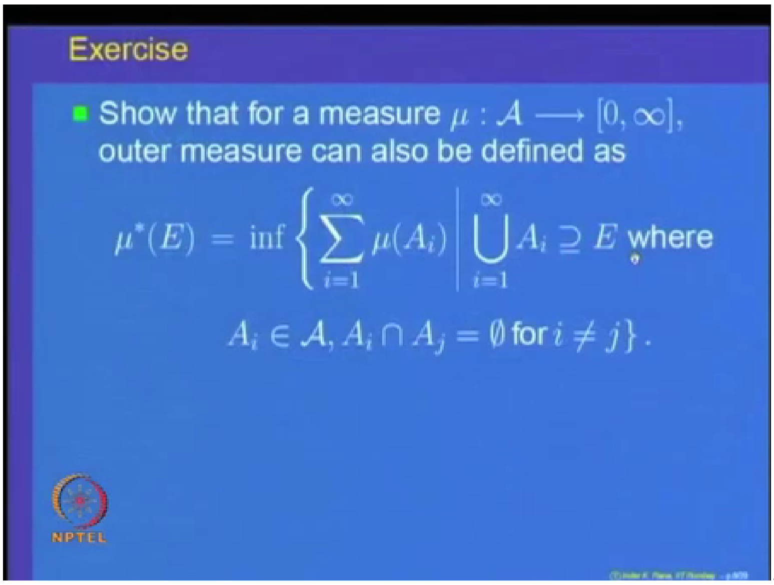

Before we go over to that process, let me give you an exercise which all of you should try; namely if mu is A measure on an algebra, then we define the outer measure as Infimum. Look at all coverings, and look at summation mu of A i right. There was no condition on the input on the sets A i and the exercise says, if you put take only those coverings of E by elements of the algebra, which are pair by disjoints, and then take the infimum over only such coverings that also will give you the outer measure.

So, the exercise is that outer measure for a set E, can be defined in terms of countable disjoint coverings of E; namely take coverings of E by elements of the algebra, and the elements of the covering R, pair wise disjoint. look at the sum mu of A i and take the infimum over such covering. So, infimum is taken over all countable disjoint coverings in the original definition. We do not put this condition, and the exercise is both of this are same, and the answer lies in the simple observation, that mean measure mu is defined on an algebra, and you look at the union of elements in the algebra. any union can also be written as A union of pair wise disjoints sets. So, union of elements of the algebra can be represented in terms of pair wise disjoint sets. That fact, we have used earlier also. So, using that you can try to prove this exercise.

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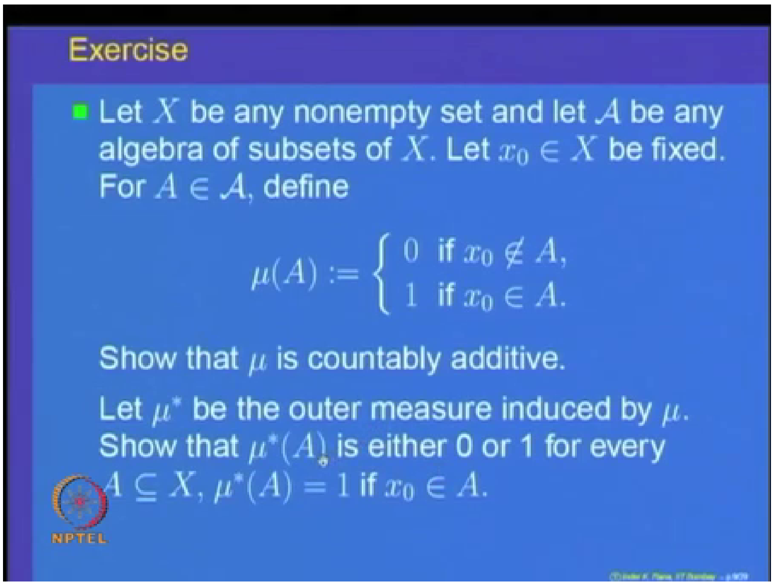

Here is another exercise which you should try to prove, to get familiarize with the concept of how to measure. Let us, let take X any non empty set A is an algebra of sub sets of a set X, and let us fix any element X 0 in X any arbitrary element. Now given any sub set A in the algebra either X naught will belong to A or X naught will not belong to A 2 2 possibility. So, if X naught does not belong to A, that particular element that you have fixed, does not belong to A, put mu of A define mu of A to be equal to 0, and define mu of A equal to 1, if X 0 belongs to a.

So, whenever X 0 is in A the measure of the set A is 1; otherwise it is equal to 0, show that this is accountable additive set function; namely it is A measure, because mu or empty set is automatically 0. So, would like you to characterize that show that the outer measure look at the outer measure induced by this measure, show that the outer measure has the property that outer measure of every set is either 0 or 1 again, and outer measure of a set is equal to 1, if the element X naught belongs to A; that is the property of mu also. So, we want you to show that mu star of A is equal to 1 if X 0 belongs to A.

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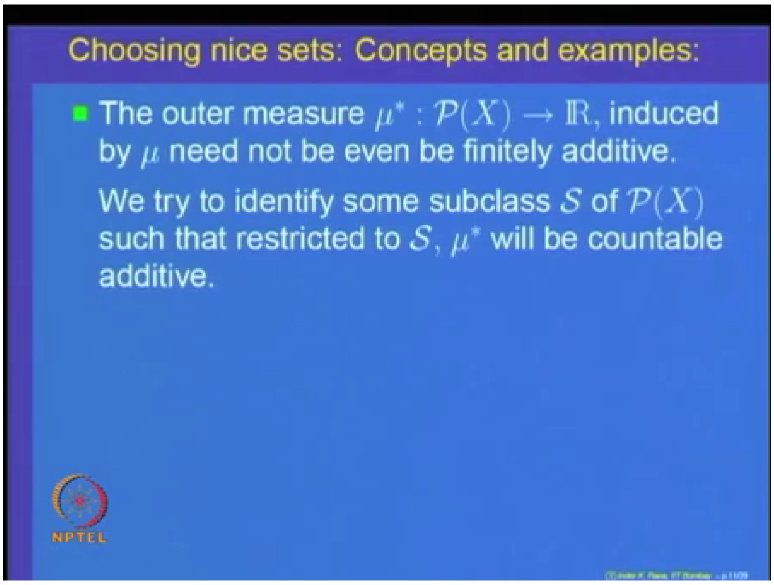

Now, can you say that the converse is true; namely we would like to show also that mu star of A is equal to 1, implies X 0 belongs to a, but that will be, you will need the condition that if X 0 belongs to A, because X 0 the single ton X 0 may not be in the algebra. So, that makes the difference. So, check look at this exercise, and try to prove the facts, ask for that will help you to understand, what is an outer measure, how does the outer measure of A set change with given conditions. Let us look at now the problem you are given A measure mu on an algebra A of sub sets of a set X, you had defined the outer measure which in general is countably sub additive. So, how to pick up sets, how to pick up those sub sets of X; such that mu star restricted to them will become countably additive as add point in the previous example that we discuss.

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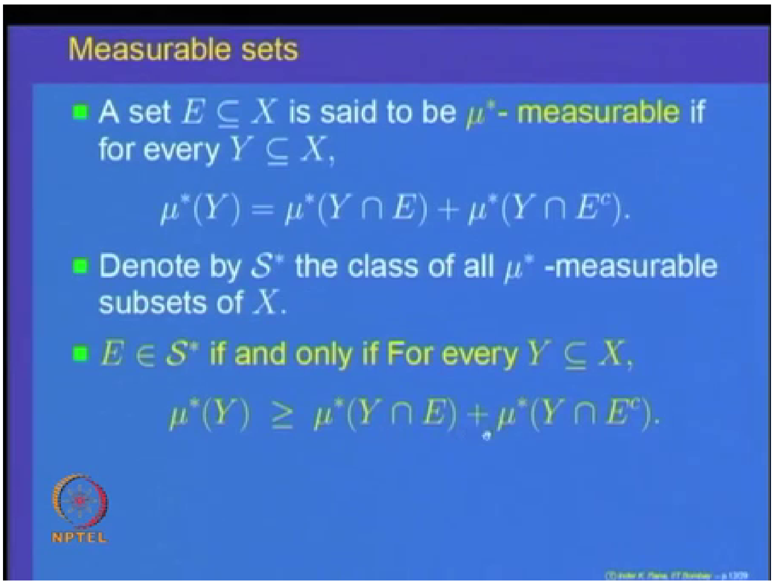

So, for that, that is what is called the concept of A measurable set, a sub set E of x. So, mu is fixed mu on the algebra is fixed, mu star is defined where that measure mu. So, mu is fixed. So, we are saying that A set E is mu star measurable mu star is outer measure induced by that measure is called measurable, if for every Y in X mu star of Y can be written as mu star of Y intersection E plus mu star of Y intersection E compliment. So, be careful this says, we are saying E is measurable. So, E measurable means take the set E right, which you want to test, whether it is measurable or not. So, divide any set Y in to 2 parts Y intersection E and Y intersection E compliment, then measure of the two piece is should add up to give you the measure the size of A mu star of Y. So, this is the picture here that you have got.

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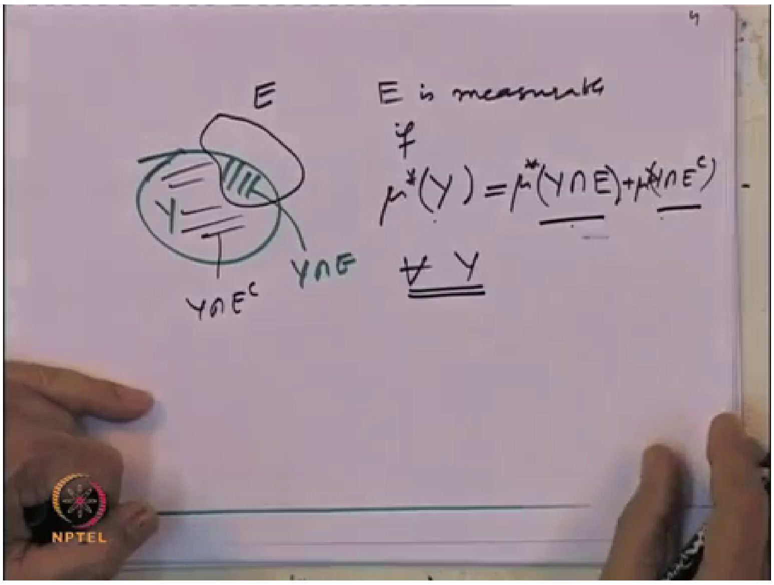

This is my set E, I want to check whether it is measurable or not. So, take this E. So, take any set Y. So, this is my Y. So, that gives me this piece which is Y intersection E, and this is the part; that is Y intersection E compliment. So, the requirement is we are saying that E is measurable if using E Y is the set, cut it in to two parts Y intersection E and Y intersection E compliment. So, they are two dissolved pieces of y, and we want you should have mu star of Y is equal to mu star of Y intersection E plus mu star of the other part that should happen every, for every y, it should happen for every y; that is important. So, that is called the we say that E is measurable, if for any sub set Y divide Y in to two parts Y intersection E and Y intersection E compliment. We want measure of outer measure of Y should be sum of these two outer measures.

So, that is when we say A set E is measurable. So, let us look at some. So, let S will denote by S star, the collection of all mu star measurable sub sets. So, whenever A set E satisfies this condition, we say that is measurable, and let us put all the measurable sets in a collection, and call that as S upper star. So, as upper star is a collection of all mu star measurable sub sets of X. Here is 1 observation that a set is measurable, if and only if mu star of Y we want definitions says it should be equal to the sum of the pieces, but it is enough to say that mu star of Y is bigger than and equal to mu star of Y intersection E plus mu star of Y intersection E compliment; that is because for every set Y, Y is equal to Y intersection E plus union Y intersection E compliment, and mu star is always sub additive. So; that means, mu star of Y is always less than or equal to mu star of Y intersection E plus mu star. So, the inequality less than or equal to, is always true, because mu is monotone. So, to verify whether a set is measurable or not, one has to check only that mu star of Y should be bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E compliment for every set Y. So, only bigger than or equal to has to be checked and. So, we have to check only bigger than or equal to, and in case mu star of reset is infinity, then this inequality is obvious. So; that means, one has to check this inequality only for sets for which mu star of Y is finite.

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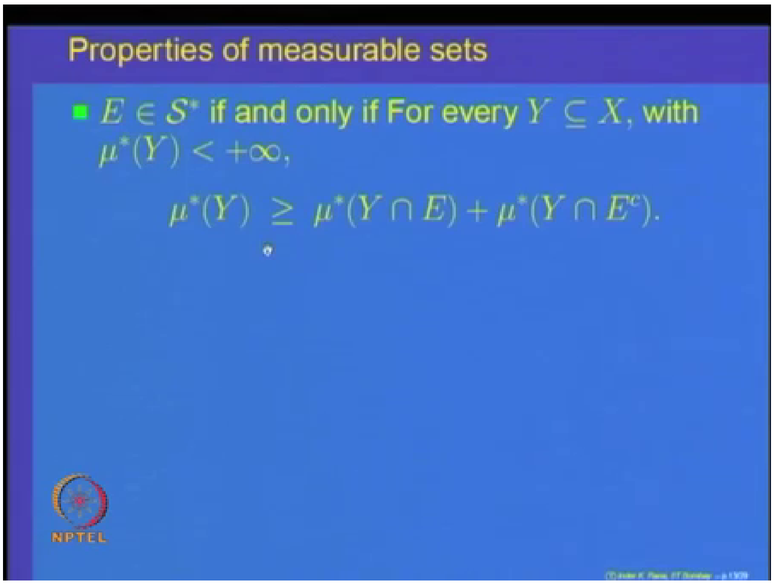

So, that gives us a very simplification saying that a set E is measurable, if and only if for every sub set Y of X with the property that mu of mu star. If Y is finite, one has to verify that mu star of Y is bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E compliment for every sub set Y with mu star of Y finite. So, this is the condition we are going to use again and again to prove whether a set or check whether a set E is mu star measurable or not.